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A RESULT EXTENDED FROM GROUPS TO HOPF ALGEBRAS

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We work over an algebraically closed field k of characteristic 0.

The aim of this short note is to prove the following theorem, which is an extension of a well-known fact on finite groups to finite dimensional semisimple Hopf algebras.

THEOREM. *Let p be an odd prime which is congruent to* 2 *modulo* 3. *Then a semisimple Hopf algebra of dimension* $3p$ *is isomorphic to the group-like Hopf algebra* kC_{3p} *of the cyclic group* C_{3p} *of order* $3p$.

This adds a result to the classification lists of semisimple Hopf algebras obtained recently by Larson-Radford [LR3], Zhu [Z], Masuoka [M1-3] and Fukuda [F].

First we show the following:

PROPOSITION 1. *Let* p, q *be primes such that* $p < q$ *and* $q \not\equiv 1$ *modulo* p. *Suppose that a semisimple Hopf algebra of dimension* pq *has a non-trivial group-like. Then* A *is isomorphic to* kC_{pq}.

PROOF. By the Nichols-Zoeller Theorem [NZ, Thm.7] the order of the group $G(A)$ of the group-likes in A devides the dimension $\dim A$ of A. Hence it follows by assumption that there is a Hopf subalgebra K of A isomorphic to either kC_p or kC_q. Let e_A (resp. e_K) be the primitive idempotent in A (resp. in K) sent to 1 by the counit ε_A of A (resp. ε_K of K). These idempotents e_A, e_K are contained in the character ring $C_k(A^*)$ of the dual Hopf algebra $A^* = \mathrm{Hom}_k(A, k)$, which is defined to be the subalgebra of A spanned by the characters of A^* [Z, Page 54]. Hence we have $e_K = e_A + e_2 + \cdots + e_r$, a sum of orthogonal primitive idempotents in $C_k(A^*)$. Note $\dim e_A A = 1$. Since A^* is also semisimple by [LR1, Thm. 3.3], we can apply [Z, Thm.1] to have that

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$\dim e_i A$ divides $pq = \dim A$ for each $2 \le i \le r$. Since $(1 - e_K)A = K^+ A$ where $K^+ = \operatorname{Ker} \varepsilon_K$, one has $e_K A \simeq \bar{A} := A/K^+ A$. Hence it follows by [S, Thm.2.4] that $\dim e_K A = \dim A/\dim K$.

Suppose $K \simeq kC_p$. Then $\dim e_K A = q$. Since $q \not\equiv 1$ modulo p by assumption, we have by counting dimensions that $r = q$ and $\dim e_i A = 1$ for each $2 \le i \le q$. Hence $\bar{A}$ is a quotient algebra of A, or in other words K is a normal Hopf subalgebra. Furthermore this is a quotient Hopf algebra of dimension q, which is isomorphic to kC_q by [Z, Thm.2]. Note that kC_p is selfdual, namely $kC_p \simeq (kC_p)^*$. Then one has a short exact sequence of finite dimensional Hopf algebras,

$$1 \to (kC_p)^* \to A \to kC_q \to 1. \tag{1}$$

It follows from [S, Thm.2.4; DT, Thm.11] that the algebra A is isomorphic to the crossed product $R * C_q$ of C_q over $R = (kC_p)^*$ with the action $\rightharpoonup : C_q \times R \to R$ implemented innerly by a convolution-invertible, right kC_q-colinear section of (1). The actions

$$\triangleright : C_p \times C_q \to C_q, \quad \triangleleft : C_p \times C_q \to C_p$$

which make (C_p, C_q) a matched pair of groups [T, Def.2.1] are both trivial, since a group of order pq is abelian. Hence the the action $\rightharpoonup$, which is induced naturally from some $\triangleleft$ by [M3, Lemma 1.2], is trivial, so that R is included in the center of A. Since the crossed product $R * C_q$ has a free R-basis of the form 1, u, $u^2, \dots, u^{q-1}$ with a unit u in A, A is commutative, so that the Hopf algebra A is isomorphic to $(kC_{pq})^* \simeq kC_{pq}$. (See the proof of [M1, Thm.2].)

Suppose $K \simeq kC_q$. The conclusion follows by exchanging p and q in the proof of the preceding case. ∎

In the same way of showing in the proof above that K is normal, we obtain the following result, which is an extension of the Ore Theorem [Su, Exersise 3(b), Page 34] on finite groups.

PROPOSITION 2. *Let A be a finite dimensional semisimple Hopf algebra and $K \subset A$ a Hopf subalgebra. If the fraction* $\dim A/\dim K$, *which is in fact an integer by* [NZ, Thm.7], *is the smallest prime divisor of* $\dim A$, *then K is a normal Hopf subalgebra.*

PROOF OF THE THEOREM. Let A be a semisimple Hopf algebra of dimension $3p$ with such an odd prime p as in the Theorem.

By proposition 1 it suffices to show that either A or A^* has a non-trivial group-like. We suppose contrary that the groups $G(A)$, $G(A^*)$ of the group-likes are both trivial to see a contradiction. In order for $G(A^*)$ to be trivial, there is no one dimensional ideal in A other than ke_A. Hence by [Z, Thm.1] we have an expression

$$1 = e_A + e_1 + \cdots + e_s + e_{s+1} \tag{2}$$

of 1 as a sum of orthogonal primitive idempotents in $C_k(A^*)$, where $s = (2p-1)/3$, $\dim e_i A = 3 (1 \leq i \leq s)$, $\dim e_{s+1} A = p$. The right ideals $e_i A (1 \leq i \leq s)$ are minimal, since otherwise A would contain a one dimensional ideal other than ke_A. Hence the number n of the minimal (two-sided) ideals in A of dimension 9 satisfies the inequality

$$n \geq \frac{s}{3} = \frac{2p-1}{9}. \tag{3}$$

The number m of all minimal ideals in A satisfies

$$m \geq s + 2 = \frac{2p+5}{3}, \tag{4}$$

since there is a similar expression as (2) in the character ring $C_k(A) (\subset A^*)$ of A, whose dimension $\dim C_k(A)$ equals m. Exclude ke_A and the n minimal ideals of dimension 9. Then there remain $m - n - 1$ minimal ideals of dimension at least 4. Hence we have

$$\begin{aligned} 3p &= \dim A \geq 1 + 9n + 4(m - n - 1) \\ &= 4m + 5n - 3 \\ &\geq \frac{34p + 28}{9} \quad (\text{by } (3), (4)). \end{aligned}$$

This yields immediately a contradiction, which completes the proof. ■

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