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ON GLOBAL HYPOELLIPTICITY ON THE TORUS

By

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Summary: We use Fourier series and continued fractions to study the property of regularity of the global solutions of certain partial (or pseudo-) differential equations on the torus.

1. Introduction

Our main purpose in this paper is to study global hypoellipticity for a class of pseudo-differential operators on the n-Torus, T^n, $n \geq 2$, of the form

$$P = p(D_1^2) + e^{imx_1} + ae^{-imx_1},$$

where $a = \pm 1$, $m \in N$, $D_1 = (1/i)(\partial/\partial x_1)$ and p is a classical symbol satisfying the additional conditions:

$$p(0) = 0; \quad |p(1)| \geq 1; \quad |p(t^2)| > 2, \quad t \in N, \quad t \geq 2. \tag{1}$$

We recall that an operator P is said to be **globally hypoelliptic** (GH) on T^n if the properties $u \in \mathscr{D}'(T^n)$ and $Pu \in C^\infty(T^n)$ imply $u \in C^\infty(T^n)$.

Under hypothesis (1), we present a **necessary and sufficient** condition for the operators in (1) to be (GH). Our examples show, in particular, that in the case when $p(t) = \lambda t^2, 1 < \lambda < 2$, the situation $m > 1$ is different from the case $m = 1$, (see [5]); namely, when $m > 1$, the operator may fail to be (GH).

Other related works dealing with global hypoellipticity are [6], [7], [1]. In [6] the operators $D_1^2 + 2\cos x_1 - \lambda, \lambda \in C$, are considered; in [7] this result is extended to cover more general operators with the same perturbation of order zero. In [1], the effect of perturbations by terms of order zero is considered only in the case of constant coefficients. Further related recent works are [2], [3].

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2. The Main Theorem and Examples

We will use the notations: T^n, the n-dimensional torus, $n \geq 2$, $(T^n \simeq \boldsymbol{R}^n/\ (2\pi \boldsymbol{Z}^n))$; $\mathscr{D}'(T^n)$, the space of distributions on T^n; $C^\infty(T^n)$, the space of C^∞, complex valued functions on T^n; $x = (x_1, \dots, x_n)$, the variable in T^n; if $k = (k_1, \dots, k_n) \in \boldsymbol{Z}^n, |k| = |k_1| + \cdots + |k_n|$; and the continued fractions:

$$K_{j=1}^{\infty}((-1)^s/a_j) = \cfrac{(-1)^s}{a_1 + \cfrac{(-1)^s}{a_2 + \cfrac{(-1)^s}{a_3 + \cfrac{(-1)^s}{\vdots}}}}, \quad \text{where } a_j \in \boldsymbol{C}, s = 0 \text{ or } s = 1.$$

THEOREM 1. *Consider the pseudo-differential operator* $P = p(D_1^2) + e^{imx_1} + ae^{-imx_1}$, $m \in \boldsymbol{N} = \{1, 2, 3, \dots\}$, *acting on* $\mathscr{D}'(T^n)$ *where* $a = \pm 1$ *and* $p = p(t)$, $t \in \boldsymbol{Z}$, *is a classical symbol satisfying*

$$p(0) = 0; \quad |p(1)| \geq 1; \quad |p(t^2)| > 2, \quad t \geq 2, \quad t \in \boldsymbol{N}. \tag{1}$$

Let $a_{l,j} = p((mj + l)^2)/\sqrt{-a}$; $\tilde{a}_{l,j} = p((mj - l)^2)/\sqrt{-a}$, $j = 1, 2, \dots$; $t_l = K_{j=1}^{\infty}(1/a_{l,j})$, *if* $l = 0, 1, \dots, m-1$, *and* $\tilde{t}_l = K_{j=1}^{\infty}(1/\tilde{a}_{l,j}), g_l = t_l + \tilde{t}_l + p(l^2)/\sqrt{-a}$, *if* $l = 1, 2, \dots, m-1$.

Then P is globally hypoelliptic on T^n *if and only if*

$$g_1 g_2 \cdots g_{m-1} \neq 0. \tag{2}$$

In view of this result a question appears: what kind of operators satisfy condition (2)? In [5], the case $m = 1$ is dealt with: there, this condition is empty. However, when $m \geq 2$ it may be valid or not, as the following examples show.

EXAMPLE 1. Here we analyze some cases where we take a simple polynomial, but we put the perturbations $e^{imx_1} + ae^{-imx_1}, m \geq 2, a = \pm 1$. If we take $p(t) = t$ and $a = -1$, we have the operators $D_1^2 + 2i\,sin(mx_1)$, $m \geq 2$. In this case, $t_l + \tilde{t}_l + a_{l,0} > 0$, for each $l = 1, 2, \dots, m-1$. By taking $p(t) = t$ and $a = 1$, we have the operators $D_1^2 + 2\,cos(mx_1)$, $m \geq 2$, and it is easy to see that $t_l + \tilde{t}_l + a_{l,0} \neq 0$, for all $l = 1, 2, \dots, m-1$. Therefore, they are all (GH) on T^n.

EXAMPLE 2. Now we take $p = p(t)$ a **real** symbol that satisfies (1) and the additional condition: $|p(1)| > 2$ or $|p(1)| = 1$. (*)

Then, we have the operators $p(D_1^2) + e^{2ix_1} + ae^{-2ix_1}, a = \pm 1$, which are (GH) on T^n; indeed, we can show that $0 < |t_1| \leq 1$ and this implies $t_1^2 + 2a_{1,0}t_1 + a_{1,0}^2 + 1 \neq 0$, which, in turn, is shown to be equivalent to $t_1 + \bar{t}_1 + a_{1,0} \neq 0$. (In fact, when $a = -1$, condition (*) is not necessary, but it is quite sharp when $a = 1$, as will be seen later).

This last example implies, in particular, that $D_1^2 + 2\,cos(2x_1)$ is (GH). In the next one, we will analyze the polynomial $p(t) = \lambda t$, when $\lambda \in \boldsymbol{R}$, $1 < |\lambda| < 2$. In [5], it was shown that $\lambda D_1^2 + e^{ix_1} + ae^{-ix_1}$, $a = \pm 1, 1 < |\lambda| < 2$, is a globally hypoelliptic operator, but this is not always true for the operators $\lambda D_1^2 + e^{2ix_1} + ae^{-2ix_1}$, when $1 < |\lambda| < 2$.

Example 3. There exist λ_1, $\lambda_2 \in \boldsymbol{R}$, $1 < \lambda_1 < 2$, $-2 < \lambda_2 < -1$, such that the operators $Q_j = D_1^2 + (2/\lambda_j)\,cos(2x_1), j = 1, 2$, are not (GH) on T^n. To prove this, we show that $g_1(\lambda_j) = 0$, for some $\lambda_1 \in (-2, -1), \lambda_2 \in (1, 2)$.

This follows from the facts:

(a) $t_1(\lambda) = ih_1(\lambda)$, where $h_1(\lambda) = 1/\{9\lambda + \sum_{j=2}^{\infty}((-1)/[\lambda(2j+1)^2])\}$, and $g_1(\lambda) = i[h_1(\lambda) + 1/(\lambda - h_1(\lambda)) - \lambda]$;

(b) if we put $H(\lambda) = -ig_1(\lambda)$, since the polynomial $p(t) = \lambda t, 1 \leq \lambda \leq 2$, satisfies conditions (1), we can see that $H(\lambda)$ is a well defined and continuous functions of the variable λ on $[-2, -1] \cup [1, 2]$;

(c) we show that $H(1) > 0$ and $H(2) < 0$, and $H(-2) = -H(2), H(-1) = -H(1)$.

We remark that the result contained in Example 3 can be extended to include pseudo-differential operators $\lambda p(D_1^2) + 2\cos(2x_1)$, provided the symbol $p(t)$ satisfies $p(1) = 1$ and $p(3^2) > 0$, in addition to (1). Note that here we have $h_1(\lambda) = 1/\{\lambda p(3^2) + K_{j=2}^{\infty}((-1)/[\lambda p((2j+1)^2)])\}$, while $g_1(\lambda)$ is the same.

In fact $g_1(\lambda) = 0$ if and only if $h_1(\lambda) - \lambda = \pm 1$. Setting $G_1(\lambda) = h_1(\lambda) - \lambda + 1$, one can see that $G_1(1) > 0$ and $G_1(2) < 0$, hence there exists $\lambda_1 \in (1, 2)$ with $G_1(\lambda_1) = 0$, or $g_1(\lambda_1) = 0$. Similarly, set $G_2(\lambda) = h_1(\lambda) - \lambda - 1$, and get $G_2(-2) > 0$ and $G_2(-1) < 0$, and once again we are done.

3. Proof of the Theorem

Proof of Sufficiency: Let $u \in \mathscr{D}'(T^n)$ and $f \in C^\infty(T^n)$ satisfy $Pu = f$. We take the Fourier series: $u = \sum_{k \in Z^n} \hat{u}(k)e_k; f = \sum_{k \in Z^n} \hat{f}(k)e_k$, where $e_k(x) = e^{ik \cdot x}$,

$x \in T^n$. By substituting then in the equation above, we have:

$$p(k_1^2)\hat{u}(k) + \hat{u}(k - me_1) + a\hat{u}(k + me_1) = \hat{f}(k), \qquad k \in \boldsymbol{Z}^n, \tag{3}$$

where $e_1 = (1, 0, \dots, 0) \in \boldsymbol{Z}^n$. We separate $\boldsymbol{Z}^n$ in **m** different regions defined by $k_1 \equiv l(mod\, m)$, for $l = 0, 1, \dots, m-1$.

FIRST REGION: if $k_1 = mj$, $j \in \boldsymbol{Z}$, equation (3) corresponds to:

$$\begin{aligned} &p(m^2j^2)\hat{u}(mj;k') + \hat{u}(m(j-1);k') + a\hat{u}(m(j+1);k') = \hat{f}(mj;k'), \\ &\forall j \in \boldsymbol{Z}, \quad \forall k' = (k_2 \dots, k_n) \in \boldsymbol{Z}^{n-1}. \end{aligned} \tag{3'}$$

We denote: $a_{0,j} = p(m^2j^2)/\sqrt{-a}$; $f_{0,j} = \hat{f}(mj;k').(\sqrt{-a})^j$ and $v_{0,j} = \hat{u}(m(j-1);k').(\sqrt{-a})^j$, $j \in \boldsymbol{Z}$. Then, (3′) becomes:

$$v_{0,j+2} = a_{0,j}v_{0,j+1} + v_{0,j} - f_{0,j}, \quad j \in \boldsymbol{Z}. \tag{4}$$

Solving (4) for $j \geq 1$, we put the initial conditions $v_{0,1} = \alpha_0$, $v_{0,2} = \beta_0$ (which will be determined later), and we have the solution:

$$v_{0,j} = \alpha_0 p_{0,j} + (\beta_0 - \gamma_{0,1})q_{0,j} + r_{0,j}, \tag{5}$$

where $p_{0,j}, q_{0,j}$ are given (as in [5]) by:

$$\begin{cases} p_{0,1} = 1; p_{0,2} = 0; p_{0,j+2} = a_{0,j}p_{0,j+1} + p_{0,j} \\ q_{0,1} = 0; q_{0,2} = 1; q_{0,j+2} = a_{0,j}q_{0,j+1} + q_{0,j}, \qquad j = 1, 2, \dots \end{cases} \tag{6}$$

and

$$r_{0,j} = T_{0,1} + T_{0,2} - f_{0,j-2}, \tag{7}$$

where:

$$\begin{cases} T_{0,1} = (p_{0,j} - t_0 q_{0,j}) \sum_{\nu=1}^{j-3} f_{0,\nu}(-1)^\nu q_{0,\nu+1} \\ T_{0,2} = q_{0,j}\{\gamma_{0,1} - \sum_{\nu=1}^{j-3} f_{0,\nu}(-1)^\nu (p_{0,\nu+1} - t_0 q_{0,\nu+1})\}, \end{cases} \tag{8}$$

$$\gamma_{0,1} = \sum_{\nu=1}^{\infty} f_{0,\nu}(-1)^\nu (p_{0,\nu+1} - t_0 q_{0,\nu+1}) \tag{9}$$

$$t_0 = \sum_{j=2}^{\infty} (-1)^j/(q_{0,j}q_{0,j+1}) = \lim_{j\to\infty}(p_{0,j}/q_{0,j}). \tag{10}$$

We can show that t_0 is a well defined non-zero number and, under conditions (1), $p_{0,j}, q_{0,j}$ and t_0 satisfy (as in [5]):

(A_0) $p_{0,j+1}q_{0,j} - p_{0,j}q_{0,j+1} = (-1)^j, j = 1,2,\dots$

(B_0) $\exists K_0 > 1$; $|q_{0,j}| \geq K_0^{j-3}$; $|p_{0,j}| \geq K_0^{j-3}$, $\forall j \geq 4$.

(C_0) $\exists C_1 > 0$, $\bar{j}_0 \geq 1$, independent of j, so that $|p_{0,j} - t_0 q_{0,j}| \leq C_1 |q_{0,j}|^{-1}$, $j \geq \bar{j}_0$.

(D_0) $t_0 \neq 0$.

We can verify that $T_{0,1}$, $T_{0,2}$ are rapidly decreasing as $j, |k'| \to \infty$ by the same arguments as in [5], and we conclude that $r_{0,j}$ is rapidly decreasing too. Since $u \in D'(T^n)$, $v_{0,j}$ has polynomial growth as $j \to \infty$. So, there exists C_2, $s > 0$ such that:

$$\frac{v_{0,j}}{q_{0,j}} = \alpha_0 \frac{p_{0,j}}{q_{0,j}} + (\beta_0 - \gamma_{0,1}) + \frac{r_{0,j}}{q_{0,j}} \leq \frac{C_2 j^s}{q_{0,j}}.$$

By letting $j \to \infty$, from (B_0), (C_0), we have:

$$\alpha_0 t_0 + \beta_0 - \gamma_{0,1} = 0. \tag{11}$$

Now we shall solve (4) for $j \leq 0$ by changing $j \leftrightarrow -j$ in (4). Since $p(0) = 0$, we have $v_{0,0} = \beta_0 + f_{0,0}$ and we define: $w_{0,j} = (-1)^{2-j} v_{0,2-j}$. Then, equation (4) becomes:

$$w_{0,j+2} = a_{0,-j} w_{0,j+1} - w_{0,j} - g_{0,j}, \quad j = 1,2,\dots, \tag{12}$$

where $g_{0,j} = (-1)^{j+1} f_{0,-j}$, with $w_{0,1} = -\alpha_0; w_{0,2} = \beta_0 + f_{0,0}$. This last problem has its solution as in (5):

$$w_{0,j} = -\alpha_0 p_{0,j} + (\beta_0 + f_{0,0} + \gamma_{0,2}) q_{0,j} + \tilde{r}_{0,j},$$

where $\gamma_{0,2} = \sum_{v=1}^{\infty} f_{0,-v}(p_{0,v+1} - t_0 q_{0,v+1})$, and $r_{0,j}$ is defined in an analogous way to $r_{0,j}$ (see (7)). As before, we can show that $\tilde{r}_{0,j}$ is rapidly decreasing as $j, |k'| \to \infty$, and it follows that

$$-\alpha_0 t_0 + \beta_0 + f_{0,0} + \gamma_{0,2} = 0. \tag{13}$$

From (11) and (13), we get:

$$\alpha_0 = \frac{(\gamma_{0,1} + \gamma_{0,2} + f_{0,0})}{2t_0}; \quad \beta_0 = \frac{(\gamma_{0,1} - \gamma_{0,2} - f_{0,0})}{2}. \tag{14}$$

We can prove that $\gamma_{0,1}$ and $\gamma_{0,2}$ are rapidly decreasing as $|k'| \to \infty$. Since $f_{0,0} = \hat{f}(0,k')$ has the same property, α_0 decreases rapidly when $|k'| \to \infty$. Taking (5) into account, it remains to prove that $M = [\alpha_0 p_{0,j} + (\beta_0 - \gamma_{0,1})q_{0,j}]$ is also rapidly decreasing. In fact, we have:

$$\left|\alpha_0 \frac{p_{0,j}}{q_{0,j}} + \beta_0 - \gamma_{0,1}\right| \overset{(11)}{=} |\alpha_0| \left|\frac{p_{0,j}}{q_{0,j}} - t_0\right| \overset{(B_0)(C_0)}{\Rightarrow} |\alpha_0 p_{0,j} + (\beta_0 - \gamma_{0,1})q_{0,j}| \leq |\alpha_0| \frac{C_1}{K_1^{j-3}},$$

if $j \geq \bar{j}_0 \geq 4$. Since α_0 is rapidly decreasing as $|k'| \to \infty$ and the same occurs with C_1/K_0^{j-3} as $j \to \infty$, it follows that M decreases rapidly as $j, |k'| \to \infty$, and the same is true for $v_{0,j}$, $j \geq 1$. By an analogous argument, we show that $w_{0,j} = (-1)^{2-j} v_{0,2-j}$, i.e. $v_{0,-j}$, is rapidly decreasing as $j, |k'| \to \infty$.

OTHER REGIONS: we fix $l \in \{1, 2, \dots, m-1\}$; if $k_1 = mj + l$, $j \in \boldsymbol{Z}$, equation (3) corresponds to:

$$\begin{aligned} &p((mj+l)^2)\hat{u}(mj+l;k') + \hat{u}(m(j-1)+l;k') + a\hat{u}(m(j+1)+l;k') \\ &\quad = \hat{f}(mj+l;k'), \quad \forall j \in \boldsymbol{Z}, \quad \forall k' \in \boldsymbol{Z}^{n-1}. \end{aligned} \tag{15}$$

We will solve (15) for $j \geq 1$ and denote:

$$a_{l,j} = p((mj+l)^2)/\sqrt{-a}; \quad f_{l,j} = \hat{f}(mj+l;k')(\sqrt{-a})^j,$$

$$v_{l,j} = \hat{u}(m(j-1)+l;k')(\sqrt{-a})^j, \quad j = 1, 2, 3, \dots$$

Thus, (15) becomes:

$$v_{l,j+2} = a_{l,j} v_{l,j+1} + v_{l,j} - f_{l,j}, \qquad j = 1, 2, \dots \tag{16}$$

We put $v_{l,1} = \alpha_l$; $v_{l,2} = \beta_l$ and we will have the solution:

$$v_{l,j} = \alpha_l p_{l,j} + (\beta_l - \gamma_{l,1}) q_{l,j} + r_{l,j}, \tag{17}$$

where $p_{l,j}$, $q_{l,j}$ are given as in (6), but now with new $a_{l,j}$ instead of $a_{0,j}$. The numbers $r_{l,j}$ have the same expression as before, with $f_{l,v} = \hat{f}(mv+l;k')(\sqrt{-a})^v$ and $\gamma_{l,1} = \sum_{v=1}^{\infty} f_{v,0}(-1)^v (p_{l,v+1} - t_l q_{l,v+1})$, where $t_l = \sum_{j=2}^{\infty} (-1)^j/(q_{l,j} q_{l,j+1}) = \lim_{j \to \infty} (p_{l,j}/q_{l,j})$. We can show that $T_{l,1}$, $T_{l,2}$ (defined as before) are rapidly decreasing as $j, |k'| \to \infty$. Since $u \in \mathscr{D}'(T^n)$, we get as in the first region:

$$\alpha_l t_l + \beta_l - \gamma_{l,1} = 0. \tag{18}$$

We have $v_{l,2} = a_{l,0} v_{l,1} + v_{l,0} - f_{l,0}$, where $a_{l,0} = p(l^2)/\sqrt{-a} \neq 0$ and $f_{l,0} = \hat{f}(l;k')$. This implies $v_{l,0} = \beta_l - a_{l,0}\alpha_l + f_{l,0}$. Now, if $j \leq 0$, by changing $j \leftrightarrow -j$ in

(16) and defining $w_{l,j} = (-1)^{2-j} v_{l,2-j}$, (16) becomes:

$$\begin{cases} w_{l,j+2} = a_{l,-j} w_{l,j+1} + w_{l,j} - (-1)^{j+1} f_{l,-j}, \quad \text{with} \\ w_{l,1} = -\alpha_l; w_{l,2} = v_{l,0} = \beta_l - a_{l,0}\alpha_l + f_{l,0}. \end{cases} \tag{19}$$

We define new $\tilde{p}_{l,j}$, $\tilde{q}_{l,j}$, as in (6), by putting $a_{l,-j}$ instead of $a_{0,j}$.

The solution of (19) is given by:

$$w_{l,j} = -\alpha_l \tilde{p}_{l,j} + (\beta_l + f_{l,0} - a_{l,0}\alpha_l + \gamma_{l,2})\tilde{q}_{l,j} + \tilde{r}_{l,j}$$

where $\tilde{r}_{l,j}$ is the same as before, but considering now $\tilde{p}_{l,j}$, $\tilde{q}_{l,j}$ and $\tilde{t}_l = \lim_{j\to\infty} (\tilde{p}_{l,j}/\tilde{q}_{l,j})$, $\gamma_{l,2} = \sum_{v=l}^{\infty} f_{l,-v}(\tilde{p}_{l,v+1} - \tilde{t}_l \tilde{q}_{l,v+1})$.

Notice that:

$$\frac{p_{l,j}}{q_{l,j}} = \cfrac{1}{a_{l,1} + \cfrac{1}{a_{l,2} + \cfrac{1}{\ddots + \cfrac{1}{a_{l,j-2}}}}} \quad \text{and} \quad \frac{\tilde{p}_{l,j}}{\tilde{q}_{l,j}} = \cfrac{1}{\tilde{a}_{l,1} + \cfrac{1}{\tilde{a}_{l,2} + \cfrac{1}{\ddots + \cfrac{1}{\tilde{a}_{l,j-2}}}}}.$$

t_l and $\tilde{t}_l$ can be written respectively as $K_{j=1}^{\infty}(1/a_{l,j})$ and $K_{j=1}^{\infty}(1/\tilde{a}_{l,j})$, which is in accordance with the statement of theorem 1.

Since $w_{l,j}$ has polynomial growth, it follows as before, that:

$$-\alpha_l(\tilde{t}_l + a_{l,0}) + \beta_l + \gamma_{l,2} + f_{l,0} = 0. \tag{20}$$

Since we have the hypothesis $a_{l,0} + t_l + \tilde{t}_l \neq 0$, α_l and β_l are well determined and, from (18), (20), we get:

$$\alpha_l = \frac{f_{l,0} + \gamma_{l,2} + \gamma_{l,1}}{a_{l,0} + t_l + \tilde{t}_l}, \quad \beta_l = \gamma_{l,1} - \frac{(f_{l,0} + \gamma_{l,2} + \gamma_{l,1})t_l}{a_{l,0} + t_l + \tilde{t}_l}. \tag{21}$$

With this expression we can show, like in the first region, that α_l is rapidly decreasing when $|k'| \to \infty$, and $v_{l,j}$, $v_{l,-j}$ are rapidly decreasing as $j, |k'| \to \infty$. From the results obtained in the first and in the other $(m-1)$ regions, we conclude that $\hat{u}(j,k')$, $j \in \boldsymbol{Z}$, is rapidly decreasing as $|j|; |k'| \to \infty$, and $u \in C^\infty(T^n)$.

Proof of Necessity: Suppose that $g_{\bar{l}} = 0$ for some $\bar{l} \in \{1, 2, \ldots, m-1\}$. We will construct a solution $u \in \mathscr{D}'(T^n) \setminus C^\infty(T^n)$ to the equation $Pu = 0$, by describing its Fourier coefficients $\hat{u}(k)$.

We put $\hat{u}(mj+l;k')=0$, $\forall l \in \{0,1,\dots,\bar{l}-1,\bar{l}+1,\dots,m-1\}$, $\forall j \in \boldsymbol{Z}$, $k' \in \boldsymbol{Z}^{n-1}$. The others, $\hat{u}(mj+\bar{l};k')$, must satisfy:

$$\begin{aligned}&p(mj+\bar{l};k')\hat{u}(mj+\bar{l};k')+\hat{u}(m(j-1)+\bar{l};k')+a\hat{u}(m(j+1)+\bar{l};k')=0,\\ &\forall j \in \boldsymbol{Z},\quad k' \in \boldsymbol{Z}^{n-1}.\end{aligned} \tag{22}$$

If we define $v_{\bar{l},j}=\hat{u}(m(j-1)+\bar{l};k')(\sqrt{-a})^j$; $a_{\bar{l},j}=p((mj+\bar{l})^2)/\sqrt{-a}$, then (22) will be equivalent to:

$$v_{\bar{l},j+2}=a_{\bar{l},j}v_{\bar{l},j+1}+v_{\bar{l},j},\quad j \in \boldsymbol{Z}. \tag{23}$$

We first solve (23) for $j \geq 1$, by putting the initial conditions: $v_{\bar{l},1}=-1/t_{\bar{l}}=\alpha_{\bar{l}}$ and $v_{\bar{l},2}=1=\beta_{\bar{l}}$, where $t_{\bar{l}}=\sum_{j=2}^{\infty}(-1)^j/(q_{\bar{l},j}q_{\bar{l},j+1})(p_{\bar{l},j}, q_{\bar{l},j}$, given as in (6), depend only on the symbol p). Thus, the solution is

$$v_{\bar{l},j}=(-1/t_{\bar{l}})p_{\bar{l},j}+q_{\bar{l},j},\quad \forall j \geq 1. \tag{24}$$

We can see that $\alpha_{\bar{l}}$ and $\beta_{\bar{l}}$ satisfy:

$$\alpha_{\bar{l}}t_{\bar{l}}+\beta_{\bar{l}}=0. \tag{25}$$

Notice that $v_{\bar{l},j}$ has polynomial growth as $j, |k'| \to \infty$; (indeed, p satisfies (1), $p_{\bar{l},j}, q_{\bar{l},j}$ still satisfy some conditions like in (A_0), (B_0), (C_0), and we can write: $\dfrac{|v_{\bar{l},j}|}{|q_{\bar{l},j}|} \leq |\alpha_{\bar{l}}|\left|\dfrac{p_{\bar{l},j}}{q_{\bar{l},j}}-t_{\bar{l}}\right|$. Thus, we conclude that, in fact, $v_{\bar{l},j}$ is bounded).

Now, solving (23) for $j \leq 0$, by changing again $j \leftrightarrow -j$ and putting $w_{\bar{l},j}=(-1)^{2-j}v_{\bar{l},2-j}$, $j \geq 1$, equation (23) becomes:

$$w_{\bar{l},j+2}=a_{\bar{l},j}w_{\bar{l},j+1}+w_{\bar{l},j},\quad j \geq 1, \tag{26}$$

and its solution is $w_{\bar{l},j}=(1/t_{\bar{l}})\tilde{p}_{\bar{l},j}+(1+a_{\bar{l},0}/t_{\bar{l}})\tilde{q}_{\bar{l},j}$, $\forall j \geq 1$ (recall that $\tilde{p}_{\bar{l},j}, \tilde{q}_{\bar{l},j}$ are defined as in (6) putting $a_{\bar{l},-j}$ instead of $a_{0,j}$).

Since $g_{\bar{l}}=0$, using (25), we get:

$$-\alpha_{\bar{l}}(\tilde{t}_{\bar{l}}+a_{\bar{l},0})+\beta_{\bar{l}}=0. \tag{27}$$

By arguments analogous to the ones above for $\tilde{p}_{\bar{l},j}$, $\tilde{q}_{\bar{l},j}$ and using (27), we conclude that $w_{\bar{l},j}$ has polynomial growth as $j, |k'| \to \infty$ (in fact, they are bounded too).

So, we have found a solution $u \in \mathscr{D}'(T^n)$, given by $\hat{u}(mj + l, k') = 0, \forall l \in \{0, 1, \dots, \bar{l} - 1, \bar{l} + 1, \dots, m - 1\}, \forall j \in \boldsymbol{Z}$, and by $\hat{u}(m(j-1) + \bar{l}; k') = (\sqrt{-a})^{-j}[(-1/t_{\bar{l}})p_{\bar{l},j} + q_{\bar{l},j}]$ and $\hat{u}(m(1-j) + \bar{l}; k') = (-1)^{j-2}(\sqrt{-a})^{j-2}[(1/t_{\bar{l}})\tilde{p}_{\bar{l},j} + (1 + a_{\bar{l},0}/t_{\bar{l},j})\tilde{q}_{\bar{l},j}]$, $\forall j \geq 1, k' \in \boldsymbol{Z}^{n-1}$.

Finally, we note that $\hat{u}(m + \bar{l}; k') = (\sqrt{-a})^{-2} \cdot v_{\bar{l},2} = (\sqrt{-a})^{-2}$, $\forall k'$. Hence, $\hat{u}$ does not decrease rapidly, and so $u \notin C^\infty(T^n)$.

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