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SEMISIMPLE HOPF ALGEBRAS OF DIMENSION 12

By

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Abstract. We determine the isomorphic classes of 12-dimensional semisimple Hopf algebras over an algebraically closed field k whose characteristic ch $k \neq 2, 3$.

Introduction

Recently the project of classifying semisimple Hopf algebras over an algebraically closed field is in progress. For example, in [M2, M3, M4, M5] Masuoka has classified semisimple Hopf algebras of dimensions $2p$, p^2 and p^3 for a prime p in characteristic zero, and found some self-dual Hopf algebras of dimension p^3 which are neither commutative nor cocommutative. Apart from these, little "non-trivial" semisimple Hopf algebras seem to be known. In this paper we classify all semisimple Hopf algebras of dimension 12. As a conclusion, there exists only two (up to isomorphism) Hopf algebras which are neither commutative nor cocommutative, and these are self-dual. For proving the results, we take advantage of the methods of [M2] and [M3].

NOTATION. For a Hopf algebra A over a field k, we denote by $\Delta_A : A \rightarrow A \otimes A$, $\Delta_A(a) = \sum a_{(1)} \otimes a_{(2)}$, $\varepsilon_A : A \rightarrow k$ and $S_A : A \rightarrow A$ the comultiplication, the counit and the antipode of A, respectively. We further denote by $G(A)$ the group of the group-like elements in A. For a finite group G, kG denotes the group-like Hopf algebra of G, and k^G means the dual Hopf algebra $(kG)^*$ of kG. C_n stands for the cyclic group of order n.

Throughout let A be a semisimple Hopf algebra of dimension 12 over an algebraically closed field k whose characteristic ch $k \neq 2, 3$. It follows from [LR, Prop. 4.6] that A is involutory, that is, $S_A \circ S_A = id_A$. Therefore by [LR, Prop.1.3(a)] A^* is semisimple, too.

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1. If A is commutative or cocommutative, then A is isomorphic to a group-like Hopf algebra or its dual. In order to classify all A's that are neither commutative nor cocommutative, we first show:

LEMMA 1.1. *Each of the orders* $|G(A)|$, $|G(A^*)|$ *equals either* 3, 4 *or* 12.

PROOF. By the Nichols-Zoeller theorem [NZ, Thm.7], $|G(A)|$ divides dim A. Further A^* is isomorphic to a direct product of some matrix algebras since it is semisimple. Note that the number of the one-dimensional ideals of A^* equals $|G(A)|$. By counting dimensions, one sees that A^* is isomorphic to one of following:

$$k \times k \times k \times M_3(k), \quad k \times k \times k \times k \times M_2(k) \times M_2(k), \quad \overbrace{k \times \cdots \times k}^{12 \text{ times}},$$

where $M_n(k)$ is the algebra of all $n \times n$ matrices. Thus it follows that $|G(A)| = 3, 4$ or 12. Similarly we have $|G(A^*)| = 3, 4$ or 12. □

PROPOSITION 1.2. *If* $|G(A^*)| = 3$, *then* A *is cocommutative.*

COROLLARY 1.3. *If* A *is neither commutative nor cocommutative, then both the orders* $|G(A)|$, $|G(A^*)|$ *equal* 4.

We devote Sections 2, 3 to the proof of Proposition 1.2. For this, we suppose in these sections that $|G(A^*)| = 3$. It follows from Lemma 1.1 that there exists a subgroup G of $G(A)$ such that $|G| = 3$ or 4.

2. First in this section we prove the following proposition by means of the method of [M2, Sect.1].

PROPOSITION 2.1. *Suppose that* $|G(A^*)| = 3$. *If* $G(A)$ *has a subgroup of order* 3, *then* A *is cocommutative.*

Throughout this section we suppose that $|G(A^*)| = 3$, and that $G(A)$ has a subgroup G of order 3. We fix a generator g of $G(\cong C_3)$. Let $H = kG$.

LEMMA 2.2. *The inclusion map* $i : H \to A$ *has a Hopf algebra retraction* $\pi : A \to H$, *that is, a Hopf algebra map such that* $\pi \circ i = id_H$.

PROOF. (Similar to the proof of [M2, Prop.1.2]) by dualizing the inclusion map $kG(A^*) \hookrightarrow A^*$, we obtain the Hopf quotient map $p: A \to D = k^{G(A^*)}$. Let

$$B = \left\{ a \in A \,\middle|\, \sum a_{(1)} \otimes p(a_{(2)}) = a \otimes p_{(1)} \right\},$$

the left coideal subalgebra of the right D-coinvariants. By [Sch, Thm.2.4] we have

$$A \cong B \otimes D \qquad \text{(left } B\text{-modules and right } D\text{-comodules)}.$$

This implies that $\dim B = 4$. If $p(g) = 1$, equivalently $H \subset B$, then $\dim H$ divides $\dim B$ by the Nichols-Zoeller theorem. This is a contradiction. Therefore $p(g) \neq 1$. Since $D \cong kC_3$, one sees easily that $H \cong D$ via p. Thus this lemma follows. □

We can view B as a quotient coalgebra of A via the isomorphism $B \cong A/AH^+$, $b \mapsto \bar{b}$, where $H^+ = \operatorname{Ker} \varepsilon_H$. By [R, Thm.3], B is a left H-module algebra with the action

$$h \rightharpoonup b = \sum h_{(1)} b S_H(h_{(2)}) \quad (h \in H, b \in B),$$

and a left H-comodule coalgebra with the coaction

$$\rho(b) = (\pi \otimes id_B) \circ \Delta_A(b) \quad (b \in B).$$

Following [R], we denote by $B \times H$ the *biproduct* constructed from $(B, H, \rightharpoonup, \rho)$.

LEMMA 2.3 [R, Thm.3(c)]. *As a Hopf algebra A is isomorphic to the biproduct $B \times H$.*

PROOF. By Lemma 2.2 this follows directly from [R, Thm.3]. □

LEMMA 2.4. (1) *As an algebra B is isomorphic to $k \times k \times k \times k$.*
(2) *B is spanned by group-like elements in B.*

PROOF. (1) As in the proof of [M2, Lemma 1.4], it follows that B is semisimple. Since B has the non-trivial (two-sided) ideal $\operatorname{Ker}(\varepsilon_A|_B)$, B must not be isomorphic to the algebra of all 2×2 matrices. Thus Part (1) follows.
(2) Apply Part (1) to B^*. □

By lemma 2.4(2), we can write as

$$B = k1 \oplus kx_0 \oplus kx_1 \oplus kx_2,$$

where x_i is a group-like element in B for each i. Let ω be a primitive 3rd root of 1. Then we have a symmetric, non-degenerate Hopf pairing $\langle\ ,\ \rangle$: $H\times H\to k, \langle g^i, g^j\rangle = \omega^{ij}$. For each i, we denote by $e_i(\in H)$ the dual basis of $g^i(\in H^*)$ with respect to this pairing.

LEMMA 2.5. *Suppose that A is not cocommutative.*

(1) *The H-module algebra action $\rightharpoonup$ on B is determined by*

$$g\rightharpoonup 1 = 1,\quad g\rightharpoonup x_0 = x_1,\quad g\rightharpoonup x_1 = x_2,\quad g\rightharpoonup x_2 = x_0$$

for a suitable indexing.

(2) *The H-comodule coalgebra coaction ρ of B is determined by*

$$\rho(1) = 1\otimes 1,\quad \rho(x_i) = e_{-i}\otimes x_0 + e_{-i+1}\otimes x_1 + e_{-i+2}\otimes x_2\quad (i=0,1,2)$$

for a suitable choice of $\omega = \langle g, g\rangle$.

(3) *We have*

$$\begin{cases} \dfrac{x_0}{x_1} + \omega\dfrac{x_1}{x_2} + \omega^2\dfrac{x_2}{x_0} = 0 \\ \dfrac{x_0}{x_2} + \omega\dfrac{x_2}{x_1} + \omega^2\dfrac{x_1}{x_0} = 0. \end{cases}$$

(4) *We have*

$$\Delta_B(x_0^2) = \frac{1}{3}\sum_{0\le i,j\le 2}\omega^{-ij}x_0x_i\otimes x_0x_j.$$

PROOF. Since B and H is commutative (resp. cocommutative), it follows from [R, Prop.1] that, if the action $\rightharpoonup$ (resp. the coaction ρ) is trivial, A is commutative (resp. cocommutative). Thus both $\rightharpoonup$ and ρ must be non-trivial.

(1) By [R, Thm.1], the automorphism $g\rightharpoonup : B\to B$ of order 3 is a coalgebra map fixing 1. Thus Part (1) follows.

(2) Since B is a left $H(=H^*)$-comodule coalgebra, B is a right H-module coalgebra with the action

$$b\leftharpoonup h = \sum\langle h, b_H\rangle b_B\quad (b\in B, h\in H),$$

where $\rho(b) = \sum b_H\otimes b_B$. Note that the automorphism $\leftharpoonup g$ of B is a coalgebra map of order 3 which fixes 1. As in Part (1), $\leftharpoonup$ can be determined. Part (2) follows, if one sees that the e_i with respect to the pairing $\langle g, g\rangle = \omega$ equals the e_{-i} with respect to the pairing $\langle g, g\rangle = \omega^{-1}$.

(3) Note that there exists a convolution-inverse S_B of id_B by [R, Prop.2]. As in the proof of [M2, Lemma 1.6(3)], we have

$$\sum S_H(b_{(2)H}) \rightharpoonup (S_B(b_{(2)B})b_{(1)}) = \varepsilon_B(b) \quad (b \in B).$$

Put $b = x_0$ in the above equation. One can verify that

$$e_2 \rightharpoonup \frac{x_0}{x_1} + e_1 \rightharpoonup \frac{x_0}{x_2} = 0. \tag{2.6}$$

Apply $e_2 \rightharpoonup$ (resp. $e_1 \rightharpoonup$) to the equation (2.6), we obtain the upper (resp. lower) equation in Part (3).

(4) From [R, Thm.1(b)], one sees that

$$\Delta_B(bb') = \sum b_{(1)}(b_{(2)H} \rightharpoonup b'_{(1)}) \otimes b_{(2)B}b'_{(2)} \quad (b, b' \in B).$$

Put $b = b' = x_0$, we obtain the equation in Part (4). □

Now we are ready to prove Proposition 2.1

Proof of Proposition 2.1. Suppose that A is not cocommutative. We will prove that this supposition leads to a contradiction.

From lemma 2.4(1), B is isomorphic as an algebra to $k \times k \times k \times k$. Let e be the unique primitive idempotent such that $\varepsilon_A(e) = 1$. We can assume that $e = (1,0,0,0)$. Put $u_0 = (0,1,0,0)$, $u_1 = (0,0,1,0)$, $u_2 = (0,0,0,1)$. Since the non-trivial action $g \rightharpoonup$ of g is an algebra automorphism of B, the action $\rightharpoonup$ is determined by

$$g \rightharpoonup e = e, \quad g \rightharpoonup u_0 = u_1, \quad g \rightharpoonup u_1 = u_2, \quad g \rightharpoonup u_2 = u_0$$

for a suitable indexing. Note that $\varepsilon_A(x_0) = 1$, and that x_0 is a unit in B. We can put $x_0 = (1, c_0, c_1, c_2)$, where $c_i \neq 0$ for each i. From Lemma 2.5(1) and the equations in Lemma 2.5(3), we have

$$\begin{cases} \dfrac{c_0}{c_1} + \omega \dfrac{c_1}{c_2} + \omega^2 \dfrac{c_2}{c_0} = 0 \\ \dfrac{c_0}{c_2} + \omega \dfrac{c_2}{c_1} + \omega^2 \dfrac{c_1}{c_0} = 0. \end{cases} \tag{2.7}$$

These equations imply that $c_0^3 = c_1^3 = c_2^3$. Hence x_i's are described as

$$x_0 = (1, c, \lambda c, \mu c), \quad x_1 = (1, \mu c, c, \lambda c), \quad x_2 = (1, \lambda c, \mu c, c),$$

where $c (\in k)$ is non-zero, and λ, μ are 3rd roots of 1. But it cannot happen that $\mu = \lambda^{-1}$, for x_0, x_1, x_2 are linearly independent. If $(\lambda, \mu) = (1, \omega)$ or $(\omega, 1)$, it contradicts the equation (2.7). We show a contradiction in the other cases, to complete the proof. Suppose that $(\lambda,\mu)=(\omega,\omega),(1,\omega^2)$ or $(\omega^2,1)$. Since $x_0^2\in B$, we write as $x_0^2=\alpha 1+\beta x_0+\gamma x_1+\delta x_2$ $(\alpha,\beta,\gamma,\delta\in k)$. Then

$$\Delta_B(x_0^2)=\alpha(1\otimes 1)+\beta(x_0\otimes x_0)+\gamma(x_1\otimes x_1)+\delta(x_2\otimes x_2). \tag{2.8}$$

By comparing the coefficients of $u_i\otimes u_i (i=0,1,2)$ in the right-hand side of the equation (2.8) and that in Lemma 2.5(4), we have

$$\begin{pmatrix} 1 & \mu^2 & \lambda^2 \\ \lambda^2 & 1 & \mu^2 \\ \mu^2 & \lambda^2 & 1 \end{pmatrix}\begin{pmatrix}\beta\\ \gamma\\ \delta\end{pmatrix}=t\begin{pmatrix}1\\1\\1\end{pmatrix},$$

where $t=\frac{1}{c^2}((1+\lambda+\mu)c^4-\alpha)$. This equation shows that $\beta=\gamma=\delta$. Then it is seen easily that $\Delta_B(x_0^2)=\Delta_B(g\rightharpoonup x_0^2)=\Delta_B(x_1^2)$, so that $x_0^2=x_1^2$. In the case $(\lambda,\mu)=(1,\omega^2)$ or $(\omega^2,1)$ (resp. (ω,ω)), one can verify that $c^2=\omega c^2$ (resp. $c^2=\omega^2c^2$), which gives a contradiction to the fact that $c\neq 0$. □

3. Next in this section we prove the following proposition, to complete the proof of Proposition 1.2. For this, we adopt the method of [M3].

PROPOSITION 3.1. *Suppose that* $|G(A^*)|=3$. *If* $G(A)$ *has a subgroup of order* 3, *then* A *is cocommutative.*

Throughout this section we suppose that $|G(A^*)|=3$, and that $G(A)$ has a subgroup G of order 4. As in Section 2, we obtain a Hopf quotient map $\pi: A\to k^{G(A^*)}(\cong kC_3)$. We can regard $k^G=kG\subset A$, for G is an abelian group.

LEMMA 3.2. *The short sequence*

$$1\to k^G\xrightarrow{i}A\xrightarrow{\pi}kC_3\to 1$$

of Hopf algebras is exact [M1, Def.1.3], *where* i *is the inclusion map.*

PROOF. We claim that $\pi(x)=1$ for any $x\in G(k^G)$. Otherwise, the order of $\pi(x)$ equals 2 or 4, for $|G(k^G)|=4$. This contradicts that $\pi(x)\in G(kC_3)(=C_3)$. Hence the condition (in [M1, Lemma 1.2]) that

$$k^G=\left\{a\in A\,\middle|\,\sum a_{(1)}\otimes\pi(a_{(2)})=a\otimes\pi(1)\right\}$$

holds. In other words, the sequence is exact. □

Let $K = k^G$, $H = kC_3$. Now we obtain a Hopf algebra extension

$$1 \to K \xrightarrow{i} A \xrightarrow{\pi} H \to 1. \tag{3.3}$$

As mentioned in [M3, Sect.1], such an extension has a *section*, that is, a unit and counit-preserving convolution-invertible integral $\phi : H \to A$ and a retraction, that is, a unit and counit-preserving convolution-invertible cointegral $\gamma : A \to K$. Furthermore there is a $1-1$ correspondence between all sections and all retractions. Then ϕ causes a left H-module algebra action on K

$$\rightharpoonup : H \otimes K \to K, \quad i(h \rightharpoonup c) = \sum \phi(H_{(1)}) i(c) \phi^{-1}(h_{(2)}) \quad (h \in H, c \in K),$$

and γ causes a right K-comodule coalgebra coaction of H

$$\rho : H \to H \otimes K, \quad \rho(\pi(a)) = \sum \pi(a_{(2)}) \otimes \gamma^{-1}(a_{(1)}) \gamma(a_{(3)}) \quad (a \in A).$$

Since H is commutative and K is cocommutative, such an action $\rightharpoonup$ and a coaction ρ are independent of the choice of ϕ and ρ. (See [M3, Sect.1].) Then A is isomorphic to the *bicrossed product* with the action $\rightharpoonup$ and the coaction ρ [H, Sect.3]. (We need not know the cocycle and the dual cocycle.) In terms of [P], A is as an algebra *crossed product* $K * C_3$ with the action $\rightharpoonup$, and A^* is as an algebra $H^* * G$ with the action ρ^*. Notice that A has the K-basis $\{1, \phi(g), \phi(g^2)\}$, and that A^* has the H^*-basis $\{\gamma^*(x) | x \in G\}$.

LEMMA 3.4. *Suppose that the Hopf algebra extension* (3.3) *causes a pair* $(\rightharpoonup, \rho)$ *described above.*

(1) $G \cong C_2 \times C_2$.

(2) *The right K-comodule coalgebra coaction ρ of H is trivial.*

(3) *The left H-module algebra action $\rightharpoonup$ on K is determined by*

$$g \rightharpoonup e_{ij} = e_{j-i,i},$$

where g is a generator of C_3, and $e_{ij} (\in K)$ is a dual basis of $s^i t^j$ $(\in kG = k(\langle s \rangle \times \langle t \rangle))$ for each i, j.

PROOF. Since $(H, K, \rightharpoonup, \rho)$ is a *abelian matched pair* of hopf algebras by [H, Prop.3.8], it follows by [M3, Lemma 1.2] that the pair $(\rightharpoonup, \rho)$ is corresponding to a pair $(\triangleright, \triangleleft)$ which makes (G, C_3) a matched pair of groups, where $\triangleright : G \times C_3 \to C_3$, $\triangleleft : G \times C_3 \to G$ are group actions. The correspondence is as follows:

$$(y \rightharpoonup f)(x) = f(x \triangleleft y), \quad \rho(y) = \sum_{x \in G} (x \triangleright y) \otimes e_x,$$

where $x \in G$, $y \in C_3$, $f \in K$ and $e_x(\in K)$ is a dual basis of $x(\in kG)$. So in order to determine the pair $(\rightharpoonup, \rho)$, we will determine the corresponding pair $(\triangleright, \triangleleft)$. Denote by $C_3 \bowtie G$ the group constructed from a matched pair $(G, C_3, \triangleright, \triangleleft)$. (See [T, Def.2.3].) It follows from [Sz, Page 112] that either G or C_3 must be a normal subgroup of $C_3 \bowtie G$. Then one sees easily that either $\triangleright$ or $\triangleleft$ is trivial. If $\triangleleft$ is trivial, equivalently $\rightharpoonup$ is so, then A is isomorphic to the *twisted group ring* $K^t[C_3]$ [P, Page 4]. Since a twisted group ring of cyclic group over a commutative ring is commutative, $\triangleleft$ must be non-trivial. Further it is seen easily that $\triangleleft$ is always trivial in the case $G \cong C_4$. This observation shows that $G \cong C_2 \times C_2$, and that $\triangleright$ is trivial. Then $\triangleleft g : G \to G$ is a group automorphism of order 3. Such a group action $\triangleleft$ is determined by

$$s \triangleleft g = t, \quad t \triangleleft g = st$$

for a suitable choice of generators s, t of G. The action $\rightharpoonup$ corresponding to this group action $\triangleleft$ is as in Part (2). $\square$

Now we will prove Proposition 3.1.

PROOF OF PROPOSITION 3.1. Notation as above. Let $\gamma : A \to K$ be a retraction of the extension (3.3). Note that the coaction ρ is trivial. As in the proof of [M3, Lemma 2.11(1)], we can choose a retraction γ satisfying

$$\gamma^*(s^2) = \gamma^*(t^2) = 1.$$

Note that $\phi(g)c = (g \rightharpoonup c)\phi(g)$ for any $c \in K$. Then as in the proof of [M3, Lemma 2.11(2)(3)], we have for $\xi = 1$ or -1

$$\Delta_A(\bar{g}) = \sum_{0 \leq i,j,r,s \leq 1} \xi^{jr} e_{ij}\bar{g} \otimes e_{rs}\bar{g},$$

where $\phi : H \to A$ is the section corresponding to γ, and $\bar{g} = \phi(g)$. A straightforward calculation using the above equation shows

$$\Delta_A(\bar{g}^3) = \sum_{i,j,r,s} \xi^{ir+j(r+s)} e_{ij}\bar{g}^3 \otimes e_{rs}\bar{g}^3.$$

Since $\bar{g}^3 \in K$, we write as $\bar{g}^3 = \sum c_{ij}e_{ij}$ for $C_{ij} \in k$. By comparing the coefficients of $e_{01} \otimes e_{10}$, $e_{10} \otimes e_{01}$ in $\Delta_A(\bar{g}^3)$, one verifies that

$$c_{11} = \xi c_{01}c_{10}, \quad c_{11} = c_{10}c_{01}.$$

These yield that $\xi = 1$, so that $\bar{g} \in G(A)$. Then we can check easily that $A \cong kD$, where $D = (C_2 \times C_2) \rtimes C_3$ is the unique (up to isomorphism) semi-direct product. In particular A is cocommutative. □

4. Finally we find all A's that are neither commutative nor cocommutative, to complete the classification. By the Corollary 1.3, we may suppose that $|G(A)| = |G(A^*)| = 4$. Let $G = G(A)$, $H = kG$. As in Section 2, the inclusion map $i : H \to A$ has a Hopf algebra retraction $\pi : A \to H$. Let $B = \{a \in A \mid \sum a_{(1)} \otimes \pi(a_{(2)}) = a \otimes \pi(1)\}$. By the same argument as in Section 2, B is a H-module algebra with a non-trivial action $\rightharpoonup$ and a H-comodule coalgebra with a non-trivial coaction ρ. Further we have

$$A \cong B \times H \quad \text{(as Hopf algebras)},$$

$$B = k1 \oplus kx_+ \oplus kx_-,$$

where $x_\pm$ are group-like elements in B.

Denote by $\mathfrak{S}_3$ the symmetric group of degree 3. Let σ be the cyclic permutation (123), and τ the transposition (12). We denote by ι the inner automorphism $inn(\tau)$. Notice that sgn, the signature map of $\mathfrak{S}_3$, is the unique non-trivial group-like element in $k^{\mathfrak{S}_3}$.

Definition 4.1. Denote by A_+ (resp. A_-) the $k^{\mathfrak{S}_3}$-ring generated by z with relations:

$$z^2 = 1 \quad (\text{resp. } sgn), \qquad zc = \iota^*(c)z \quad (c \in k^{\mathfrak{S}_3}).$$

Given $A_\pm$ a coalgebra structure such that the subalgebra $k^{\mathfrak{S}_3}$ is a subcoalgebra, and that z is group-like, then $A_\pm$ are bialgebras. Furthermore A_+ (resp. A_-) becomes a Hopf algebra with the antipodes S determined by

$$S(z) = z \quad (\text{resp. } (sgn)z), \qquad S(c) = S_{k^{\mathfrak{S}_3}}(c) \quad (c \in k^{\mathfrak{S}_3}).$$

We point out that $A_\pm$ are semisimple. Indeed $A_\pm \cong k \times k \times k \times k \times M_2(k) \times M_2(k)$. It is seen easily that $G(A_+) \cong C_2 \times C_2$, and that $G(A_-) \cong C_4$.

Remark 4.2. (1) As an algebra A_+^* is isomorphic to A_-^*. In fact, these are the $k\mathfrak{S}_3$-rings generated by v with relations:

$$v^2 = v, \qquad av = va \quad (a \in k\mathfrak{S}_3).$$

On the other hand, the coalgebra structures Δ, ε, and the antipode S of A_+^* (resp.

A^*_-) are determined by

$$\Delta(\sigma) = \sigma v \otimes \sigma + \sigma(1-v) \otimes \sigma^2, \quad \varepsilon(\sigma) = 1,$$
$$\Delta(\tau) = \tau \otimes \tau \text{ (resp. } \tau v \otimes \tau + \tau(1-v) \otimes \tau(2v-1)), \quad \varepsilon(\tau) = 1,$$
$$\Delta(v) = v \otimes v + (1-v) \otimes (1-v), \quad \varepsilon(v) = 1,$$
$$S(\sigma) = \sigma(1-v) + \sigma^2 v,$$
$$S(\tau) = \tau \text{ (resp. } \tau(2v-1)), \quad S(v) = v.$$

(2) $A^*_\pm$ are both self-dual, that is, $A^*_\pm \cong A_\pm$. Let ω be a primitive 3rd root of 1, ζ_+ a square root of 1, and ζ_- a primitive square root of -1. Denote by $e_{ij}(\in K^{\mathfrak{S}_3})$ the dual basis of $\sigma^i\tau^j(\in k\mathfrak{S}_3)$ for each i, j. Then the mapping $\sigma \mapsto \sum \omega^j e_{ij}$, $\tau \mapsto 1/2((1+\zeta_\pm)+(1-\zeta_\pm)sgn)z$, $v \mapsto 1/2(1+sgn)$ gives Hopf algebra isomorphisms from $A^*_\pm$ to $A_\pm$.

PROPOSITION 4.3. *Suppose that* $|G(A)| = |G(A^*)| = 4$. *Then as a Hopf algebra A is isomorphic to either A_+ or A_-.*

PROOF. Case $G \cong C_4$. We fix a generator g of G. By the same way as in Section 2, the H-module algebra action $\rightharpoonup$ on B and the H-comodule coalgebra coaction ρ of B are determined by

$$g \rightharpoonup x_\pm = x_\mp, \quad \rho(x_\pm) = \tfrac{1}{2}((1+g^2) \otimes x_\pm + (1-g^2) \otimes x_\mp).$$

Since $\rho(B) \subset k\langle g^2\rangle \otimes B$, it follows that $B \otimes k\langle g^2\rangle = B \times k\langle g^2\rangle$ is a 6-dimensional (semisimple) Hopf subalgebra of A. Denote by K this Hopf subalgebra. Note that K is commutative and not cocommutative, it follows by [M2, Thm.1.10] that $K \cong k^{\mathfrak{S}_3}$. It is clear that A is the crossed product $K * C_2$ with the K-basis $\{1, g\}$ such that $g \in G(A)$, and that g^2 is the unique non-trivial group-like element in K. We conclude that $A \cong A_-$, if one sees that ι^* is the unique (up to conjugacy) Hopf algebra automorphism of $k^{\mathfrak{S}_3}$ of order 2 with non-trivial invariants.

Case $G \cong C_2 \times C_2$. We can choose generators s, t of G so that the action $\rightharpoonup$ is determined by

$$s \rightharpoonup x_\pm = x_\mp, \quad t \rightharpoonup x_\pm = x_\pm.$$

The coaction ρ is one of following:
(i) $\rho(x_\pm) = \frac{1}{2}((1 \pm t) \otimes x_+ + (1 \mp t) \otimes x_-)$.
(ii) $\rho(x_\pm) = \frac{1}{2}((1 \pm s) \otimes x_+ + (1 \mp s) \otimes x_-)$.
(iii) $\rho(x_\pm) = \frac{1}{2}((1 \pm st) \otimes x_+ + (1 \mp st) \otimes x_-)$.

In each case, it follows as in the above case that $B \times \langle t \rangle$, $B \times \langle s \rangle$ or $B \times \langle st \rangle$ is a 6-dimensional Hopf subalgebra of A. Since this Hopf subalgebra must be commutative or cocommutative by [M2, Thm.1.10] Case (ii) or (iii) cannot happen. As in Case $G \cong C_4$, we conclude that $A \cong A_+$. □

Now we obtain the classification result.

THEOREM. *Let A be a 12-dimensional semisimple Hopf algebra over an algebraically closed field k whose characteristec $\neq 2$ or 3. Then A is isomorphic to either*

$$kG, \qquad k^G, \qquad A_+ \qquad \text{or} \qquad A_-,$$

where G is a group of order 12 and $A_\pm$ are the mutually non-isomorphic Hopf algebras defined in Definition 4.1.

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