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A NORMAL FORM FOR ARITHMETICAL DERIVATIONS IMPLYING THE ω-CONSISTENCY OF ARITHMETIC

By

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Abstract. We give a normal form theorem for arithmetical derivations. It is proved by induction up to ε_1 and implies the ω-consistency of arithmetic.

1. Introduction

Mints [6] investigated some kinds of normal form theorems for LK (cf. [10]), which can be considered as extensions of the cut elimination theorem. In order to explain his result, we shall state some notions. A variable in a derivation is said to be *redundant* if it occurs in an upper sequent of an inference I and does not occur in the lower sequent of I and is not used as the eigenvariable of I. A logical inference J in a derivation is said to be *reducible* with respect to LK if one of the auxiliary formula of J is derivable (refutable) in LK provided that it belongs to the antecedent (succedent) of the sequent in which it occurs. Then, Mints proved the following theorem:

THEOREM (Mints). *Assume that the language of LK contains at least one constant symbol. Let π be a derivation. Then we can transform π into a cut free derivation π' which satisfies the following conditions*:

(1) *The end sequent of π' is that of π.*

(2) *π' includes no redundant variables.*

(3) *π' includes no reducible inferences w.r.t. LK.*

On the other hand, normal forms for arithmetical derivations are investigated by Hinata [3], Jervell [4] and others. Hinata's normal form theorem is proved by induction up to ε_0 and implies the 1-consistency of arithmetic.

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In this paper, we shall give an extended form of Hinata's result, which can be considered as an analogue of Mints' Theorem. It is proved by induction up to ε_1 and implies the ω-consistency of arithmetic.

As for the ω-consistency of arithmetic, it is known that the ω-consistency of arithmetic is proved by induction up to ε_1 and can not be proved by induction up to $\alpha(\alpha < \varepsilon_1)$ (cf. [2], [5], [7] and [9]).

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2. Normal form theorem

In this paper, we shall consider the following system PA. The nonlogical symbols of PA consist of the following symbols:

(1) Constant symbol: 0;

(2) Function symbols:$'$ (successor) and $\bar{f}$ for each primitive recursive function f;

(3) Predicate symbol: $=$.

Let LK^* be the system obtained from LK by restricting its initial sequents to initial sequents which consist of atomic formulas and by replacing

$$\supset\text{: } right : \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B} \quad \text{by} \quad \supset\text{: } right : \frac{A, \Gamma \to \Delta}{\Gamma \to \Delta, A \supset B} \quad \text{and} \quad \frac{\Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}.$$

PA^- is the system obtained from LK^* by adding the usual initial sequents for arithmetic, which consist of atomic formulas. And PA is the system obtained from PA^- by adding the following inference rule *ind*:

$$\frac{\Gamma \to \Delta, A(0) \quad A(a), \Gamma \to \Delta, A(a') \quad A(t), \Gamma \to \Delta}{\Gamma \to \Delta},$$

where the free variable a does not occur in $A(t)$, Γ and Δ. This free variable is called the *eigenvariable*, and $A(a)$ and t is called the *induction formula* and the *induction term*, respectively. And also $A(0), A(a), A(a')$ and $A(t)$ are called *auxiliary formulas*. Ind is said to be *constant normal* if its induction formula contains at least one occurrence of its eigenvariable and its induction term contains at least one free variable.

DEFINITION 2.1. Let Γ be a sequence $A_1, \ldots, A_n$ of formulas. Let $\langle i_1, i_2, \ldots, i_k \rangle$ be a sequence of natural numbers such that $1 \le i_1 < i_2 < \cdots <$

$i_k \leq n$. Then, the sequence $A_{i_1}, \ldots, A_{i_k}$ is called a *part* of Γ. Γ^* is used to denote a part of Γ. Let $\Lambda \to \Pi$ be a sequent. Then $\Lambda^* \to \Pi^*$ is called a *part* of $\Lambda \to \Pi$.

DEFINITION 2.2. Let S be a sequent and S^* a part of S. And let π be a derivation of S and C a formula in π. Then C is said to be (S^*)-*implicit* if a descendant (cf. [10]) of C is in S^* or a cut formula or an auxiliary formula. Otherwise C is said to be (S^*)-*explicit*. An inference in π is called (S^*)-*implicit* or (S^*)-*explicit* according as its principal formula is (S^*)-implicit or (S^*)-explicit.

DEFINITION 2.3. A variable in a derivation is said to be *redundant* if it occurs in an upper sequent of an inference I and does not occur in the lower sequent of I and is not used as the eigenvariable of I.

DEFINITION 2.4. Let T be a subtheory of PA. And let π be a PA-derivation. Then a logical inference I in π is said to be *reducible with respect to T* if one of the auxiliary formulas of I is derivable (refutable) in T provided that it belongs to the antecedent (succedent) of the sequent in which it occurs.

DEFINITION 2.5. Let S be a sequent and S^* a part of S. And let π be a derivation of S. We consider the following conditions (1) $\sim$ (5) on π.

(1) There are no redundant variables.
(2) There are no cuts except inessential ones (cf. [10]).
(3) There are no inds except constant normal ones.
(4) There are no inferences which are reducible with respect to PA^-.
(5) There are no (S^*)-explicit inferences which are reducible with respect to PA.

π is said to be *irreducible* if it satisfies the conditions (1) $\sim$ (3). And π is said to be PA^--*irreducible* or (S^*)-*strongly irreducible* according as it satisfies the conditions (1) $\sim$ (4) or (1) $\sim$ (5), respectively. Especially, we say that π is *strongly irreducible* if it is $(\to)$-strongly irreducible.

DEFINITION 2.6. Let T be a theory which contains arithmetic. Then T is said to be ω-consistent if it satisfies the following condition: For any formula $A(a)$ which does not have free variables except a, if $\exists x A(x)$ is derivable in T, then there exists a numeral n such that $\neg A(n)$ is not derivable in T. Let $k \geq 1$. Then the restriction of the ω-consistency of T to formulas $A \in \Sigma_{k-1}$ is called the k-consistency of T.

As for the k-consistency of a theory which contains arithmetic, the following fact is known.

FACT (Smoryński [8]). *Let T be a theory which contains arithmetic. Then, for $k = 1, 2, T$ is k-consistent iff, for any Σ_k-sentence A, if A is derivable in T, then A is true.*

The following theorem is proved by induction up to ε_0 in [3].

THEOREM 1 (Hinata). *We can transform any derivation into an irreducible one with the same end sequent.*

The following corollaries are direct consequences of Theorem 1.

COROLLARY 1. *Let $\exists x R(x)$ be an existential sentence. Assume that $\exists x R(x)$ is derivable in PA. Then $\exists x R(x)$ is derivable in PA^-.*

COROLLARY 2. *PA is 1-consistent.*

In this paper, we shall show the following theorem by induction up to ε_1.

THEOREM 2. *We can transform any derivation into a strongly irreducible derivation with the same end sequent.*

COROLLARY 3. *PA is ω-consistent.*

PROOF. Let $A(a)$ be an arbitrary formula such that it has no free variables except a and $A(n)$ is derivable in PA for any numeral n. Then, it suffices to show that $\forall x A(x) \rightarrow$ is not derivable in PA. Assume that $\forall x A(x) \rightarrow$ is derivable in PA. Then, there exists a strongly irreducible derivation of $\forall x A(x) \rightarrow$ by Theorem 2. Let π be a strongly irreducible derivation of $\forall x A(x) \rightarrow$. Assume that π includes at least one boundary inference (cf. Definition 3.4). Note that the end-place (cf. Definition 3.4) of π contains no free variable. So, no inds belong to the boundary of π (cf. Definition 3.4). Thus each inference which belongs to the boundary of π must be of the form:

$$\frac{A(t), \Gamma \rightarrow \Delta}{\forall x A(x), \Gamma \rightarrow \Delta,}$$

where Γ consists of $\forall x A(x)$ or atomic formulas and Δ consists of atomic formulas. Because, if $\Gamma(\Delta)$ contains a formula B which includes at least one logical symbol, then B occurs in the antecedent (succedent) of the end sequent of π. Since π contains no redundant variables, t contains no free variables. Since there is a numeral n such that $t = n$ is derivable in PA, $\rightarrow A(t)$ is derivable in PA. But it contradicts our assumption. So, π includes no boundary inferences. Thus we can transform π into a derivation π' whose end sequent is a part of the end sequent of π and which includes no free variables, no weakenings, no essential cuts, no inds and no logical inferences. Since any formula in π' doesn't include logical symbols, the end sequent of π' is $\rightarrow$. But, it is clear that there is not such a derivation. ■

3. Preliminaries

In this section, we shall define some necessary notions and state some propositions, which will be used in the next section.

DEFINITION 3.1. For any formula A, the *degree* $d(A)$ of A is defined inductively as follows:

(1) $d(A) = 1$, if A is atomic;

(2) $d(B_1 \wedge B_2) = d(B_1 \vee B_2) = d(B_1 \supset B_2) = max\{d(B_1) + 1, d(B_2) + 1\}$;

(3) $d(\neg B) = d(\forall x B) = d(\exists x B) = d(B) + 1$.

DEFINITION 3.2. Let I be an inference. Then the *degree* $d(I)$ of I is defined as follows:

$$d(I) = \begin{cases} max\{d(A) | A \text{ is an auxiliary formula of } I\}, & \text{if } I \text{ is a logical inference,} \\ \text{the degree of a cut formula of } I, & \text{if } I \text{ is a cut,} \\ \text{the degree of the induction formula of } I, & \text{if } I \text{ is an ind,} \\ 0, & \text{otherwise.} \end{cases}$$

DEFINITION 3.3. Let π be a derivation and S a sequent in π. For any natural number ρ, the *height* $h_\rho(S; \pi)$ based on ρ of S in π is defined as follows:

(1) $h_\rho(S; \pi) = \rho$, if S is the end sequent of π.

(2) Let S be one of the upper sequents of an inference I in π and S' the lower sequent of I. Assume that $h_\rho(S'; \pi)$ is defined. Then,

$$h_\rho(S; \pi) = max\{h_\rho(S'; \pi), d(I)\}.$$

DEFINITION 3.4. Let π be a derivation. We say that a sequent S in π *belongs to the end-place* of π if neither a logical inference nor an ind occurs below S in π. And we say that an inference I in π *belongs to the boundary* of π or is a *boundary inference* of π if the lower sequent of I belongs to the end-place of π and the upper sequents of I do not belong to the end-place of π.

NOTATION. Let α and β be ordinals. Then $\alpha \sharp \beta$ is used to denote the natural sum of α and β. And $\alpha \times \beta$ is used to denote the natural product of α and β. Let $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ be in Cantor normal form and n a finite ordinal. Then, we have the following equations:

$$(1)\ \alpha \times n = \overbrace{\alpha \sharp \cdots \sharp \alpha}^{n \text{ times}}; \quad (2)\ \beta \times \omega = \omega^{\beta_1+1} + \cdots + \omega^{\beta_m+1}.$$

DEFINITION 3.5. Let $\check{S}$ be a sequent and $\check{S}^*$ a part of $\check{S}$. And let π be a derivation of $\check{S}$ and p a natural number. To each sequent S in π and each inference I in π, we assign ordinals $O_p(S;\pi;\check{S}^*)$, $O_p(I;\pi;\check{S}^*)$, respectively, as follows:

(1) If S is an initial sequent,

$$O_p(S;\pi;\check{S}^*) = 1.$$

(2) Let S_i $(1 \le i \le n)$ be the upper sequents of I. Assume that $O_p(S_i;\pi;\check{S}^*)$ are defined for each $1 \le i \le n$.

(2.1) If I is a weak inference,

$$O_p(I;\pi;\check{S}^*) = O_p(S_1;\pi;\check{S}^*).$$

(2.2) If I is $(\check{S}^*)$-explicit,

$$O_p(I;\pi;\check{S}^*) = \begin{cases} O_p(S_1;\pi;\check{S}^*)\sharp\varepsilon_0, & \text{if } I \text{ has one upper sequent,} \\ O_p(S_1;\pi;\check{S}^*)\sharp O_p(S_2;\pi;\check{S}^*)\sharp\varepsilon_0, & \text{if } I \text{ has two upper sequents.} \end{cases}$$

(2.3) If I is $(\check{S}^*)$-implicit,

$$O_p(I;\pi;\check{S}^*) = \begin{cases} O_p(S_1;\pi;\check{S}^*)\sharp\omega^{d(I)}, & \text{if } I \text{ has one upper sequent,} \\ O_p(S_1;\pi;\check{S}^*)\sharp O_p(S_2;\pi;\check{S}^*)\sharp\omega^{d(I)}, & \text{if } I \text{ has two upper sequents.} \end{cases}$$

(2.4) If I is a cut,

$$O_p(I;\pi;\check{S}^*) = O_p(S_1;\pi;\check{S}^*)\sharp Op(S_2;\pi;\check{S}^*).$$

(2.5) If I is an ind,

$$O_\rho(I;\pi;\check{S}^*) = O_\rho(S_1;\pi;\check{S}^*)\sharp(O_\rho(S_2;\pi;\check{S}^*)\times\omega)\sharp O_\rho(S_3;\pi;\check{S}^*)\sharp\omega^{d(I)}.$$

(3) Let S be the lower sequent of I. And let σ be the height based on ρ of an upper sequent of I and τ the height based on ρ of S. Then,

$$O_\rho(S;\pi;\check{S}^*) = \omega_{\sigma-\tau}(O_\rho(I;\pi;\check{S}^*)).$$

We define $O_\rho(\pi;\check{S}^*)$ by $O_\rho(S;\pi;\check{S}^*)$, where S is the end sequent of π.

The following propositions are proved easily.

PROPOSITION 1. *Assume that π is a derivation. Let S be a sequent in π. Let ρ and σ be natural numbers such that $\rho \le \sigma$. Then, $h_\rho(S;\pi) \le h_\sigma(S;\pi)$.*

PROPOSITION 2. *Suppose that π is a derivation of $\check{S}$. Assume that $\check{S}^*$ is a part of $\check{S}$. Let ρ and σ be natural numbers such that $\rho \le \sigma$. Let S be a sequent in π. Then, $\omega_{h_\rho(S;\pi)}(O_\rho(S;\pi;\check{S}^*)) \le \omega_{h_\sigma(S;\pi)}(O_\sigma(S;\pi;\check{S}^*))$.*

We can prove the next corollary by the same way as in Lemma 12.7 in [10], using the property that the ordinal operations $\sharp$, $\times$ and exponential are strictly increasing.

PROPOSITION 3. *Suppose that π is of the form:*

$$\begin{array}{c} \pi_1 \vdots \\ \Lambda \to \Pi \\ \vdots \\ \Gamma \to \Delta. \end{array}$$

Let π_1' be a derivation of $\Lambda, \Gamma' \to \Delta', \Pi$. Then we define π' as follows:

$$\begin{array}{c} \pi_1' \vdots \\ \Lambda, \Gamma' \to \Delta', \Pi \\ \vdots \\ \Gamma, \Gamma' \to \Delta', \Delta. \end{array}$$

Let $\Gamma^* \to \Delta^*$ be a part of $\Gamma \to \Delta$. And let Γ'^* be a part of Γ' and Δ'^* a part of Δ'.

Assume that

$$O_0(\Lambda, \Gamma' \to \Delta', \Pi; \pi'; \Gamma^*, \Gamma'^* \to \Delta'^*, \Delta^*) < O_0(\Lambda \to \Pi; \pi; \Gamma^* \to \Delta^*).$$

Then $O_0(\pi'; \Gamma^*, \Gamma'^* \to \Delta'^*, \Delta^*) < O_0(\pi; \Gamma^* \to \Delta^*)$.

4. Proof of Theorem 2

We shall prove the following Theorem 3 which clearly implies Theorem 2.

THEOREM 3. *Assume that $\check{\pi}$ is a derivation of $\check{S}$. Let $\check{S}^*$ be a part of $\check{S}$. Then we can transform $\check{\pi}$ into a derivation whose end sequent is $\check{S}$ and which is $(\check{S}^*)$-strongly irreducible.*

PROOF. We shall prove this statement by induction on $O_0(\check{\pi}; \check{S}^*)$. Assume that $\check{S}$ is of the form $\Gamma \to \Delta$ and $\check{S}^*$ is of the form $\Gamma^* \to \Delta^*$.

As usual, we transform $\check{\pi}$ into a derivation π which satisfies the following conditions:

1) π includes no redundant variables.

2) The end sequent of π is $\check{S}$.

3) If I is a weakening in the end place of π, then every inference below I is an exchange or a weakening.

4) $O_0(\pi; \check{S}^*) \leq O_0(\check{\pi}; \check{S}^*)$.

We shall classify π into some cases. When we are concerned with a case in the following, we suppose that π satisfies none of the conditions of the preceding cases.

From now on, the letter "S" in "$\Lambda \overset{S}{\to} \Pi$" is used to denote the sequent $\Lambda \to \Pi$.

(1) The case where π includes at least one $(\check{S}^*)$-explicit inference which is reducible w.r.t. *PA*.

We shall transform π into a derivation π' by the same way as in [1]. Let I be one of $(\check{S}^*)$-explicit inferences which are reducible w.r.t. *PA*. We shall consider the case that I is a $\supset$: *left*. The other cases are treated similarly.

Assume that π is of the form:

$$\dfrac{\begin{matrix} \pi_1 \vdots \\ \Lambda_1 \overset{S_1}{\to} \Pi_1, A \end{matrix} \quad \begin{matrix} \pi_2 \vdots \\ B, \Lambda_2 \overset{S_2}{\to} \Pi_2 \end{matrix}}{A \supset B, \Lambda_1, \Lambda_2 \overset{S}{\to} \Pi_1, \Pi_2} I$$
$$\vdots \qquad .$$

Assume that $h_0(S_1;\pi)=\rho$ and $h_0(S;\pi)=\sigma$. And also assume that $\Lambda_1^* \to \Pi_1^*$ is the sequent obtained from S_1 by deleting the $(\check{S}^*)$-explicit formulas in π. By our assumption, $A \to$ or $\to B$ is derivable in PA. We treat only the case that $A \to$ is derivable in PA, since the other case is similar. Let $\hat{\pi}$ be a derivation of $A \to$. Then we reduce π into the derivation π':

$$\dfrac{\dfrac{\begin{array}{cc}\pi_1 \vdots & \hat{\pi} \vdots \\ \Lambda_1 \overset{S_1}{\to} \Pi_1, A & A \overset{\hat{S}}{\to}\end{array}}{\Lambda_1 \to \Pi_1}}{\dfrac{}{A \supset B, \Lambda_1, \Lambda_2 \overset{S}{\to} \Pi_1, \Pi_2}}\begin{array}{c}\\ \vdots\end{array}$$

Then we shall prove $O_0(\pi';\check{S}^*) < O_0(\pi;\check{S}^*)$. $\Lambda_1^* \to \Pi_1^*, A$ is the sequent obtained from S_1 by deleting the $(\check{S}^*)$-explicit formulas in π' and $h_0(S;\pi')=\sigma$. Assume that $h_0(S_1;\pi')=\tau(\leq\rho)$. Then,

$$\begin{aligned}
O_0(S_1;\pi';\check{S}^*) &= O_\tau(S_1;\pi_1;\Lambda_1^* \to \Pi_1^*, A)\\
&\leq O_\tau(S_1;\pi_1;\Lambda_1^* \to \Pi_1^*)\\
&\leq \omega_{\rho-\tau}(O_\rho(S_1;\pi_1;\Lambda_1^* \to \Pi_1^*))\\
&= \omega_{\rho-\tau}(O_0(S_1;\pi;\check{S}^*)).
\end{aligned}$$

On the other hand, we have $O_0(\hat{S};\pi';\check{S}^*) < \varepsilon_0$, because every inference in $\hat{\pi}$ is (S^*)-implicit in π'. Thus,

$$\begin{aligned}
O_0(S;\pi';\check{S}^*) &= \omega_{\tau-\sigma}(O_0(S_1;\pi';\check{S}^*)\sharp O_0(\hat{S};\pi';\check{S}^*))\\
&< \omega_{\tau-\sigma}(\omega_{\rho-\tau}(O_0(S_1;\pi;\check{S}^*))\sharp\varepsilon_0)\\
&\leq \omega_{\tau-\sigma}(\omega_{\rho-\tau}(O_0(S_1;\pi;\check{S}^*)\sharp\varepsilon_0))\\
&< \omega_{\rho-\sigma}(O_0(S_1;\pi;\check{S}^*)\sharp O_0(S_2;\pi;\check{S}^*)\sharp\varepsilon_0)\\
&= O_0(S;\pi;\check{S}^*).
\end{aligned}$$

So, $O_0(\pi';\check{S}^*) < O_0(\pi;\check{S}^*)$ by Proposition 3. Thus we can transform π' into a derivation whose end sequent is $\check{S}$ and which is $(\check{S}^*)$-strongly irreducible, by induction hypothesis.

(2) The case where π includes at least one inference which is reducible w.r.t. PA^-.

We shall transform π into a derivation π' by the same way as in [1]. Let I be one of inferences which are reducible w.r.t. PA^-. Then I is $(\check{S}^*)$-implicit, because π includes no $(\check{S}^*)$-explicit inferences which are reducible w.r.t. PA. We shall consider the case that I is a $\supset$: *right*. The other cases are treated similarly.

Assume that π is of the form:

$$\dfrac{\begin{array}{c}\pi_1 \vdots \\ A, \Lambda \overset{S_1}{\rightarrow} \Pi\end{array}}{\begin{array}{c}\Lambda \overset{S}{\rightarrow} \Pi, A \supset B \\ \vdots\end{array}}\, I \quad .$$

Assume that $h_0(S_1; \pi) = \rho$ and $h_0(S; \pi) = \sigma$. And also assume that A, $\Lambda^* \rightarrow \Pi^*$ is the sequent obtained from S_1 by deleting the $(\check{S}^*)$-explicit formulas in π. By our assumption, $\rightarrow A$ is derivable in PA^-. Let $\hat{\pi}$ be a PA^--derivation whose end sequent is $\rightarrow A$ and includes no cuts except inessential ones. Then we reduce π into the derivation π':

$$\dfrac{\dfrac{\begin{array}{c}\hat{\pi} \vdots \\ \overset{\hat{S}}{\rightarrow} A\end{array} \quad \begin{array}{c}\pi_1 \vdots \\ A, \Lambda \overset{S_1}{\rightarrow} \Pi\end{array}}{\Lambda \rightarrow \Pi}}{\begin{array}{c}\Lambda \overset{S}{\rightarrow} \Pi, A \supset B \\ \vdots\end{array}} \quad .$$

Then we shall prove $O_0(\pi'; \check{S}^*) < O_0(\pi; \check{S}^*)$. $h_0(S_1; \pi') = \rho$ and $h_0(S; \pi') = \sigma$. And A, $\Lambda^* \rightarrow \Pi^*$ is the sequent obtained from S_1 by deleting the $(\check{S}^*)$-explicit formulas in π'. Then $O_0(S_1; \pi'; \check{S}^*) = O_\rho(S_1; \pi_1; A, \Lambda^* \rightarrow \Pi^*) = O_0(S_1; \pi; \check{S}^*)$. On the other hand, we have $O_0(\hat{S}; \pi'; \check{S}^*) < \omega^{d(I)}$, because every inference in $\hat{\pi}$ is (S^*)-implicit in π' and every formula in $\hat{\pi}$ is an atomic formula or a subformula of A. Thus,

$$\begin{aligned} O_0(S; \pi'; \check{S}^*) &= \omega_{\rho-\sigma}(O_0(\hat{S}; \pi'; \check{S}^*) \sharp O_0(S_1; \pi'; \check{S}^*)) \\ &< \omega_{\rho-\sigma}(\omega^{d(I)} \sharp O_0(S_1; \pi; \check{S}^*)) \\ &= O_0(S; \pi; \check{S}^*). \end{aligned}$$

So, $O_0(\pi'; \check{S}^*) < O_0(\pi; \check{S}^*)$ by Proposition 3. Thus we can transform π' into a derivation whose end sequent is $\check{S}$ and which is $(\check{S}^*)$-strongly irreducible, by induction hypothesis.

(3) The case where π includes no boundary inferences.

π consists of initial sequents, weak inferences and cuts. Note that the cut formulas in π are only inessential, since weakings do not occur above cuts in π by our assumption. Thus π is a required derivation.

(4) The case where π includes at least one ind which belongs to the boundary of π.

Assume that π is of the form:

$$\dfrac{\begin{matrix}\pi_1 \vdots \\ \Lambda \xrightarrow{S_1} \Pi, A(0)\end{matrix} \quad \begin{matrix}\pi_2(a) \vdots \\ A(a), \Lambda \xrightarrow{S_2} \Pi, A(a')\end{matrix} \quad \begin{matrix}\pi_3 \vdots \\ A(t), \Lambda \xrightarrow{S_3} \Pi\end{matrix}}{\begin{matrix}\Lambda \xrightarrow{S} \Pi \\ \vdots \\ \Gamma \to \Delta\end{matrix}}\, I \qquad ,$$

where I belongs to the boundary of π. Assume that $h_0(S_1;\pi) = \rho$ and $h_0(S;\pi) = \sigma$. Assume that $\Lambda^* \to \Pi^*$, $A(0)$ is the sequent obtained from S_1 by deleting the $(\check{S}^*)$-explicit formulas in π. Then $A(a), \Lambda^* \to \Pi^*, A(a')$ is the sequent obtained from S_2 by deleting the $(\check{S}^*)$-explicit formulas in π and $A(t), \Lambda^* \to \Pi^*$ is the sequent obtained from S_3 by deleting the $(\check{S}^*)$-explicit formulas in π.

(4.1) The case where I is not constant normal.

We assume that the induction formula $A(a)$ of I includes at least one occurrence of a, since we can treat the other case similarly. Then the induction term t of I is closed. So, there exists a numeral n such that $t = n$ is derivable in PA, and there exists a derivation $\hat{\pi}$ of $A(n) \to A(t)$ such that $\hat{\pi}$ does not include essential cuts and inds (cf. [10]). We shall reduce π into the following derivation π':

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1 \vdots \\ \Lambda \xrightarrow{S_1} \Pi, A(0)\end{matrix} \quad \begin{matrix}\pi_2(0) \vdots \\ A(0), \Lambda \xrightarrow{S_2^0} \Pi, A(1)\end{matrix}}{\Lambda, \Lambda \to \Pi, \Pi, A(1)}}{\Lambda \to \Pi, A(1)} \qquad \begin{matrix}\pi_2(1) \vdots \\ A(1), \Lambda \xrightarrow{S_2^1} \Pi, A(2)\end{matrix}}{\Lambda, \Lambda \to \Pi, \Pi, A(2)}}{\Lambda \to \Pi, A(2)}$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Lambda \to \Pi, A(n) \qquad \begin{matrix}\hat{\pi} \vdots \\ A(n) \xrightarrow{\hat{S}} A(t)\end{matrix}}{\Lambda, \Lambda \to \Pi, \Pi, A(t)}}{\Lambda \to \Pi, A(t)} \qquad \begin{matrix}\pi_3 \vdots \\ A(t), \Lambda \xrightarrow{S_3} \Pi\end{matrix}}{\Lambda, \Lambda \to \Pi, \Pi}}{\begin{matrix}\Lambda \xrightarrow{S} \Pi \\ \vdots \\ \Gamma \to \Delta\end{matrix}} \qquad .$$

Then we shall prove $O_0(\pi'; \check{S}^*) < O_0(\pi; \check{S}^*)$. We shall note that $O_0(S_i; \pi'; \check{S}^*) = O_0(S_i; \pi; \check{S}^*)$ for $i = 1, 3$ and $O_0(S_2^j; \pi'; \check{S}^*) = O_0(S_2; \pi; \check{S}^*)$ for $j = 0, \dots, n-1$. On the other hand, we have $O_0(\hat{S}; \pi'; \check{S}^*) < \omega^{d(I)}$, because every inference in $\hat{\pi}$ is (S^*)-implicit in π' and every formula in $\hat{\pi}$ is an atomic formula or a subformula of $A(n)$ or $A(t)$. Since $O_0(S_2; \pi; \check{S}^*) \times n < O_0(S_2, \pi; \check{S}^*) \times \omega$ and $O_0(\hat{S}; \pi'; \check{S}^*) < \omega^{d(I)}$, we have

$$\begin{aligned} O_0(S; \pi'; \check{S}^*) &= \omega_{p-\sigma}(O_0(S_1; \pi; \check{S}^*) \sharp (O_0(S_2; \pi; \check{S}^*) \times n) \sharp O_0(S_3; \pi; \check{S}^*) \sharp O_0(\hat{S}; \pi'; \check{S}^*)) \\ &< \omega_{p-\sigma}(O_0(S_1; \pi; \check{S}^*) \sharp (O_0(S_2; \pi; \check{S}^*) \times \omega) \sharp O_0(S_3; \pi; \check{S}^*) \sharp \omega^{d(I)}) \\ &= O_0(S; \pi; \check{S}^*). \end{aligned}$$

So, $O_0(\pi'; \check{S}^*) < O_0(\pi; \check{S}^*)$ by Proposition 3. Thus we can transform π' into a derivation whose end sequent is $\check{S}$ and which is $(\check{S}^*)$-strongly irreducible, by induction hypothesis.

(4.2) The case where I is constant normal.

Let b be a variable which does not occur in π. We shall construct the following derivations $\hat{\pi}_1$, $\hat{\pi}_2$, $\hat{\pi}_3$ from π.

$$\begin{array}{ccc}
\hat{\pi}_1 & \hat{\pi}_2 & \hat{\pi}_3 \\
\pi_1 \ \vdots & \pi_2(b) \ \vdots & \pi_3 \ \vdots \\
\Lambda \overset{S_1}{\rightarrow} \Pi, A(0) & A(b), \Lambda \overset{S_2}{\rightarrow} \Pi, A(b') & A(t), \Lambda \overset{S_3}{\rightarrow} \Pi \\
\hline\hline
\Lambda \overset{S^1}{\rightarrow} A(0), \Pi & \Lambda A(b) \overset{S^2}{\rightarrow} A(\bar{\mathscr{S}}(b')), \Pi & \Lambda, A(t) \overset{S^3}{\rightarrow} \Pi \\
\vdots & \vdots & \vdots \\
\Gamma \rightarrow A(0), \Delta & \Gamma, A(b) \rightarrow A(b'), \Delta & \Gamma, A(t) \rightarrow \Delta \\
\hline\hline
\Gamma \rightarrow \Delta, A(0) & A(b), \Gamma \rightarrow \Delta, A(b') & A(t), \Gamma \rightarrow \Delta
\end{array}$$

Then we shall prove $O_0(\hat{\pi}_2; A(b), \Gamma^* \rightarrow \Delta^*, A(b')) < O_0(\pi; \check{S}^*)$. $h_0(S_2, \hat{\pi}_2) = \sigma$ and $A(b), \Lambda^* \rightarrow \Pi^*$, $A(b')$ is the sequent obtained from S_2 by deleting the $(A(b), \Gamma^* \rightarrow \Delta^*, A(b'))$-explicit formulas in $\hat{\pi}_2$. So,

$$\begin{aligned} O_0(S_2; \hat{\pi}_2; A(b), \Gamma^* \rightarrow \Delta^*, A(b')) &= O_\sigma(S_2; \pi_2; A(b), \Lambda^* \rightarrow \Pi^*, A(b')) \\ &\leq \omega_{p-\sigma}(O_p(S_2; \pi_2; A(b), \Lambda^* \rightarrow \Pi^*, A(b'))) \\ &= \omega_{p-\sigma}(O_p(S_2; \pi; \check{S}^*)). \end{aligned}$$

Thus,

$$
\begin{aligned}
&O_0(S^2; \hat{\pi}_2; A(b), \Gamma^* \to \Delta^*, A(b')) \\
&\quad = O_0(S_2; \hat{\pi}_2; A(b), \Gamma^* \to \Delta^*, A(b')) \\
&\quad \le \omega_{\rho-\sigma}(O_0(S_2; \pi; \check{S}^*)) \\
&\quad < \omega_{\rho-\sigma}(O_0(S_2; \pi; \check{S}^*) \times \omega) \\
&\quad < \omega_{\rho-\sigma}(O_0(S_1; \pi; \check{S}^*) \sharp (O_0(S_2; \pi; \check{S}^*) \times \omega) \sharp O_0(S_3; \pi; \check{S}^*) \sharp \omega^{d(I)}) \\
&\quad = O_0(S; \pi; \check{S}^*).
\end{aligned}
$$

So, $O_0(\hat{\pi}_2; A(b), \Gamma^* \to \Delta^*, A(b')) < O_0(\pi; \check{S}^*)$ by Proposition 3. Similarly, we can prove $O_0(\hat{\pi}_1; \Gamma^* \to \Delta^*, A(0)) < O_0(\pi; \check{S}^*)$ and $O_0(\hat{\pi}_3; A(t), \Gamma^* \to \Delta^*) < O_0(\pi; \check{S}^*)$.

Thus, by induction hypothesis, we can transform $\hat{\pi}_1$ into a derivation π'_1 whose end sequent is $\Gamma \to \Delta, A(0)$ and which is $(\Gamma^* \to \Delta^*, A(0))$-strongly irreducible, and $\hat{\pi}_2$ into a derivation π'_2 whose end sequent is $A(b)$, $\Gamma \to \Delta$, $A(b')$ and which is $(A(b), \Gamma^* \to \Delta^*, A(b'))$-strongly irreducible, and $\hat{\pi}_3$ into a derivation π'_3 whose end sequent is $A(t), \Gamma \to \Delta$ and which is $(A(t), \Gamma^* \to \Delta^*)$-strongly irreducible. We shall define π' as follows:

$$
\frac{\begin{matrix}\pi'_1 \vdots \\ \Gamma \to \Delta, A(0)\end{matrix} \quad \begin{matrix}\pi'_2 \vdots \\ A(b), \Gamma \to \Delta, A(b')\end{matrix} \quad \begin{matrix}\pi'_3 \vdots \\ A(t), \Gamma \to \Delta\end{matrix}}{\Gamma \to \Delta}\;.
$$

Note that π includes no redundant variables, and I is constant normal and belongs to the boundary of π. So, the free variables which occur in t occur in $\Gamma \to \Delta$. Thus π' is a derivation whose end sequent is $\check{S}$ and which is $(\check{S}^*)$-strongly irreducible.

(5) The case where π includes at least one $(\to)$-explicit inference which belongs to the boundary of π.

Let I be one of $(\to)$-explicit inferences which belong to the boundary of π.

(5.1) The case where I is $(\check{S}^*)$-explicit.

We shall consider the case that I is a $\forall$: *left*. The other cases are treated similarly.

Assume that π is of the form:

$$\cfrac{\begin{array}{c}\pi_1 \vdots \\ A(t), \Lambda \xrightarrow{S_1} \Pi\end{array}}{\begin{array}{c}\forall x A(x), \Lambda \xrightarrow{S} \Pi \\ \vdots \\ \Gamma \to \Delta\end{array}}\, I \quad .$$

Assume that $h_0(S_1;\pi) = \rho$ and $h_0(S;\pi) = \sigma$. Assume that $\Lambda^* \to \Pi^*$ is the sequent obtained from S_1 by deleting the $(\check{S}^*)$-explicit formulas in π. Then we reduce π into the derivation π':

$$\cfrac{\begin{array}{c}\pi_1 \vdots \\ A(t), \Lambda \xrightarrow{S_1} \Pi \\ \hline\hline \Lambda, A(t) \to \Pi\end{array}}{\begin{array}{c}\forall x A(x), \Lambda, A(t) \xrightarrow{S} \Pi \\ \vdots \\ \Gamma, A(t) \to \Delta\end{array}} \quad .$$

Then we shall prove $O_0(\pi';\Gamma^* \to \Delta^*) < O_0(\pi;\check{S}^*)$. $\Lambda^* \to \Pi^*$ is the sequent obtained from S_1 by deleting the $(\Gamma^* \to \Delta^*)$-explicit formulas in π'. And $h_0(S_1;\pi') = h_0(S;\pi') = \sigma$. So,

$$\begin{aligned} O_0(S_1;\pi';\Gamma^* \to \Delta^*) &= O_\sigma(S_1;\pi_1;\Lambda^* \to \Pi^*) \\ &\leq \omega_{\rho-\sigma}(O_\rho(S_1;\pi_1;\Lambda^* \to \Pi^*)) \\ &= \omega_{\rho-\sigma}(O_0(S_1;\pi;\check{S}^*)). \end{aligned}$$

Thus,

$$\begin{aligned} O_0(S;\pi';\Gamma^* \to \Delta^*) &= O_0(S_1;\pi_1';\Gamma^* \to \Delta^*) \\ &\leq \omega_{\rho-\sigma}(O_0(S_1;\pi;\check{S}^*)) \\ &< \omega_{\rho-\sigma}(O_0(S_1;\pi;\check{S}^*)\sharp\varepsilon_0) \\ &= O_0(S;\pi;\check{S}^*). \end{aligned}$$

Hence $O_0(\pi';\Gamma^* \to \Delta^*) < O_0(\pi;\check{S}^*)$ by Proposition 3. Thus we can transform π' into a derivation $\hat{\pi}$ whose end sequent is $\Gamma, A(t) \to \Delta$ and which is $(\Gamma^* \to \Delta^*)$-

strongly irreducible, by induction hypothesis. So, we shall define $\tilde{\pi}$ as follows:

$$\cfrac{\cfrac{\cfrac{\hat{\pi}}{A(t), \Gamma \to \Delta}}{\forall x A(x), \Gamma \to \Delta}\, J}{\Gamma \to \Delta} \quad .$$

Note that π includes no redundant variables and I belongs to the boundary of π. So, the free variables which occur in t occur in $\Gamma \to \Delta$. Note that J is $(\check{S}^*)$-explicit inference in $\tilde{\pi}$. And $\to A(t)$ is not derivable in PA, since π includes no $(\check{S}^*)$-explicit inferences which are reducible w.r.t. PA. Thus $\tilde{\pi}$ is $(\check{S}^*)$-strongly irreducible.

(5.2) The case where I is $(\check{S}^*)$-implicit.

We shall consider the case that I is a $\forall$: *right*. The other cases are treated similarly.

Assume that π is of the form:

$$\begin{array}{c} \pi_1(a) \ \vdots \\ \cfrac{\Lambda \xrightarrow{S_1} \Pi, A(a)}{\Lambda \xrightarrow{S} \Pi, \forall x A(x)}\, I \\ \vdots \\ \Gamma \to \Delta \end{array} \quad .$$

Assume that $h_0(S_1;\pi) = \rho$ and $h_0(S;\pi) = \sigma$. And assume that $\Lambda^* \to \Pi^*$, $A(a)$ is the sequent obtained from S_1 by deleting the $(\check{S}^*)$-explicit formulas in π. Let b be a variable which does not occur in π. Then we reduce π into the derivation π':

$$\begin{array}{c} \pi_1(b) \ \vdots \\ \cfrac{\cfrac{\Lambda \xrightarrow{S_1} \Pi, A(b)}{\Lambda \to A(b), \Pi}}{\Lambda \xrightarrow{S} A(b), \Pi, \forall x A(x)} \\ \vdots \\ \Gamma \to A(b), \Delta \end{array} \quad .$$

Then we shall prove $O_0(\pi'; \Gamma^* \to A(b), \Delta^*) < O_0(\pi; \check{S}^*)$. $h_0(S_1;\pi') = h_0(S;\pi') = \sigma$. And $\Lambda^* \to \Pi^*$, $A(b)$ is the sequent obtained from S_1 by deleting the $(\Gamma^* \to A(b), \Delta^*)$-explicit formulas in π'. So,

$$\begin{aligned} O_0(S_1;\pi';\Gamma^* \to A(b), \Delta^*) &= O_\sigma(S_1;\pi_1;\Lambda^* \to \Pi^*, A(b)) \\ &\le \omega_{\rho-\sigma}(O_\rho(S_1;\pi_1;\Lambda^* \to \Pi^*, A(b))) \\ &= \omega_{\rho-\sigma}(O_0(S_1;\pi;\check{S}^*)). \end{aligned}$$

Thus,

$$\begin{aligned} O_0(S;\pi';\Gamma^* \to A(b),\Delta^*) &= O_0(S_1;\pi';\Gamma^* \to A(b),\Delta^*) \\ &\leq \omega_{p-\sigma}(O_0(S_1;\pi;\check{S}^*)) \\ &< \omega_{p-\sigma}(O_0(S_1;\pi;\check{S}^*)\sharp\omega^{d(I)}) \\ &= O_0(S;\pi;\check{S}^*). \end{aligned}$$

Hence $O_0(\pi';\Gamma^* \to A(b),\Delta^*) < O_0(\pi;\check{S}^*)$ by Proposition 3. So, we can transform π' into a derivation $\hat{\pi}$ whose end sequent is $\Gamma \to A(b)$, Δ and which is $(\Gamma^* \to A(b),\Delta^*)$-strongly irreducible, by induction hypothesis. We shall define $\tilde{\pi}$ as follows:

$$\cfrac{\cfrac{\cfrac{\hat{\pi}}{\Gamma \to \Delta, A(b)}}{\Gamma \to \Delta, \forall x A(x)}\,J}{\Gamma \to \Delta}\ .$$

Note that J is $(\check{S}^*)$-implicit in $\tilde{\pi}$. And the sequent $A(b) \to$ is not derivable in PA^-, since π includes no inferences which are reducible w.r.t. PA^-. So, $\tilde{\pi}$ is $(\check{S}^*)$-strongly irreducible.

(6) The case where all the inferences which belong to the boundary of π are $(\to)$-implicit inferences.

At first, we shall show that there exists a suitable cut (cf. [10]). We shall consider the following property $(*)$ for a sequent S in the end-place of π.

$(*)$ S includes a descendant of the principal formula of a boundary inference.

The lower sequent of a boundary inference satisfies the property $(*)$ and the end sequent doesn't satisfy the property $(*)$. So, there exists an inference whose upper sequent(s) satisfies the property $(*)$ and whose lower sequent doesn't satisfy the property $(*)$. We take one of the uppermost ones and denote it by I. It is clear that I is a cut. Let S_1 (S_2) be the left (right) upper sequent of I. Then, we can suppose that S_1 satisfies the property $(*)$. Then the cut formula which occurs in S_1 must be a descendant of the principal formula of a boundary inference and include logical symbols. If no boundary inferences occur above S_2, S_2 doesn't include a formula which contains logical symbols. Because π includes no weakenings above S_2 by our assumption. However, S_2 includes as formula which contains logical symbols. So, π must include at least one boundary inference above S_2. If S_2 doesn't satisfy the property $(*)$, there exists an inference above I whose upper sequent(s) satisfies the property $(*)$ and whose lower sequent doesn't satisfy the property $(*)$. But it contradicts our choice of I. Thus S_2 satisfies the property $(*)$. Since the lower sequent of I doesn't satisfy the property $(*)$, the cut formula of I which occurs in S_2 must be a descendant of the principal formula of a boundary inference. So, I is a suitable cut. We shall consider the case that the cut formulas of I have $\forall$ as their outermost logical symbols. The other cases are treated similarly.

Assume that π is of the form:

$$
\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\pi_1(a)\ \vdots\\ \Lambda_1 \overset{S_1^u}{\to} \Pi_1, A(a)\end{array}}{\Lambda_1 \overset{S_1^l}{\to} \Pi_1, \forall x A(x)}\,I_1\\ \vdots\\ \Lambda_3 \overset{S_3}{\to} \Pi_3, \forall x A(x)\end{array}
\qquad
\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ A(t), \Lambda_2 \overset{S_2^u}{\to} \Pi_2\end{array}}{\forall x A(x), \Lambda_2 \overset{S_2^l}{\to} \Pi_2}\,I_2\\ \vdots\\ \forall x A(x), \Lambda_4 \overset{S_4}{\to} \Pi_4\end{array}}
{\begin{array}{c}\Lambda_3, \Lambda_4 \to \Pi_3, \Pi_4\\ \vdots\\ \dfrac{\qquad\qquad}{\Gamma_1 \overset{S}{\to} \Delta_1}\,I_3\\ \vdots\\ \Gamma \to \Delta\end{array}}\,I
\qquad .
$$

Here I_1 and I_2 belong to the boundary of π. And $\Gamma_1 \to \Delta_1$ denotes the uppermost sequent below I whose height based on 0 is less than that of the upper sequents of I. Assume that $h_0(S_1^u; \pi) = \rho_{1u}$, $h_0(S_1^l; \pi) = \rho_{1l}$, $h_0(S_3; \pi) = \rho$ and $h_0(S; \pi) = \sigma$. And also assume that $\Lambda_1^* \to \Pi_1^*$, $A(a)$ is the sequent obtained from S_1^u by deleting the $(\check{S}^*)$-explicit formulas in π. Then we reduce π into the derivation π':

$$
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\pi_1(t)\ \vdots\\ \Lambda_1 \overset{S_1^u}{\to} \Pi_1, A(t)\end{array}}{\overline{\Lambda_1 \overset{S_1^l}{\to} A(t), \Pi_1, \forall x A(x)}}\\ \vdots\\ \Lambda_3 \overset{S_3^1}{\to} A(t), \Pi_3, \forall x A(x)\end{array}
\quad
\begin{array}{c}\vdots\\ \forall x A(x), \Lambda_4 \overset{S_4^1}{\to} \Pi_4\end{array}}
{\Lambda_3, \Lambda_4 \to A(t), \Pi_3, \Pi_4}\\
\vdots\\
\dfrac{\qquad\qquad}{\Gamma_1 \overset{S^1}{\to} A(t), \Delta_1}\,I_3'\\
\overline{\overline{\Gamma_1 \to \Delta_1, A(t)}}
\end{array}
\qquad
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Lambda_3 \overset{S_3^2}{\to} \Pi_3, \forall x A(x)\end{array}
\quad
\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ A(t), \Lambda_2 \overset{S_2^u}{\to} \Pi_2\end{array}}{\overline{\forall x A(x), \Lambda_2, A(t) \overset{S_2^l}{\to} \Pi_2}}\\ \vdots\\ \forall x A(x), \Lambda_4, A(t) \overset{S_4^2}{\to} \Pi_4\end{array}}
{\Lambda_3, \Lambda_4, A(t) \to \Pi_3, \Pi_4}\\
\vdots\\
\dfrac{\qquad\qquad}{\Gamma_1, A(t) \overset{S^2}{\to} \Delta_1}\,I_3''\\
\overline{\overline{A(t), \Gamma_1 \to \Delta_1}}
\end{array}
}
{\begin{array}{c}
\Gamma_1, \Gamma_1 \to \Delta_1, \Delta_1\\
\overline{\overline{\Gamma_1 \overset{S}{\to} \Delta_1}}\\
\vdots\\
\Gamma \to \Delta
\end{array}}
$$

Then we shall prove $O_0(\pi'; \check{S}^*) < O_0(\pi; \check{S}^*)$. $\Lambda_1^* \to \Pi_1^*, A(t)$ is the sequent obtained from S_1^u by deleting the $(\check{S}^*)$-explicit formulas in π'. And $h_0(S_1^u; \pi') = h_0(S_1^l; \pi') = \rho_{1l}$, $h_0(S_3^1; \pi') = \rho$ and $h_0(S; \pi') = \sigma$. Assume that $h_0(S^1; \pi') = h_0(S^2; \pi') = \tau$. Then $\sigma \leq \tau < \rho$. Since we have

$$\begin{aligned} O_0(S_1^u; \pi'; \check{S}^*) &= O_{\rho_{1l}}(S_1^u; \pi_1; \Lambda_1^* \to \Pi_1^*, A(t)) \\ &\leq \omega_{\rho_{1u}-\rho_{1l}}(O_{\rho_{1u}}(S_1^u; \pi_1; \Lambda_1^* \to \Pi_1^*, A(t))) \\ &= \omega_{\rho_{1u}-\rho_{1l}}(O_0(S_1^u; \pi; \check{S}^*)), \end{aligned}$$

we have

$$\begin{aligned} O_0(S_1^l; \pi'; \check{S}^*) &= O_0(S_1^u; \pi'; \check{S}^*) \\ &\leq \omega_{\rho_{1u}-\rho_{1l}}(O_0(S_1^u; \pi; \check{S}^*)) \\ &< \omega_{\rho_{1u}-\rho_{1l}}(O_0(S_1^u; \pi; \check{S}^*) \sharp \omega^{d(I_1)}) \\ &= O_0(S_1^l; \pi; \check{S}^*). \end{aligned}$$

Thus $O_0(I_3'; \pi'; \check{S}^*) < O_0(I_3; \pi; \check{S}^*)$. Similarly, we have $O_0(I_3''; \pi'; \check{S}^*) < O_0(I_3; \pi; \check{S}^*)$. Then

$$O_0(S^1; \pi'; \check{S}^*) = \omega_{\rho-\tau}(O_0(I_3'; \pi'; \check{S}^*)) < \omega_{\rho-\tau}(O_0(I_3; \pi; \check{S}^*)),$$

$$O_0(S^2; \pi'; \check{S}^*) = \omega_{\rho-\tau}(O_0(I_3''; \pi'; \check{S}^*)) < \omega_{\rho-\tau}(O_0(I_3; \pi; \check{S}^*)).$$

Thus, $O_0(S^1; \pi'; \check{S}^*) \sharp O_0(S^2; \pi'; \check{S}^*) < \omega_{\rho-\tau}(O_0(I_3; \pi; \check{S}^*))$, because $\rho - \tau > 0$. Hence,

$$\begin{aligned} O_0(S; \pi'; \check{S}^*) &= \omega_{\tau-\sigma}(O_0(S^1; \pi'; \check{S}^*) \sharp O_0(S^2; \pi'; \check{S}^*)) \\ &< \omega_{\tau-\sigma}(\omega_{\rho-\tau}(O_0(I_3; \pi; \check{S}^*))) \\ &= \omega_{\rho-\sigma}(O_0(I_3; \pi; \check{S}^*)) \\ &= O_0(S; \pi; \check{S}^*). \end{aligned}$$

So, $O_0(\pi'; \check{S}^*) < O_0(\pi; \check{S}^*)$ by Proposition 3. Thus we can transform π' into a derivation whose end sequent is $\check{S}$ and which is $(\check{S}^*)$-strongly irreducible, by induction hypothesis. ■

5. Appendix

We can prove the following theorem by induction up to ε_0.

THEOREM 4. *Assume that π is a derivation of S. Then we can transform π into a PA^--irreducible derivation with the same end sequent.*

PROOF. We can prove this statement by a method similar to that in Theorem 3. Note that then we use induction on $O_0(\pi; S)$. ■

COROLLARY 4. *PA is 2-consistent.*

PROOF. Let $\exists x A(x)$ be a Σ_2-sentence. Then we can assume that $A(a)$ is a Π_1-formula. Suppose that $\exists x A(x)$ is derivable in PA. Then we shall show that $\exists x A(x)$ is true. Assume that $\exists x A(x)$ is not true. Let t be a closed term. Then, $\neg A(t)$ is true. Since $\neg A(t)$ is a Σ_1-sentence, $\to \neg A(t)$ is derivable in PA^- by Σ_1-completeness. So, we have the statement $(*)$ that $A(t) \to$ is derivable in PA^- for any closed term t.

On the other hand, there is a PA^--irreducible derivation π of $\exists x A(x)$ by our assumption and Theorem 4. Assume that π includes at least one boundary inference. Since the end-place of π includes no free variable, no inds belong to the boundary of π. Thus, every boundary inference must be of the form:

$$\frac{\Gamma \to \Delta, A(t')}{\Gamma \to \Delta, \exists x A(x)},$$

where Γ consists of atomic formulas and Δ consists of atomic formulas or $\exists x A(x)$. Since π includes no redundant variables, t' is closed. Since π is a PA^--irreducible derivation, $A(t') \to$ is not derivable in PA^-. But, this contradicts $(*)$. Thus, π includes no boundary inferences. Then we can transform π into a derivation of $\to$ which includes no free variables, no essential cuts, no inds and no logical inferences. But there is not such a derivation. Thus $\exists x A(x)$ is true. ■

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