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ALMOST KÄHLER MANIFOLDS OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

By

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1. Introduction

An almost Hermitian manifold $M=(M,J,g)$ is called an almost Kähler manifold if the corresponding Kähler form Ω is closed, or equivalently,

$$g((\nabla_X J)Y,Z)+g((\nabla_Y J)Z,X)+g((\nabla_Z J)X,Y)=0,$$

for all smooth vector fields X, Y, Z on M.

Concerning the integrability of the almost complex structure of an almost Kähler manifold, the following conjecture of S. I. Goldberg is well-known ([1]):

The almost complex structure of a compact Einstein almost Kähler manifold is integrable (and therefore the manifold is Kähler).

As concerns this conjecture, some progress has been made under some curvature conditions: K. Sekigawa proved that the above conjecture is true when the scalar curvature is non-negative [8, 9]. A complete almost Kähler manifold of constant sectional curvature is a flat Kähler manifold [4, 5, 10]. A 4-dimensional compact almost Kähler manifold which is Einstein and $*$-Einstein is a Kähler manifold [10].

In connection with these results, the author proved that a compact almost Kähler manifold of constant holomorphic sectional curvature $c \geqq 0$ which satisfies the curvature condition (b) is Kähler [7]. (The condition (b) will be given in the next section.)

In this note, we shall show that, in the case where the dimension of the manifold is four, the above statement can be improved. Namely, we shall prove that a four-dimensional compact almost Kähler manifold of constant holomorphic sectional curvature which satisfies the RK-condition is a Kähler manifold.

Throughout this paper, we assume that the manifold under consideration to be connected and of class C^∞.

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2. Preliminaries

Let $M=(M,J,g)$ be an $m(=2n)$-dimensional almost Hermitian manifold with an almost Hermitian structure (J,g). We denote by Ω and N the Kähler form and the Nijenhuis tensor of M defined respectively by $\Omega(X,Y)=g(X,JY)$ and $N(X,Y)=[JX,JY]-[X,Y]-J[JX,Y]-J[X,JY]$ for $X,Y\in\mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of all smooth vector fields on M. The Nijenhuis tensor N satisfies

$$N(JX,Y)=N(X,JY)=-JN(X,Y),\quad X,Y\in\mathfrak{X}(M).$$

Further we denote by $\nabla, R=(R_{ijk}{}^{l}), \rho=(\rho_{ij}), \tau, \rho^{*}=(\rho^{*}_{ij})$ and τ^{*} the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor, the scalar curvature, the Ricci $*$-tensor and the $*$-scalar curvature of M, respectively:

$$\begin{aligned}
R(X,Y)Z &= [\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z,\\
R(X,Y,Z,W) &= g(R(X,Y)Z,W),\\
\rho(x,y) &= \text{trace of } [z\mapsto R(z,x)y],\\
\tau &= \text{trace of } \rho,\\
\rho^{*}(x,y) &= \text{trace of } [z\mapsto R(x,Jz)Jy],\\
\tau^{*} &= \text{trace of } \rho^{*}.
\end{aligned}$$

where $X,Y,Z,W\in\mathfrak{X}(M)$, $x,y,z\in T_pM$, $p\in M$. The Ricci $*$-tensor ρ^{*} satisfies

$$\rho^{*}(JX,JY)=\rho^{*}(Y,X),\qquad X,Y\in\mathfrak{X}(M).$$

An almost Hermitian manifold M is called a *weakly $*$-Einstein manifold* if it satisfies $\rho^{*}=\lambda^{*}g$ for some function λ^{*} on M. In particular, if λ^{*} is constant on M, then M is called a *$*$-Einstein manifold*.

We define the tensor field $G=(G_{ijkl})$ by

$$G(X,Y,Z,W)=R(X,Y,Z,W)-R(X,Y,JZ,JW).$$

Then we have

$$\sum_{i=1}^{m}G(E_i,X,Y,E_i)=\rho(X,Y)-\rho^{*}(X,Y), \tag{2.1}$$

where $\{E_i\}_{i=1,\cdots,m}$ is a local orthonormal frame of M. We shall consider the following three conditions on G for an almost Hermitian manifold M:

(a) $G=0$,

(b) $G(JX, Y, JZ, W) = G(X, Y, Z, W)$, for $X, Y, Z, W \in \mathfrak{X}(M)$,

(c) $G(X, Y, Z, W) = G(Z, W, X, Y,)$, for $X, Y, Z, W \in \mathfrak{X}(M)$.

It is easily shown that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). We remark that the condition (a) is equivalent to

$$R(X, Y) \circ J = J \circ R(X, Y), \tag{2.2}$$

for $X, Y \in \mathfrak{X}(M)$. The condition (b) is equivalent to

$$\begin{aligned} &R(X, Y, Z, W) \\ &= R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW), \end{aligned} \tag{2.3}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$. The condition (c) is equivalent to

$$R(JX, JY, JZ, JW) = R(X, Y, Z, W) \tag{2.4}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$. An almost Hermitian manifold M satisfying (2.4) (and so, the condition (c)) is called an *RK-manifold* ([11]).

In an RK-manifold M, we have easily the following

$$G(JX, JY, JZ, JW) = G(X, Y, Z, W), \tag{2.5}$$

$$\rho(JX, JY) = \rho(X, Y), \tag{2.6}$$

$$\rho^*(X, Y) = \rho^*(Y, X), \tag{2.7}$$

$$\rho^*(JX, JY) = \rho^*(X, Y), \tag{2.8}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$.

Now we assume that $M = (M, J, g)$ is an almost Kähler manifold. Then we have

$$2g((\nabla_X J)Y, Z) = g(JX, N(Y, Z)).$$

It is well-known that ([2])

$$\begin{aligned} &G(X, Y, Z, W) + G(JX, JY, JZ, JW) \\ &\quad + G(JX, Y, JZ, W) + G(X, JY, Z, JW) \\ &\quad = 2g((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z). \end{aligned} \tag{2.9}$$

By (2.1) and (2.9), we have

$$\begin{aligned} &\rho(X, Y) + \rho(JX, JY) - \rho^*(X, Y) - \rho^*(JX, JY) \\ &\quad = -\sum_{i=1}^{m} g((\nabla_{E_i} J)X,\ (\nabla_{E_i} J)Y), \end{aligned} \tag{2.10}$$

$$\tau - \tau^* = -\frac{1}{2}\|\nabla J\|^2 = -\frac{1}{8}\|N\|^2. \tag{2.11}$$

In the sequel, we shall adopt the following notational convention:

$$\nabla_{\bar{i}} J_{jk} = J_i{}^a \nabla_a J_{jk}, \quad \nabla_{\bar{i}} J_{\bar{j}\bar{k}} = J_i{}^a J_j{}^b \nabla_a J_{bk},$$
$$N_{\bar{i}jk} = J_i{}^a N_{ajk}, \quad N_{\bar{i}\bar{j}k} = J_i{}^a J_j{}^b N_{abk},$$
$$R_{ijk\bar{l}} = J_l{}^a R_{ijka}, \quad G_{ij\bar{k}\bar{l}} = J_k{}^a J_l{}^b G_{ijab}, \qquad \text{etc.}$$

Then it is easy to see that

$$\nabla_i J_{jk} + \nabla_{\bar{i}} J_{\bar{j}k} = 0,$$
$$\nabla_{\bar{i}} J_{jk} = \nabla_i J_{\bar{j}k} = \nabla_i J_{j\bar{k}},$$
$$N_{ijk} = -2\nabla_{\bar{i}} J_{jk}, \quad 2\nabla_i J_{jk} = N_{\bar{i}jk}.$$

By the Ricci identity, we have

$$\begin{aligned} 2G_{ijkl} &= \frac{1}{2}N^t{}_{ij}N_{tkl} - \nabla_i N_{j\bar{k}\bar{l}} + \nabla_j N_{i\bar{k}\bar{l}} \\ &= \nabla_i N_{jkl} - \nabla_j N_{ikl} + \frac{1}{2}(N_{ila}N_{jk}{}^a - N_{ika}N_{jl}{}^a). \end{aligned} \tag{2.12}$$

3. The pointwise constancy of holomorphic sectional curvature

Let $M=(M,J,g)$ be an $m(=2n)$-dimensional almost Kähler manifold of pointwise constant holomorphic sectional curvature $c=c(p)$ $(p \in M)$. Then, taking account of [6] and (2.12), we have

$$\begin{aligned} R_{ijkl} &= \frac{c}{4}\{g_{il}g_{jk} - g_{ik}g_{jl} + J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl}\} \\ &\quad + \frac{1}{16}\{N^t{}_{ik}N_{tjl} - N^t{}_{il}N_{tjk} + 2N^t{}_{ij}N_{tkl}\} \\ &\quad + \frac{1}{96}Q_{ijkl}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} Q_{ijkl} = &-13\{\nabla_i N_{j\bar{k}\bar{l}} - \nabla_j N_{i\bar{k}\bar{l}} + \nabla_k N_{l\bar{i}\bar{j}} - \nabla_l N_{k\bar{i}\bar{j}}\} \\ &+3\{\nabla_{\bar{i}} N_{jk\bar{l}} - \nabla_{\bar{j}} N_{ik\bar{l}} + \nabla_{\bar{k}} N_{li\bar{j}} - \nabla_{\bar{l}} N_{ki\bar{j}}\} \\ &-\frac{13}{2}\{\nabla_i N_{\bar{k}\bar{j}l} - \nabla_k N_{\bar{i}\bar{j}l} + \nabla_j N_{\bar{l}\bar{i}k} - \nabla_l N_{\bar{j}\bar{i}k} \\ &\quad - \nabla_i N_{\bar{l}\bar{j}k} + \nabla_l N_{\bar{i}\bar{j}k} - \nabla_j N_{\bar{k}\bar{i}l} + \nabla_k N_{\bar{j}\bar{i}l}\} \\ &+\frac{3}{2}\{\nabla_{\bar{i}} N_{kj\bar{l}} - \nabla_{\bar{k}} N_{ij\bar{l}} + \nabla_{\bar{j}} N_{li\bar{k}} - \nabla_{\bar{l}} N_{ji\bar{k}} \end{aligned}$$

$$
\begin{aligned}
&-\nabla_{\bar i}N_{lj\bar k}+\nabla_{\bar l}N_{ij\bar k}-\nabla_{\bar j}N_{ki\bar l}+\nabla_{\bar k}N_{ji\bar l}\}\\
&+2\{\nabla_i N_{j\bar k\bar l}+\nabla_{\bar i}N_{\bar jkl}+\nabla_j N_{ikl}+\nabla_{\bar j}N_{\bar i\bar k\bar l}\}\\
&+\nabla_i N_{k\bar j\bar l}+\nabla_{\bar i}N_{\bar kjl}+\nabla_k N_{ijl}+\nabla_{\bar k}N_{\bar i\bar j\bar l}\\
&-\nabla_i N_{\bar l\bar j\bar k}-\nabla_i N_{\bar ljk}-\nabla_l N_{ijk}-\nabla_{\bar l}N_{\bar i\bar j\bar k}.
\end{aligned}
$$

We put

(3.2)
$$S_{ij}=N_{iab}N_j{}^{ab},$$

and

(3.3)
$$T_{ij}=N_{abi}N^{ab}{}_j.$$

Then, from (3.1), we have

(3.4)
$$
\begin{aligned}
\rho_{ij}&=\frac{n+1}{2}cg_{ij}-\frac{1}{8}S_{ij}+\frac{1}{32}T_{ij}+\frac{1}{4}(\nabla^a N_{ija}+\nabla^a N_{jia})\\
&=\frac{\tau}{2n}g_{ij}+\frac{3}{64n}\|N\|^2 g_{ij}-\frac{1}{8}S_{ij}+\frac{1}{32}T_{ij}+\frac{1}{4}(\nabla^a N_{ija}+\nabla^a N_{jia}),
\end{aligned}
$$

(3.5)
$$\tau=n(n+1)c-\frac{3}{32}\|N\|^2,$$

(3.6)
$$
\begin{aligned}
\rho^*_{ij}&=\frac{n+1}{2}cg_{ij}+\frac{1}{32}T_{ij}-\frac{1}{4}(\nabla^a N_{ija}-\nabla^a N_{jia})\\
&=\frac{\tau^*}{2n}g_{ij}-\frac{1}{64n}\|N\|^2 g_{ij}+\frac{1}{32}T_{ij}-\frac{1}{4}(\nabla^a N_{ija}-\nabla^a N_{jia}),
\end{aligned}
$$

(3.7)
$$\tau^*=n(n+1)c+\frac{1}{32}\|N\|^2,$$

(3.8)
$$\rho_{ij}-\rho^*_{ij}=-\frac{1}{8}S_{ij}+\frac{1}{2}\nabla^a N_{ija},$$

(3.9)
$$\tau+3\tau^*=m(m+2)c.$$

Now we assume that both ρ and ρ^* are J-invariant. Then $\nabla^a N_{ija}$ is also J-invariant and symmetric with respect to the indices i,j by (3.8). It follows that

$$\rho^*_{ij}=\frac{\tau^*}{2n}g_{ij}-\frac{1}{64n}\|N\|^2 g_{ij}+\frac{1}{32}T_{ij}.$$

Hence, M is weakly $*$-Einstein if

(3.10)
$$T_{ij}=\frac{1}{2n}\|N\|^2 g_{ij}.$$

Moreover, by (2.10) we have

(3.11)
$$\rho_{ij}-\rho^*_{ij}=-\frac{1}{2}(\nabla_a J_{bi})\nabla^a J^b_j=-\frac{1}{8}T_{ij}.$$

Then we have from (3.10) and (2.11)

$$\rho_{ij} = \rho^*_{ij} - \frac{1}{8}T_{ij} = \frac{1}{2n}\left(\tau^* - \frac{\|N\|^2}{8}\right)g_{ij} = \frac{\tau}{2n}g_{ij},$$

which shows that M is Einstein.

Since the RK-condition (c) implies that ρ and ρ^* are J-invariant (see (2.6), (2.8)), we get the following

PROPOSITION 3.1. *Let M be an almost Kähler manifold of pointwise constant holomorphic sectional curvature. If M satisfies the condition* (c) *and* $T = (1/2n)\|N\|^2 g$, *then M is an Einstein and weakly $*$-Einstein manifold.*

We have the following theorem which is analogous to the result of Watanabe and Takamatsu [12] for nearly Kähler manifolds.

THEOREM 3.2. *Let M be an almost Kähler manifold of pointwise constant holomorphic sectional curvature satisfying*

$$\rho - \rho^* = \frac{\tau - \tau^*}{2n}g. \tag{3.12}$$

Then M is an Einstein and weakly $*$-Einstein manifold.

PROOF. By (3.12), ρ^* is symmetric and therefore J-invariant. So ρ is also J-invariant. Then from (3.11), (3.12) and (2.11), we have $T = \frac{1}{2n}\|N\|^2 g$. Thus theorem follows from Proposition 3.1. ■

For the 4-dimensional case, we get the following

THEOREM 3.3. *Let M be a 4-dimensional compact almost Kähler manifold of constant holomorphic sectional curvature. If M satisfies the condition* (c), *then M is a Kähler manifold.*

PROOF. For an orthonormal basis $\{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ of T_pM, we have

$$\begin{aligned} T_{11} &= \sum_{a,b=1}^{4} N^a{}_{b1}N^a{}_{b1} = \sum_{a=1}^{4}(N^a{}_{31}N^a{}_{31} + N^a{}_{41}N^a{}_{41}) \\ &= \|N(e_3,e_1)\|^2 + \|N(e_4,e_1)\|^2 = 2\|N(e_1,e_3)\|^2 \\ &= \frac{1}{4}\|N\|^2, \end{aligned}$$

$$
\begin{aligned}
T_{12} &= \sum_{a,b=1}^{4} N^a{}_{b1} N^a{}_{b2} = \sum_{a=1}^{4} (N^a{}_{31} N^a{}_{32} + N^a{}_{41} N^a{}_{42}) \\
&= g(N(e_3,e_1), N(e_3,Je_1)) + g(N(e_4,e_1), N(e_4,Je_1)) \\
&= -g(N(e_3,e_1), JN(e_3,e_1)) - g(N(e_4,e_1), JN(e_4,e_1)) \\
&= 0.
\end{aligned}
$$

Similarly, we have

$$T_{22} = T_{33} = T_{44} = \frac{1}{4}\|N\|^2,$$

and

$$T_{13} = T_{14} = T_{23} = T_{24} = T_{34} = 0.$$

Consequently, we have

$$T_{ij} = \frac{1}{4}\|N\|^2 g_{ij}.$$

By Proposition 3.1, we see that M is an Einstein and weakly $*$-Einstein manifold. Since c and τ are constant on M, τ^* is also constant by (3.9), that is, M is $*$-Einstein. Then, taking account of the theorem of Sekigawa and Vanhecke [10], we can conclude that M is Kählerian. ■

REMARK. From the result of U. K. Kim, I-B. Kim and J-B. Jun [3], it will be also obtained that M is Einstein and weakly $*$-Einstein.

COROLLARY 3.4. *Let M be a 4-dimensional compact almost Kähler manifold of constant holomorphic sectional curvature satisfying*

$$\rho - \rho^* = \frac{\tau - \tau^*}{4} g$$

Then M is a Kähler manifold.

PROOF. This follows from Theorem 3.2 and Theorem 3.3. ■

COROLLARY 3.5. *Let M be a 4-dimensional compact almost Kähler manifold of pointwise constant holomorphic sectional curvature. If M satisfies the condition (b), then M is a Kähler manifold.*

PROOF. Under the condition (b), we can see that the function c is constant on M ([6]). Hence this follows immediately from Theorem 3.3. ■

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