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INDUCED MAPPINGS ON HYPERSPACES

By

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Abstract. Let $f: X \to Y$ be a mapping between continua. Then f induces two mappings $C(f): C(X) \to C(Y)$ and $2^f: 2^X \to 2^Y$ in the natural way. In this paper, we shall study about the following question: Dose the correspondences $f \to C(f)$ and $f \to 2^f$ preserve or reverse what classes of mappings? When Y is locally connected, many classes of mappings are preserved by these correspondences. We shall consider the classes of monotone, open, OM, confluent, quasi-monotone and weakly monotone mappings.

1. Introduction

In this paper, continua are compact connected metric spaces, mappings are continuous functions. Throughout this paper, the letters X and Y will always denote nondegenerate continua and a mapping $f: X \to Y$ is always onto. We shall use the letter d for the metric function for both spaces X and Y. The *hyperspaces* of X are the metric spaces $2^X = \{K \subset X : K$ is nonempty and compact$\}$ and $C(X) = \{K \in 2^X : K$ is connected$\}$ with the Hausdorff metric H_d (see [8] for the definition of the Hausdorff metric and basic properties of hyperspaces). A mapping $f: X \to Y$ induces mappings $C(f): C(X) \to C(Y)$ and $2^f: 2^X \to 2^Y$ naturally. If $g: Y \to Z$ is an another mapping, then $C(g \circ f) = C(g) \circ C(f)$ and $2^{g \circ f} = 2^g \circ 2^f$ hold. Clearly 2^f is onto (since we always assume that $f: X \to Y$ is onto) but $C(f)$ is onto if and only if f is weakly confluent.

The following three statements for a mapping $f: X \to Y$ are equivalent:

(1) f is a homeomorphism;

(2) $C(f)$ is a homeomorphism;

(3) 2^f is a homeomorphism.

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We shall study in the sections below the relations about the above type between the mappings f, $C(f)$ and 2^f.

Some of the results are improvement of those partially appeared in [2] and [3]. But for completeness, we shall describe their proofs.

2. Definitions and Notations

We shall give the list of definitions for mappings treated hereafter. A mapping $f : X \to Y$ is said to be

(1) *monotone* if for each $y \in Y$, $f^{-1}(y)$ is connected; equivalently, if for each subcontinuum L of Y, $f^{-1}(L)$ is connected;

(2) *open* if f maps every open set in X onto an open set in Y;

(3) *an OM-mapping* (resp. *an MO-mapping*) if there are mappings g and h, where g is open and h is monotone, such that $f = g \circ h$ (resp. $f = h \circ g$);

(4) *confluent* if for each subcontinuum L of Y, each component of $f^{-1}(L)$ is mapped by f onto L;

(5) *quasi-monotone* if for each subcontinuum L of Y with a nonempty interior, the set $f^{-1}(L)$ has a finite number of components and f maps each of them onto L;

(6) *weakly monotone* if for each subcontinuum L of Y with a nonempty interior, each component of the set $f^{-1}(L)$ is mapped by f onto L.

For the implications between these classes of mappings, see p28 in [7].

Let $\mathscr{H}$ denote either $C(X)$ or 2^X. A *Whitney map* $\mu : \mathscr{H} \to [0,1]$ is a mapping such that $\mu(\{x\}) = 0$ for each $x \in X$, $\mu(X) = 1$ and if $A, B \in \mathscr{H}$ with $A \subset B \neq A$, then $\mu(A) < \mu(B)$. Such a mapping always exists ([9] or [8]). Let $A_0, A_1 \in \mathscr{H}$. A mapping $\sigma : [0,1] \to \mathscr{H}$ is said to be a *segment with respect to the Whitney map* μ *from* A_0 to A_1 provided that $\sigma(0) = A_0$, $\sigma(1) = A_1$, $\mu[\sigma(t)] = (1-t)\mu(A_0) + t\mu(A_1)$ for each $t \in [0,1]$ and if $0 \leq t_1 \leq t_2 \leq 1$, then $\sigma(t_1) \subset \sigma(t_2)$. When we use a segment, we will consider it with respect to some fixed Whitney map. A condition of the existence of a segment is as follows:

LEMMA 2.1 ([4] or [8]). *Let A_0, $A_1 \in \mathscr{H}$, where $\mathscr{H}$ denotes either $C(X)$ or 2^X. Then there exists a segment from A_0 to A_1 if and only if*

(2.1.1) *$A_0 \subset A_1$ if $\mathscr{H} = C(X)$,*

(2.1.2) *$A_0 \subset A_1$ and each component of A_1 intersects A_0 if $\mathscr{H} = 2^X$.*

Let A_1, $A_2, \ldots$ be a sequence of nonempty subsets of X. Then lim inf A_n and lim sup A_n are defined by lim inf $A_n = \{x \in X$: if U is a neighborhood of x in X, then $U \cap A_n \neq \phi$ for allmost all $n\}$, lim sup $A_n = \{x \in X$: if U is a neighborhood of x in X, then $U \cap A_n \neq \phi$ for infinitely many $n\}$. If lim inf $A_n =$ lim sup $A_n = A$, then we say that $\{A_n\}_{n=1}^{\infty}$ converges to A and wright it by lim $A_n = A$. Following is known:

Lemma 2.2 [8]. *Let $A_1, A_2, \ldots$ be a sequence in 2^X (resp. $C(X)$). Then* lim $A_n = A$ *in the sense above if and only if it converges to A with respect to the Hausdorff metric for 2^X (resp. $C(X)$).*

When we say a sequence $\{A_n\}_{n=1}^{\infty}$ converges in 2^X or $C(X)$, we will mean in a convenient sense of one of the two senses. We shall wright $\bar{A}$, int A for the closure of A, the interior of A respectively. If $\mathscr{A}$ is a subset of a hyperspace $\mathscr{H}$, then we shall wright Int $\mathscr{A}$ for the interior of $\mathscr{A}$ in $\mathscr{H}$.

For a subset A of a space, we say that $A = A_1 \cup A_2$ is a *separation of A* if $A_1 \neq \phi \neq A_2$ and $\bar{A}_1 \cap A_2 = A_1 \cap \bar{A}_2 = \phi$.

Lemma 2.3 [10]. *If A and B are nonempty disjoint closed subsets of a compact set K such that no component of K intersects both A and B, then there exists a separation $K = K_a \cup K_b$ of K such that $A \subset K_a$ and $B \subset K_b$.*

Furthere we shall use the following notation. For any collection $\mathscr{A}$ of subsets of a space, $\mathscr{A}^*$ denotes the union of all members contained in $\mathscr{A}$.

3. Monotone Mappings

If $\mathscr{K}$ is a subcontinuum of 2^X and $\mathscr{K} \cap C(X) \neq \phi$, then $\mathscr{K}^*$ is connected [8]. This is generalized as follows:

Lemma 3.1. *Let $\mathscr{K}$ be a subcontinuum of 2^X and $K \in \mathscr{K}$. Then each component of $\mathscr{K}^*$ intersects K.*

Proof. On the contrary, suppose there is a component C of $\mathscr{K}^*$ such that $C \cap K = \phi$. Then by lemma 2.3, there is a separation $\mathscr{K}^* = A \cup B$ of $\mathscr{K}^*$ such that $K \subset A$ and $C \subset B$. Put $\mathscr{K}_0 = \{L \in \mathscr{K} : L \subset A\}$ and $\mathscr{K}_1 = \{L \in \mathscr{K} : L \cap B \neq \phi\}$. Then we have a separation $\mathscr{K} = \mathscr{K}_0 \cup \mathscr{K}_1$ of $\mathscr{K}$. This contradicts to the connectedness of $\mathscr{K}$.

Theorem 3.2. *Let $f : X \to Y$ be a mapping. Then, the following three statements are equivalent:*

(3.2.1) *f is a monotone mapping*;

(3.2.2) *$C(f)$ is a monotone mapping*;

(3.2.3) *2^f is a monotone mapping.*

PROOF. (3.2.1) $\Rightarrow$ (3.2.2): Suppose that f is monotone and let L be an arbitrary element of $C(Y)$. Put $M = f^{-1}(L)$ and let K be an arbitrary element of $[C(f)]^{-1}(L)$. Then, since f is monotone, M is a subcontinuum of X and contains K. Therefore, by lemma 2.1, there is a segment σ from K to M in $C(X)$. It is evident that the image of σ is contained in $[C(f)]^{-1}(L)$. Thus, in particular, $[C(f)]^{-1}(L)$ is arcwise connected.

(3.2.2) $\Rightarrow$ (3.2.3): Suppose that $C(f)$ is monotone and let B be an arbitrary element of 2^Y. Put $A = f^{-1}(B)$. Then $A \in [2^f]^{-1}(B)$. Let K be a component of A considered as a subset of X. Since $C(f)$ is monotone, $[C(f)]^{-1}(f(K))^*$ is connected and contained in $f^{-1}(f(K))$ and hence is equal to K. Therefore every component of A intersects each element of $[2^f]^{-1}(B)$. It follows by lemma 2.1 that $[2^f]^{-1}(B)$ is arcwise connected.

(3.2.3) $\Rightarrow$ (3.2.1): Suppose that 2^f is monotone and let $y \in Y$. Then by lemma 3.1, $[2^f]^{-1}(\{y\})^* = f^{-1}(y)$ is connected.

REMARK. If f is monotone and $\mathscr{B}$ is an arcwise connected subcontinuum of 2^Y (resp. $C(Y)$), then $[2^f]^{-1}(\mathscr{B})$ (resp. $[C(f)]^{-1}(\mathscr{B})$) is arcwise connected.

4. Open Mappings

The following lemma is a characterization of open mappings. The equivalence (4.1.1.) $\Leftrightarrow$ (4.1.2) is appeared in [7], p.14 without proof (see also [5], pp. 67–68).

LEMMA 4.1. *Let $f: X \to Y$ be a mapping. Then the following three statements are equivalent*:

(4.1.1) *f is an open mapping*;

(4.1.2) *for each sequence $\{y_n\}_{n=1}^\infty$ in Y such that* $\lim y_n = y$, $\limsup f^{-1}(y_n) = f^{-1}(y)$;

(4.1.3) *for each sequence $\{y_n\}_{n=1}^\infty$ in Y such that* $\lim y_n = y$, $\{f^{-1}(y_n)\}_{n=1}^\infty$ *converges to* $f^{-1}(y)$.

PROOF. The implication (4.1.3) $\Rightarrow$ (4.1.2) is evident.

(4.1.1) $\Rightarrow$ (4.1.3): Suppose f is open and let $\{y_n\}_{n=1}^\infty$ be a sequence in Y such that $\lim y_n = y$. Since the continuity of f implies $\limsup f^{-1}(y_n) \subset f^{-1}(y)$, it is sufficient to show that $f^{-1}(y) \subset \liminf f^{-1}(y_n)$. Let $x \in f^{-1}(y)$ and U an open neighborhood of x in X. Since $f(U)$ is a neighborhood of y, there is an integer n_0 such that $y_n \in f(U)$ and hence $f^{-1}(y_n) \cap U \neq \phi$ for each $n \geq n_0$. Therefore $x \in \liminf f^{-1}(y_n)$ and hence we have $f^{-1}(y) \subset \liminf f^{-1}(y_n)$.

For any collection $U_1, U_2, \ldots, U_n$ of open sets in X, let $\langle U_1, U_2, \ldots, U_n \rangle = \{A \in 2^X : A \subset \bigcup_{i=1}^n U_i$ and $A \cap U_i \neq \phi$ for each $i = 1, 2, \ldots, n\}$. It is known that:

LEMMA 4.2 [8]. *The collection of all subsets of 2^X of the form $\langle U_1, U_2, \ldots, U_n \rangle$ is a base for the Hausdorff metric topology for 2^X.*

THEOREM 4.3. *Let $f : X \to Y$ be a mapping. Consider the following three statements*:

(4.3.1) *f is an open mapping*;

(4.3.2) *$C(f)$ is an open mapping*;

(4.3.3) *2^f is an open mapping.*

Then (4.3.1) *and* (4.3.3) *are equivalent and* (4.3.2) *implies* (4.3.1).

PROOF. (4.3.1) $\Rightarrow$ (4.3.3): Suppose f is open and let $\{B_n\}_{n=1}^{\infty}$ be a sequence in 2^Y such that $\lim B_n = B$. Since 2^f is continuous, $\limsup[2^f]^{-1}(B_n)$ is contained in $[2^f]^{-1}(B)$. Let A be an arbitrary element of $[2^f]^{-1}(B)$ and let $U_1, U_2, \ldots, U_r$ be open sets in X such that $A \in \langle U_1, U_2, \ldots, U_r \rangle$. Since A is compact, there are open sets $V_1, V_2, \ldots, V_r$ of X such that $\overline{V}_i \subset U_i$ for each $i = 1, 2, \ldots, r$ and $A \in \langle V_1, V_2, \ldots, V_r \rangle$. Since f is open, $\langle f(V_1), f(V_2), \ldots, f(V_r) \rangle$ is an open neighborhood of $f(A) = B$ in 2^Y. Therefore there is an integer n_0 such that $B_n \in \langle f(V_1), f(V_2), \ldots, f(V_r) \rangle$ for each $n \geq n_0$. Put $A_n = f^{-1}(B_n) \cap [\bigcup_{i=1}^r \overline{V}_i]$. Then it is easy to see that $A_n \in [2^f]^{-1}(B_n) \cap [\langle U_1, U_2, \ldots, U_r \rangle]$ and hence by lemma 4.2, we have $A \in \liminf[2^f]^{-1}(B_n)$. It follows from lemma 4.1, that 2^f is an open mapping.

(4.3.3) $\Rightarrow$ (4.3.1): Suppose 2^f is an open mapping. Let U be an open set in X and let $x \in U$. Since $\langle U \rangle$ is an open neighborhood of $\{x\} \in 2^X, 2^f(\langle U \rangle) = \langle f(U) \rangle$ is an open neighborhood of $\{f(x)\} \in 2^Y$. Therefore $f(U)$ is a neighborhood of $f(x)$. Since x is an arbitrary element of $U, f(U)$ is open in Y.

The proof of the implication (4.3.2) $\Rightarrow$ (4.3.1) is similar.

Note that in general, $C(f)([\langle U_1, U_2, \ldots, U_n \rangle] \cap C(X))$ is not equal to $[\langle f(U_1), f(U_2), \ldots, f(U_n) \rangle] \cap C(Y)$ even though $n = 1$. Following is an example where f is open, X and Y are locally connected but $C(f)$ is not open.

EXAMPLE Let X, Y be plane continua defined by

$$Y = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\},$$
$$X = \{(x, y) : (x, y) \in Y \text{ or} (-x, -y) \in Y\}.$$

Define $f : X \to Y$ by

$$f(x, y) = \begin{cases} (x, y) & \text{if } (x, y) \in Y \\ (-x, -y) & \text{if } (x, y) \notin Y. \end{cases}$$

for each $(x, y) \in X$. Let $K = \{(x, y) \in X : x = 0 \text{ or } y = 0\}$. Then f is open but $C(f)$ is not open at $K \in C(X)$.

5. OM-mappings

In [6], A. Lelek and D. R. Read had given a characterization of OM-mappings as follows:

LEMMA 5.1 [6]. *A mapping $f : X \to Y$ is an OM-mapping if and only if for each $y \in Y$ and each sequence $\{y_n\}_{n=1}^{\infty}$ in Y, $\lim y_n = y$ implies that $\limsup f^{-1}(y_n)$ meets each component of $f^{-1}(y)$.*

We always saw that the correspondence $f \to C(f)$ does not preserve the class of open mappings. Nevertheless it preserves the class of OM-mappings.

THEOREM 5.2. *For a mapping $f : X \to Y$, the following three statements are equivalent*:

(5.2.1) *f is an OM-mapping*;

(5.2.2) *$C(f)$ is an OM-mapping*;

(5.2.3) *2^f is an OM-mapping.*

PROOF. The implication (5.2.1) $\Rightarrow$ (5.2.3) follows from Theorems 3.2 and 4.3.

(5.2.1) $\Rightarrow$ (5.2.2): Suppose f is an OM-mapping and $\{L_n\}_{n=1}^{\infty}$ is a sequence in $C(Y)$ which converges to $L \in C(Y)$. Let $\mathscr{K}$ be a component of $[C(f)]^{-1}(L)$. We must show that $\limsup [C(f)]^{-1}(L_n) \cap \mathscr{K} \neq \phi$. Choose a point $x \in \mathscr{K}^*$ and put $y = f(x)$. There is a point $y_n \in L_n$ for each $n = 1, 2, \ldots$ such that $\lim y_n = y$. Let C be the component of $f^{-1}(y)$ containing x. Since f is an OM-mapping, there is a point $x_n \in f^{-1}(y_n)$ such that some subsequence of $\{x_n\}_{n=1}^{\infty}$ converges to some point of C. We may assume $\lim x_n = x_0 \in C$. Let K_n be the component of

$f^{-1}(L_n)$ containing x_n for each $n = 1, 2, \ldots$. Since OM-mappings are confluent, we have $K_n \in [C(f)]^{-1}(L_n)$. We may assume that $\{K_n\}_{n=1}^{\infty}$ converges to K_0 for some $K_0 \in C(X)$. It is easy to see that K_0 and $K_0 \cup C$ are elements of $C(X)$ contained in the same component of $[C(f)]^{-1}(L)$. Let K be an element of $\mathscr{K}$ such that $x \in K$. Then K and $K_0 \cup C$ are in the same component of $[C(f)]^{-1}(L)$. Thus $K_0 \in \mathscr{K}$ and hence we have $\limsup [C(f)]^{-1}(L_n) \cap \mathscr{K} \neq \phi$. Therefore by lemma 5.1, $C(f)$ is an OM-mapping.

(5.2.2) $\Rightarrow$ (5.2.1): Suppose $C(f)$ is an OM-mapping and $\{y_n\}_{n=1}^{\infty}$ is a sequence in Y which converges to $y \in Y$. Clearly the sequence $\{\{y_n\}\}_{n=1}^{\infty}$ considered as a sequence in $C(Y)$, converges to $\{y\} \in C(Y)$. Let K be a component of $f^{-1}(y)$. Then $C(K)$, considered as a subset of $C(X)$, is a component of $[C(f)]^{-1}(\{y\})$. By the assumption and lemma 5.1, there is $K_n \in [C(f)]^{-1}(\{y_n\})$ for each n such that some subsequence of $\{K_n\}_{n=1}^{\infty}$ converges to an element of $C(K)$. Since $K_n \subset f^{-1}(y_n)$, this implies that $\limsup f^{-1}(y_n) \cap K \neq \phi$. Therefore applying lemma 5.1 again, we have that f is an OM-mapping.

The implication (5.2.3) $\Rightarrow$ (5.2.1) is similarly proved.

THEOREM 5.3. *If $f : X \to Y$ is an MO-mapping, then 2^f is also an MO-mapping.*

This follows directly from Theorems 3.2 and 4.3.

6. Confluent mappings

First we prove a special case.

LEMMA 6.1. *Let $f : X \to Y$ be a confluent mapping.*

(6.1.1) *If $\mathscr{L}$ is an arc in $C(Y)$, then each component of $[C(f)]^{-1}(\mathscr{L})$ is mapped by $C(f)$ onto $\mathscr{L}$.*

(6.1.2) *If $\mathscr{L}$ is an arc in 2^Y, then each component of $[2^f]^{-1}(\mathscr{L})$ is mapped by 2^f onto $\mathscr{L}$.*

PROOF. We only prove (6.1.2) since (6.1.1) is more simple. Let $\mathscr{L}$ be an arc in 2^Y and $\alpha : [0,1] \to \mathscr{L}$ a homeomorphism. Let $\mathscr{K}$ be a component of $[2^f]^{-1}(\mathscr{L})$. Without loss of generality, we may assume $\alpha(0) \in 2^f(\mathscr{K})$. It is sufficient to show that $\alpha(1) \in 2^f(\mathscr{K})$. On the contrary, suppose that $\alpha(1) \notin 2^f(\mathscr{K})$. Then by lemma 2.3, there is a separation $[2^f]^{-1}(\mathscr{L}) = \mathscr{K}_0 \cup \mathscr{K}_1$

such that $\mathscr{K} \subset \mathscr{K}_0$ and $[2^f]^{-1}(\alpha(1)) \subset \mathscr{K}_1$. Put $t_0 = \sup\{t : \alpha(t) \in 2^f(\mathscr{K}_0)\}$. Then by compactness of $\mathscr{K}_0$, $t_0 < 1$ and there is $K \in \mathscr{K}_0$ such that $2^f(K) = \alpha(t_0)$. Let M be the union of all components C of $f^{-1}(\alpha(t_0))$ such that $C \cap K \neq \phi$. Note that $M \in \mathscr{K}_0$ since K and M are joind by a segment in $[2^f]^{-1}(\alpha(t_0))$. Let M_t be the union of all components of $f^{-1}(\alpha([t_0, t])^*)$ intersecting M for each $t \in [t_0, 1]$. Choose a sequence $t_1, t_2, \ldots$ in $[t_0, 1]$ such that $1 > t_1 > t_2 > \cdots$, and $\lim t_n = t_0$. For each $n = 1, 2, \ldots$, put $K_n = f^{-1}(\alpha(t_n)) \cap M_{t_n}$. Since f is confluent, each component of M_{t_n} is mapped by f onto a component of $\alpha([t_0, t_n])^*$. Therefore, by lemma 3.1, it is not so difficult to see that $K_n \in [2^f]^{-1}(\alpha(t_n))$ and each component of M_{t_n} intersects K_n. We may assume that $\lim K_n = K_0$ for some $K_0 \in 2^X$. Then $K_0 \subset \bigcap_{n=1}^{\infty} M_{t_n} = M$ and each component of M intersects K_0. Therefore by lemma 2.1, there is a segment from K_0 to M whose image is clearly contained in $[2^f]^{-1}(\alpha(t_0))$. Therefore $K_0 \in \mathscr{K}_0$. On the other hand, $K_n \in \mathscr{K}_1$ for each $n = 1, 2, \ldots$ Hence we have a contradiction since $H_d(\mathscr{K}_0, \mathscr{K}_1) > 0$.

Corollary 6.2. *Let $f : X \to Y$ be a confluent mapping.*

(6.2.1) *If $\mathscr{L}$ is an arcwise connected subcontinuum of $C(Y)$, then each component of $[C(f)]^{-1}(\mathscr{L})$ is mapped by $C(f)$ onto $\mathscr{L}$.*

(6.2.2) *If $\mathscr{L}$ is an arcwise connected subcontinuum of 2^Y, then each component of $[2^f]^{-1}(\mathscr{L})$ is mapped by 2^f onto $\mathscr{L}$.*

Proof. Let $\mathscr{L}$ be an arcwise connected subcontinuum of $C(Y)$ and let $\mathscr{K}$ be a component of $[C(f)]^{-1}(\mathscr{L})$. Choose an element $K \in \mathscr{K}$. Then for any $L \in \mathscr{L} - \{f(K)\}$, there is an arc $\mathscr{B}$ in $\mathscr{L}$ with the end points $f(K)$ and L. Let $\mathscr{A}$ be the component of $[C(f)]^{-1}(\mathscr{B})$ containing K. Then clearly $\mathscr{A} \subset \mathscr{K}$, lemma 6.1 implies $L \in C(f)(\mathscr{K})$. (6.2.2) is similarly proved.

Theorem 6.3. *Let $f : X \to Y$ be a mapping. Consider the following three statements*:

(6.3.1) *f is a confluent mapping*;

(6.3.2) *$C(f)$ is a confluent mapping*;

(6.3.3) *2^f is a confluent mapping.*

Then the implications (6.3.2) $\Rightarrow$ (6.3.1) *and* (6.3.3) $\Rightarrow$ (6.3.1) *hold. If Y is locally connected, then they are equivalent.*

Proof. (6.3.3) $\Rightarrow$ (6.3.1): Let L be a subcontinuum of Y and K a component of $f^{-1}(L)$. Let $\mathscr{L}$ and $\mathscr{K}$ be subcontinua of 2^Y and 2^X respectively defined by $\mathscr{L} = \{\{y\} : y \in L\}$, $\mathscr{K} = \{\{x\} : x \in K\}$. Let $\mathscr{M}$ be a component of $[2^f]^{-1}(\mathscr{L})$ such that $\mathscr{K} \cap \mathscr{M} \neq \phi$. Then it is clear that $\mathscr{M}^* = K$. Since $2^f(\mathscr{M}) = \mathscr{L}$, we have $f(K) = L$.

The implication $(6.3.2) \Rightarrow (6.3.1)$ is similarly proved.

Now suppose that f is confluent and Y is locally connected. We shall only prove that 2^f is confluent and omit the proof for $C(f)$ to be confluent. Let $\mathscr{L}$ be a subcontinuum of 2^X and $\mathscr{K}$ a component of $[2^f]^{-1}(\mathscr{L})$. Since 2^Y is locally connected ([1] or [8]), there are locally connected subcontinua $\mathscr{L}_1, \mathscr{L}_2, \ldots$ of 2^Y such that $\mathscr{L}_1 \supset \mathscr{L}_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} \mathscr{L}_n = \mathscr{L}$ (see [5], p.260). Let $\mathscr{K}_n$ be the component of $[2^f]^{-1}(\mathscr{L}_n)$ containing $\mathscr{K}$ for each $n = 1, 2, \ldots$. It follows evidently that $\mathscr{K}_1 \supset \mathscr{K}_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} \mathscr{K}_n = \mathscr{K}$. Since by corollary 6.2 and continuity of 2^f, $2^f(\mathscr{K}) = 2^f(\bigcap_{n=1}^{\infty} \mathscr{K}_n) = \bigcap_{n=1}^{\infty} 2^f(\mathscr{K}_n) = \bigcap_{n=1}^{\infty} \mathscr{L}_n = \mathscr{L}$.

The following example shows that there is a confluent mapping f such that neither $C(f)$ nor 2^f is weakly confluent.

Example. In the Euclidean plane with polar coordinates (r, θ), let S be the unite circle $S = \{(r, \theta) : r = 1 \text{ and } 0 \leq \theta < 2\pi\}$ and let A_1, A_2, B_1, B_2 be spaces each homeomorphic to the half open interval $[0, 1)$, defined by

$$A_1 = \left\{(r, \theta) : \theta = \frac{\pi}{2} \sin \frac{1}{1 - r}, 1 < r \leq 2\right\},$$

$$A_2 = \left\{(r, \theta) : \theta = \frac{\pi}{2}\left(2 + \sin \frac{1}{1 - r}\right), \frac{1}{2} \leq r < 1\right\},$$

$$B_1 = \left\{(r, \theta) : \theta = \pi \sin \frac{1}{1 - r}, 1 < r \leq 2\right\},$$

$$B_2 = \left\{(r, \theta) : \theta = \pi\left(2 + \sin \frac{1}{1 - r}\right), \frac{1}{2} \leq r < 1\right\}.$$

Define X, Y and $f : X \to Y$ by $X = S \cup A_1 \cup A_2$, $Y = S \cup B_1 \cup B_2$ and $f(r, \theta) = (r, 2\theta)$ for all $(r, \theta) \in X$. Then f is cofluent and weakly monotone. Let $K_t = \{(r, \theta) : r = 1 \text{ and } (\pi/2)(t - 1) \leq \theta \leq (\pi/2)(2t - 1)\}$ for $t \in [0, 1]$ and $L_t = \{(r, \theta) : r = 1 \text{ and } (\pi/2)(t + 1) \leq \theta \leq (\pi/2)(2t + 1)\}$ for $t \in [0, 1]$ The sets $\mathscr{K} = \{K_t : t \in [0, 1]\}$ and $\mathscr{L} = \{L_t : t \in [0, 1]\}$ are disjoint arcs in $C(X)$ such that $C(f)(\mathscr{K}) = C(f)(\mathscr{L})$. There exist subsets $\mathscr{M}$, $\mathscr{N}$ of $C(X)$ such that $\mathscr{M}$ and $\mathscr{N}$ are both homeomorphic to the half open interval, each element of $\mathscr{M}$ (resp. $\mathscr{N}$) is contained in A_1 (resp. A_2), $\overline{\mathscr{M}} - \mathscr{M} = \mathscr{K}$ and $\overline{\mathscr{N}} - \mathscr{N} = \mathscr{L}$. To see this, let $g : S \cup A_1 \to S$ be the retraction defined by $g(r, \theta) = (1, \theta)$ for each $(r, \theta) \in S \cup A_1$. We consider $C(X)$ and $C(S \cup A_1)$ as subsets of $C(X)$. Put $\mathscr{M}_0 = [C(g)]^{-1}(\mathscr{K})$.

Note that $[C(g)]^{-1}(K_1) - \{K_1\}$ is a disjoint union of countably many arcs in $C(S \cup A_1)$ and $[C(g)]^{-1}(K_t)$ is a countable set with one limits element K_t for $0 \leq t < 1$. Define $\mathscr{M} = \mathscr{M}_0 - \mathscr{K}$. Similarly we can fined a described set $\mathscr{N}$. Put $\mathscr{A}_1 = \mathscr{M} \cup \mathscr{K}$, $\mathscr{A}_2 = \mathscr{N} \cup \mathscr{L}$ and $\mathscr{B} = C(f)(\mathscr{A}_1 \cup \mathscr{A}_2)$. Then $\mathscr{B}$ is a subcontinuum of $C(Y)$ and $[C(f)^{-1}(\mathscr{B})$ has just two components $\mathscr{A}_1$ and $\mathscr{A}_2$. But neither of them is mapped by $C(f)$ onto $\mathscr{B}$.

7. Quasi-monotone and Weakly monotone mappings

LEMMA 7.1. *If $f : X \to Y$ is weakly monotone and Y is locally connected, then f is confluent.*

PROOF. Let L be a subcontinuum of Y and K a component of $f^{-1}(L)$. Since Y is locally connected, there are subcontinua $L_n (n = 1, 2, \dots)$ of Y such that $L_1 \supset L_2 \supset L_3 \supset \dots$, $\bigcap_{n=1}^{\infty} L_n = L$ and $\operatorname{int} L_n \neq \phi$ for each $n = 1, 2, \dots$ (see [5] or [10]). Let K_n be a component of $f^{-1}(L_n)$ containing K for each $n = 1, 2, \dots$. Then clearly $K = \bigcap_{n=1}^{\infty} K_n$ and hence $f(K) = \bigcap_{n=1}^{\infty} f(K_n) = \bigcap_{n=1}^{\infty} L_n = L$.

By Theorem 6.3, we have:

COROLLARY 7.2. *If $f : X \to Y$ is weakly monotone and Y is locally connected, then both of the mappings $C(f)$ and 2^f are confluent.*

THEOREM 7.3. *Let $f : X \to Y$ be a mapping. Consider the following three statements:*

(7.3.1) *f is a quasi-monotone (resp. a weakly monotone) mapping;*

(7.3.2) *$C(f)$ is a quasi-monotone (resp. a weakly monotone) mapping;*

(7.3.3) *2^f is a quasi-monotone (resp. a weakly monotone) mapping.*

Then one of (7.3.2) *and* (7.3.3) *implies* (7.3.1). *If Y is locally connected, then they are equivalent.*

PROOF. We shall only prove for the class of quasi-monotone mappings. The proof of the implication that (7.3.2) or (7.3.3) implies (7.3.1) is similar as the proof of Theorem 4.3.

Now suppose f is quasi-monotone and Y is locally connected. Let $\mathscr{L}$ be a subcontinuum of $C(Y)$ such that $\operatorname{Int} \mathscr{L} \neq \phi$. Choose $L_0 \in \operatorname{Int} \mathscr{L}$ and $y \in L_0$. Since $L_0 \in \operatorname{Int} \mathscr{L}$ and Y is locally connected, there is a small closed connected neighborhood V of y in Y such that $L = V \cup L_0 \in \mathscr{L}$. Since f is quasi-monotone and $\operatorname{int} L \neq \phi$, $f^{-1}(L)$ has a finite number of components, say $K_1, K_2, \dots, K_r$, each of them is mapped by f onto L. Since quasi-monotone mappings are weakly monotone, corollary 7.2 implies that $C(f)$ is confluent. Let $\mathscr{K}_0$ be a component of $[C(f)]^{-1}(\mathscr{L})$. Then $C(f)(\mathscr{K}_0) = \mathscr{L}$. Thus there is $K \in \mathscr{K}_0$ such that $C(f)(K) = L$. Therefore $K \subset K_i$ for some $i \in \{1, 2, \ldots, r\}$. Then by lemma 2.1, it is easy to see that $K_i \in \mathscr{K}_0$. Therefore the number of components of $[C(f)]^{-1}(\mathscr{L})$ is at most r. This and corollary 7.2 implies that $C(f)$ is quasi-monotone.

Next, suppose that $f : X \to Y$ is quasi-monotone and Y is locally connected (the case for weakly monotone mappings are follows from corollary 7.2). By Theorem 6.3, 2^f is confluent. Let $\mathscr{B}$ be a subcontinuum of 2^Y with a nonempty interior and $B \in \operatorname{Int}\mathscr{B}$. There is a positive number ε such that if $L \in 2^Y$ and $H_d(B, L) < \varepsilon$, then $L \in \mathscr{B}$. Since Y is uniformly locally connected, there are $\delta > 0$ and $M \in 2^Y$ such that $V_\delta(B) \subset M \subset V_\varepsilon(B)$ and each component of M intersects B, where $V_\gamma(B)$ is the γ-neighborhood of B in Y for each $\gamma > 0$ (see [10], pp. 20–22). The number of components of M is finite because let $\{M_\alpha : \alpha \in \Omega\}$ be the set of components of M, choose a point $y_\alpha \in M_\alpha \cap B$ for each $\alpha \in \Omega$, then the set $\{y_\alpha : \alpha \in \omega\}$ is discrete and hence a finite set. Let $M_1, M_2, \ldots, M_r$ be the components of M. Since f is quasi-monotone and $\operatorname{int} M_i \neq \phi$,. $f^{-1}(M_i)$ has finitely many, say $n(i)$, components for each $i = 1, 2, \ldots, r$. Then, as the proof of (7.3.1) $\Rightarrow$ (7.3.2), the number of components of $[2^f]^{-1}(\mathscr{B})$ is at most $n(1) \cdot n(2) \ldots n(r)$.

8. Problems

There is an open mapping f such that $C(f)$ is not open (the example in section 4).

1. Is there an open mapping $f : X \to Y$ such that $C(f)$ is open but $C(C(f))$ is not open?

2. Does the correspondence $f \to C(f)$ preserve or reverse the class of MO-mappings? If f is open, then is $C(f)$ an MO-mapping? If 2^f is an MO-mapping, then is f an MO-mapping?

3. For a cofluent mapping $f : X \to Y$, is it true that if 2^f is confluent, then $C(f)$ is confluent?

A continuum X is said to have property [K] if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $a, b \in X$, $d(a, b) < \delta$ and $a \in A \in C(X)$, then there exists $B \in C(X)$ such that $b \in B$ and $H_d(A, B) < \varepsilon$.

It is easy to see that if $\mathscr{A}$ is a subcontinuum of 2^X and $\operatorname{Int}\mathscr{A} \neq \phi$, then $\operatorname{int}\mathscr{A}^* \neq \phi$. If X has property [K], then for a subcontinuum $\mathscr{K}$ of $C(X)$,

$\mathrm{Int}\,\mathscr{K} \neq \phi$ implies $\mathrm{int}\,\mathscr{K}^* \neq \phi$. But if X does not have property [K], $\mathrm{int}\,\mathscr{K}^*$ may be empty.

EXAMPLE. In the Euclidean plan, let us denote xy the straight line segment with the end points x, y. Let $p = (1,0)$, $q = (-1,0)$ and $a_n = (0, 1/n)$ for each $n = 1, 2, \ldots$. Let $A_n = a_{2n}p$, $B_n = a_{2n+1}q$ for $n = 1, 2, \ldots$ and $C = pq$. Let $X = C \cup [\bigcup_{n=1}^{\infty} A_n] \cup [\bigcup_{n=1}^{\infty} B_n]$ and $\mathscr{K} = \{p_s p_t : s - t = 1 \text{ and } 1/3 \leq t \leq 2/3\}$, where $p_s = (s, 0) \in X$, then $\mathscr{K}$ is a subcontinuum of $C(X)$ such that $\mathrm{Int}\,\mathscr{K} \neq \phi$ but $\mathrm{int}\,\mathscr{K}^* = \phi$.

4. In Theorem 6.3, can the condition "Y is locally connected" be weakend?

Added in proof H. Kato announced me that by adding countably many disjoint half open lines on the continua of the example in section 6 of this paper, it is possible to construct continua having property [K] and a confluent mapping between them whose induced mappings are not weakly confluent.

Recently A. Illames answered Problem 1 affirmatively. He showed that if $C(C(f))$ is open, then f is a homeomorphism.

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