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# ON A TYPE OF REAL HYPERSURFACES IN COMPLEX PROJECTIVE SPACE

By

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**Abstract.** We give a classification of real hypersurfaces in complex projective space under assumptions that the structure vector $\xi$ is principal, the focal map has constant rank and that $\nabla_{\xi}C=0$, where $C$ is the Weyl conformal curvature tensor of the real hypersurface.

## 1. Introduction

Let $M^{n}(c)$ denote an n-dimensional complex space form with constant holomorphic sectional curvature $c$. It is well known that a complete and simply connected complex space form is either complex projective space $PC^{n}$, complex Euclidean space $C^{n}$ or complex hyerbolic space $HC^{n}$, according as $c>0$, $c=0$ or $c<0$.

In this paper we consider a real hypersurface $M$ of $PC^{n}$. The induced almost contact metric structure and the Weyl conformal curvature tensor of the real hypersurface $M$ in $PC^{n}$ are respectively denoted by $(\varphi, \xi, \eta, g)$ and $C$. Many differential geometers have studied $M$ by using the structure $(\varphi, \xi, \eta, g)$. Typical examples of real hypersurfaces in complex projective space $PC^{n}$ are homogeneous ones and one of the first researches is the classification of these by Takagi [12]. He proved that all homogeneous hypersurfaces of $PC^{n}$ could be divided into six types which are said to be $A_{1}, A_{2}, B, C, D$ and $E$ (see Theorem A). This result was generalized by Kimura [4], who classified real hypersurfaces of $PC^{n}$ with constant principal curvatures and for which the structure vector $\xi$ is principal. Now, there exist many studies of real hypersurfaces in $PC^{n}$. Some hypersurfaces in $PC^{n}$ are characterized by conditions on the shape operator (or principal curvatures) and one of the structure tensors. On the other hand, some

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studies about the nonexistence of real hypersurfaces under natural linear conditions imposed on the Ricci tensor $S$ or $\nabla S$ or the Weyl conformal curvature tensor $C$ or $\nabla C$ have been made by many researchers. Many results for real hypersurfaces of complex projective space have been obtained by Cecil and Ryan [1], Kimura [4], [5], Kon [7], S. Maeda [8], [9], Okumura [11], Takagi [12], [13] and so on (for more details see [8]). In particular, it is well known that there exist no Einstein real hypersurfaces $M$ in $PC^n$ for $n \geq 3$ (cf. [7]). Also $PC^n (n \geq 3)$ does not admit real hypersurfaces $M$ with parallel Ricci tensor $S$ [2]. Recently S. Maeda [9], classified real hypersurfaces $M$ in $PC^n$ satisfying $\nabla_\xi S = 0$ (that is the Ricci tensor $S$ is parallel in the direction of the structure vector $\xi = -JN$, where $N$ is a unit normal vector field on $M$) under the conditions that $\xi$ is a principal curvature vector of $M$ and that the focal map has constant rank on $M$.

On the other hand U. H. Ki, H. Nakagawa and Y. J. Suh [3] have proved that $PC^n$ does not admit real hypersurfaces $M$ with harmonic Weyl tensor $C$. So $PC^n$ does not admit real hypersurfaces $M$ with parallel Weyl tensor $C$ (that is $\nabla_X C = 0$ for each vector $X$ tangent to $M$). This is perhaps natural since $\nabla C = 0$ is not a conformal invariant. However one might impose a weaker condition utilizing some additional structure eventhough one might not have conformal invariance. Thus we investigate real hypersurfaces $M$ by using the condition $\nabla_\xi C = 0$ (on the derivative of $C$) which is weaker than $\nabla C = 0$.

The purpose of this paper is to classify real hypersurfaces $M$ in $PC^n$ satisfying $\nabla_\xi C = 0$ under the condition that $\xi$ is a principal curvature vector of $M$ and that the focal map has constant rank on $M$.

THEOREM. *Let $M$ be a real hypersurface of $PC^n (n \geq 3)$ on which $\xi$ is a principal curvature vector with principal curvature $\alpha = 2\cot 2r$ and the focal map $\phi_r$ has constant rank. If for the Weyl conformal curvature tensor $C$ we have $\nabla_\xi C = 0$, then $M$ is locally congruent to one of the following:*

(1) *a homogeneous real hypersurface which lies on a tube of radius $r$ over a totally geodesic $PC^k (1 \leq k \leq n-1)$, where $0 < r < \pi/2$,*

(2) *a non-homogeneous real hypersurface whch lies on a tube of radius $\pi/4$ over a Kaehler submanifold $\tilde{N}$ with nonzero principal curvature $\neq \pm 1$.*

(3) *a non-homogeneous real hypersurface which lies on a tube of radius $r$ over a k-dimensional Kaehler submanifold $\tilde{N}$ on which the rank of each shape operator is not greater than 2 with nonzero principal curvature $\neq \pm\sqrt{(n-k-1)/(k-1)}$ and $\cot^2 r = (n-k-1)/(k-1)$, where $k = 2, \cdots, n-1$.*

REMARK 1. Since case (3) in the Theorem is a new example which is different from case (7) in Maeda's theorem in [9], it is essential to guarantee the existence of the Kaehler submanifold $\tilde{N}^k (k \geq 2)$ such that the rank of each shape operator is not greater than 2 in $PC^n$. The following example $\tilde{N}^{n-1}$ is a complex hypersurface (with singularity) in $PC^n$ such that the rank of each shape operator is not greater than 2 in $PC^n$.

EXAMPLE. Let $\gamma$ be a non-totally-geodesic complex curve in $PC^n$ and let $\phi_{\pi/2}(\gamma)$ be a tube of radius $\pi/2$ around the curve $\gamma$, that is $\phi_{\pi/2}(\gamma) = \bigcup_{x\in\gamma} \{exp_x(\pi/2)\upsilon, \upsilon$ is a unit normal vector of $\gamma$ at $x\}$. Then $\phi_{\pi/2}(\gamma)$ is an $(n$-$1)$-dimensional complex hypersurface in $PC^n$ (with singularity). Let $\pm cot\theta$ be the eigenvalues of the shape operator $A_\upsilon$ with respect to a unit normal vector $\upsilon$ of $\gamma$. Then the principal curvatures of $\phi_{\pi/2}(\gamma)$ at $exp_x(\pi/2)\upsilon$ are given by (see Proposition 3.1 in [1]) $cot(\pi/2+\theta)$ with multiplicity 1, $cot(\pi/2-\theta)$ with multiplicity 1 and 0 with multiplicity $2n-4$.

## 2. Preliminaries.

First we briefly describe the basic properties of real hypersurfaces of a complex projective space. Let $M$ be an orientable real hypersurface of $PC^n (n \geq 3)$ with the Fubini-Study metric of constant holomorphic sectional curvature 4. On a neighborhood of each point of $M$, we denote by $N$ a local unit normal vector field of $M$ in $PC^n$. It is well known that $M$ admits an almost contact metric structure induced from the complex structure $J$ on $PC^n$. Namely, for the Riemannian metric $g$ of $M$ induced from the Fubini-Study metric $\tilde{g}$ of $PC^n$, we define a tensor field $\varphi$ of type (1,1), a vector fiels $\xi$ and a 1-form $\eta$ on $M$ by $g(\varphi X, Y) = \tilde{g}(JX, Y), g(\xi, X) = \eta(X) = \tilde{g}(JX, N)$ for any vector fields $X$, $Y$ on $M$. Then we have

$$\varphi^2 X = -X + \eta(X)\xi, \ g(\xi,\xi) = 1, \ \varphi(\xi) = 0. \tag{2.1}$$

The Riemannian connections $\tilde{\nabla}$ of $PC^n$ and $\nabla$ of $M$ are related by the following formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \ \ \tilde{\nabla}_X N = -AX \tag{2.2}$$

where $A$ is the shape operator of $M$ in $PC^n$.

Now it follows from (2.2) that the structure $(\varphi, \xi, \eta, g)$ satisfies

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \ \ \nabla_X \xi = \varphi AX. \tag{2.3}$$

Let $\tilde{R}$ and $R$ be the curvature tensors of $PC^n$ and $M$, respectively. Since the curvature tensor $\tilde{R}$ has a nice form, namely $PC^n$ is of constant holomorphic sectional curvature 4, the Gauss and Codazzi equations are respectively

$$(2.4)\quad \begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y \\ &\quad -2g(\varphi X,Y)\varphi Z + g(AY,Z)AX - g(AX,Z)AY, \\ (\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\varphi Y - 2g(\varphi X,Y)\xi \end{aligned}$$

By (2.1), (2.3) and (2.4) we get

$$(2.5)\quad QX = (2n+1)X - 3\eta(X)\xi + hAX - A^2X$$

where $h = traceA$ and $Q$ denotes the Ricci operator of $M$ defined from the Ricci tensor $S$, i.e. $S(X,Y) = g(QX,Y)$. The Weyl conformal curvature tensor $C$ of $M$ is given by

$$(2.6)\quad \begin{aligned} C(X,Y)Z = R(X,Y)Z &+ \frac{1}{2n-3}[g(QX,Z)Y - g(QY,Z)X + g(X,Z)QY \\ &- g(Y,Z)QX] - \frac{\tau}{2(n-1)(2n-3)}(g(X,Z)Y - g(Y,Z)X) \end{aligned}$$

where $\tau$ is the scalar curvature of $M$.

An eigenvector $X$ of the shape operator $A$ is called a principal curvature vector and an eigenvalue $\lambda$ is called a principal curvature. From now on, we assume that the structure vector field $\xi$ is principal, and $\alpha$ is the principal curvature associated with $\xi$, that is, $A\xi = \alpha\xi$. Then it has been shown that $\alpha$ is constant (see [14]). Also for a principal curvature vector $X$ orthogonal to $\xi$ and the associated principal curvature $\lambda$ we have (see [10])

$$(2.7)\quad AX = \lambda X \ \ and \ \ A\varphi X = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\varphi X$$

Now we recall without proof the following in order to prove our Theorem.

THEOREM A ([12]). *Let $M$ be a homogeneous real hypersurface of $PC^n$. Then $M$ is a tube of radius $r$ over one of the following Kaehler submanifolds:*

($A_1$) *hyperplane* $PC^{n-1}$, *where* $0 < r < \pi/2$,

($A_2$) *totally geodesic* $PC^k (1 \le k \le n-2)$, *where* $0 < r < \pi/2$,

($B$) *complex quadric* $Q_{n-1}$, *where* $0 < r < \pi/4$,

($C$) $PC^1 \times PC^{(n-1)/2}$, *where* $0 < r < \pi/4$ *and* $n$ $(\ge 5)$ *is odd,*

($D$) *complex Grasmannian* $G_{2,5}(C)$, *where* $0 < r < \pi/4$ *and* $n = 9$,

($E$) *Hermitian symmetric space SO(10)/U(5), where* $0 < r < \pi/4$ *and*

$n=15$.

THEOREM B ([4]). *Let $M$ be a real hypersurface of $PC^n$. Then $M$ has constant principal curvatures and $\xi$ is a principal vector if and only if $M$ is locally congruent to a homogeneous real hypersurface.*

THEOREM C ([6]). *Let $M$ be a real hypersurface of $PC^n$. If $\nabla_\xi A=0$, then $\xi$ is a principal curvature vector; in addition, except for the null set on which the focal map $\phi_r$ degenerates, $M$ is locally congruent to one of the following:*

(i) *a homogeneous real hypersurface which lies on a tube of radius $r$ over a totally geodesic $PC^k(1\le k\le n-1)$, where $0<r<\pi/2$.*

(ii) *a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kaehler submanifold $N$ with nonzero principal curvatures $\neq \pm 1$.*

THEOREM D ([1]). *Let $M$ be a connected orientable real hypersurface (with unit normal vector $N$) in $PC^n$ on which $\xi$ is a principal curvature vector with principal curvature $\alpha=2cot2r$ and the focal map $\phi_r$ has constant rank on $M$. Then the following hold:*

(i) *$M$ is a tube of radius $r$ around a certain Kaehler submanfild $\tilde{N}$ in $PC^n$.*

(ii) *For $x\in M$, let $cot\theta$ be a principal curvature of the shape operator at $exp_x rN$ of $\tilde{N}$, $N$ being the inward normal at $x$. Then the real hypersurface $M$ has a principal curvature equal to $cot(\theta-r)$ at $x$.*

REMARK 2. For later use, we note that from the Theorem A, the homogeneous real hypersurfaces $M$ of type $A_1, A_2, B, C, D$, and $E$ have distinct principal curvatures $\xi_i$ with multiplicities $m(\xi_i)$ which we can read as follows (cf. [12]).

$A_1$: $\xi_1=cotr,\ m(\xi_1)=2(n-1),\ \xi_2=2cot2r,\ m(\xi_2)=1$

$A_2$: $\xi_1=cotr,\ m(\xi_1)=2k,\ \xi_2=-tanr,\ m(\xi_2)=2(n-k-1),$
$\xi_3=2cot2r,\ m(\xi_3)=1$

$B$: $\xi_1=cot(r-(\pi/4)),\ m(\xi_1)=n-1,\ \xi_2=-tan(r-(\pi/4)),\ m(\xi_2)=n-1,$
$\xi_3=2cot2r,\ m(\xi_3)=1$

$C$: $\xi_i=cot(r-(\pi i/4))\ (i=1,2,3,4),\ m(\xi_i)=n-3,$ for $i=2,4$
and $m(\xi_i)=2$, for $i=1,3$ $\xi_5=2cot2r,\ m(\xi_5)=1$

$D$: $\xi_i=cot(r-(\pi i/4)),\ m(\xi_i)=4\ (i=1,2,3,4),$
$\xi_5=2cot2r,\ m(\xi_5)=1$ and $\dim M=17$

$E$: $\xi_i=cot(r-(\pi i/4)),\ (i=1,2,3,4),\ m(\xi_i)=8$ for $i=2,4$ and
$m(\xi_i)=6$ for $i=1,3,$ $\xi_5=2cot2r,$ and $m(\xi_5)=1$ and $\dim M=29$.

It is easy to see that if $\xi$ is a principal curvature vector with principal curvature $\alpha$, then

$$(\nabla_\xi A)X = \alpha\varphi AX - A\varphi AX + \varphi X. \tag{2.8}$$

Indeed, from (2.4) for $Y = \xi$ we have $(\nabla_\xi A)X = \alpha\nabla_X\xi - A\nabla_X\xi - \varphi X$ and then using (2.3) we obtain (2.8).

Finally we complete our preliminaries with the following two lemmas:

LEMMA 1. *If $\xi$ is a principal curvature vector and $\nabla_\xi C = 0$, then $\xi\tau = 0$.*

PROOF. From (2.6) by using (2.4) and (2.5) we get

$$\begin{aligned} C(X,Y)Z = {} & \frac{1}{2n-3}\left(\frac{\tau}{2(n-1)} - 2n - 5\right)(g(Y,Z)X - g(X,Z)Y) + g(\varphi Y,Z)\varphi X \\ & -g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z + g(AY,Z)AX - g(AX,Z)AY \\ & + \frac{1}{2n-3}[3\eta(Z)(\eta(Y)X - \eta(X)Y) + h(g(AX,Z)Y - g(AY,Z)X) \\ & + g(A^2Y,Z)X - g(A^2X,Z)Y + 3(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi \\ & + h(g(X,Z)AY - g(Y,Z)AX) + g(Y,Z)A^2X - g(X,Z)A^2Y] \end{aligned}$$

We note that the condition $\nabla_\xi C = 0$ is equivalent to

$$\nabla_\xi(C(X,Y)Z - C(\nabla_\xi X,Y)Z - C(X,\nabla_\xi C)Z - C(X,Y)\nabla_\xi Z = 0. \tag{2.9}$$

Now for simplicity we put

$$U_X = \alpha\varphi AX - A\varphi AX + \varphi X, \quad V_X = U_{AX} + AU_X. \tag{2.10}$$

Then by a straightforward calculation and using (2.3) and (2.8) we obtain

$$\begin{aligned} (\nabla_\xi C)(X,Y,Z) = {} & \frac{1}{2(n-1)(2n-3)}(\xi\tau)(g(Y,Z)X - g(X,Z)Y) \\ & + g(U_Y,Z)AX - g(U_X,Z)AY + g(AY,Z)U_X - g(AX,Z)U_Y \\ & + \frac{1}{2n-3}[h(g(U_X,Z)Y - g(U_Y,Z)X) + g(V_Y,Z)X - g(V_X,Z)Y \\ & + h(g(X,Z)U_Y - g(Y,Z)U_X) + g(Y,Z)V_X - g(X,Z)V_Y] \end{aligned} \tag{2.11}$$

If we choose $X$ orthogonal to $\xi$ and $AX = \lambda X$, then

$$U_X = (\alpha\lambda - \lambda\mu + 1)\varphi X, \quad V_X = (\lambda + \mu)(\alpha\lambda - \lambda\mu + 1)\varphi X \tag{2.12}$$

where $\mu=(\alpha\lambda+2)/(2\lambda-\alpha)$.

Therefore putting $Z=\xi$ in (2.11) we obtain

$$\frac{1}{2(n-1)(2n-3)}(\xi\tau)\eta(Y)X + (\alpha\lambda-\lambda\mu+1)\left(\alpha+\frac{1}{2n-3}(\lambda+\mu-h)\right)\eta(Y)\varphi X=0. \tag{2.13}$$

Thus $\xi\tau=0$.

We notice that from (2.13) we also have

$$(\alpha\lambda-\lambda\mu+1)\left(\alpha+\frac{1}{2n-3}(\lambda+\mu-h)\right)=0. \tag{2.14}$$

LEMMA 2. *If $\xi$ is a principal curvature vector with principal curvature $\alpha=0$, then $\xi\tau=0$ and $\nabla_\xi C=0$.*

PROOF. From (2.5) we have $\tau=4(n^2-1)+h^2-trA^2$. Thus $\xi\tau=2h(\xi h) -tr\nabla_\xi A^2$. But from [9. Lemma 2] we know that $\xi h=0$. Also $\alpha=0$ implies $\nabla_\xi A=0$ (see [9, Lemma 1]). Thus we obtain $\xi\tau=0$.

Now from (2.10) and (2.8) we get $U_\xi=0$ and $U_X=0$ for $X$ orthogonal to $\xi$ such that $AX=\lambda X$. Thus finally we have $U_X=V_X=0$ for all $X$. Then from (2.11) we obtain $\nabla_\xi C=0$.

## 3. Proof of Theorem:

From the fact that the principal curvature $\alpha$ of the principal curvature vector $\xi$ is constant, our discussion is divided into two cases:

(i) $\alpha=0$ and (ii) $\alpha\neq 0$.

(i) $\alpha=0$.

In this case we have $\nabla_\xi A=0$. Hence by virtue of Theorem $C$ we find that $M$ is locally congruent to a homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic $PC^k(1\leq k\leq n-1)$, or congruent to a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kaehler submanifold $\tilde{N}$ with nonzero principal curvatures $\neq\pm 1$. Thus $M$ is of case (1) with $r=(\pi/4)$ or of case (2) in the Theorem. From Lemma 2 we have that these examples satisfy $\nabla_\xi C=0$.

(ii) $\alpha\neq 0$.

We will follow the method of [9] and we will prove that $M$ cannot be homogeneous of type $B, C, D$, or $E$.

From Lemma 1 and the relations (2.11) and (2.14) we have that for any principal curvature vector $X$ orthogonal to $\xi$, the principal curvature $\lambda$ must satisfy the following equation for $\lambda$

$$(3.1) \qquad (\lambda^2-\alpha\lambda-1)[2\lambda^2-2(h-(2n-3)\alpha)\lambda+h\alpha+2-(2n-3)\alpha^2]=0.$$

Since $\xi$ is a principal curvature vector and the focal map $\phi_r$ has constant rank on $M$, our hypersurface $M$ is a tube (of radius $r$) over a certain (k-dimensional) Kaehler submanifold $\tilde{N}$ in $PC^n$. So we may put $\alpha = 2cot2r(=cotr-tanr)$ (cf. Theorem D). Now from (3.1) we have $\lambda^2-\alpha\lambda-1=0$ which gives $\lambda=cotr$ and $\lambda=-tanr$, or

$$(3.2) \qquad 2\lambda^2-2(h-(2n-3)\alpha)\lambda+h\alpha+2-(2n-3)\alpha^2=0.$$

We denote by $\lambda_1, \lambda_2 (\neq cotr, -tanr)$ the solutions of (3.2).

Since

$$(3.3) \qquad \lambda_1+\lambda_2=h-(2n-3)\alpha$$

we have

$$(3.4) \qquad \frac{\alpha\lambda_1+2}{2\lambda_1-\alpha}=\lambda_2$$

Now denote by $V_\lambda$ the eigenspace of $A$ associated with the eigenvalue $\lambda$ and by $m(\lambda)$ the multiplicity of $\lambda$. Then by using (2.7) and (3.4) we obtain

$$\varphi V_{cotr}=V_{cotr},\ \varphi V_{-tanr}=V_{-tanr} \text{ and } \varphi V_{\lambda_1}=V_{\lambda_2}.$$

Thus the real hypersurface $M$ has at most five distinct principal curvatures $2cot2r$ (with multiplicity 1) $cotr$ (with multiplicity $2n-2k-2$), $-tanr$ (with multiplicity $2k-2m$), $\lambda_1$ (with multiplicity $m\geq 0$) and $\lambda_2$ (with multiplicity $m\geq 0$). Hence

$$(3.5) \qquad h=(2n-2k-1)cotr-(2k-2m+1)tanr+m(\lambda_1+\lambda_2).$$

Using (3.3), (3.4) and (3.5) we obtain

$$(3.6) \qquad (2n-2k-1)cotr-(2k-2m+1)tanr+(m-1)\left(\lambda_1+\frac{\alpha\lambda_1+2}{2\lambda_1-\alpha}\right)-(2n-3)\alpha=0.$$

Now for the multiplicity $m$ of the principal curvature $\lambda_1$, namely for the integer $m=m(\lambda_1)$ we distinguish three cases: $m=0$, $m=1$ and $m\geq 2$.

We shall prove that $m<2$.

Suppose for the moment that $m\geq 2$. From (3.6) we have that $\lambda_1$ = *constant*.

Thus our manifold $M$ is homogeneous (cf. Theorem B) and from the Remark 2 we conclude that $M$ is of type $B$, $C$, $D$ or $E$. We will check one by one that these cases cannot occur.

Let $M$ be of type $B$ (namely $M$ is a tube of radius $r$). Then $M$ has three distinct constant principal curvatures $\mu_1=(1+x)/(1-x)$, $\mu_2=(x-1)/(x+1)$, $\alpha=(x-1/x)$, where $x$ = *cotr*, with $m(\mu_1)=n-1$, $m(\mu_2)=n-1$ and $m(\alpha)=1$.

Thus

$$h=(n-1)\frac{4x}{1-x^2}+\frac{x^2-1}{x}.$$

On the other hand, from (3.3) we have

$$h=\frac{4x}{1-x^2}+(2n-3)\frac{x^2-1}{x}.$$

From the last two relations we obtain

$$(n-2)\frac{4x}{1-x^2}=2(n-2)\frac{x^2-1}{x} \quad or \quad x^4+1=0\text{, impossible.}$$

Now let $M$ be of type $C$ (which is also a tube of radius $r$). Let $x$ = *cotr*. Then $M$ has five distinct constant principal curvatures $\mu_1=(1+x)/(1-x)$ with $m(\mu_1)=2$, $\mu_2=(x-1)/(x+1)$ with $m(\mu_2)=2$, $\mu_3=x$ with $m(\mu_3)=n-3$, $\mu_4=(-1/x)$ with $m(\mu_4)=n-3$ and $\alpha=(x-1/x)$ with $m(\alpha)=1$ (cf. Remark 2). Since $\varphi V_{\mu_1}=V_{\mu_2}$, $\varphi V_{\mu_3}=V_{\mu_3}$ and $\varphi V_{\mu_4}=V_{\mu_4}$, the condition $\nabla_\xi C=0$ is equivalent to $h=\mu_1+\mu_2+(2n-3)\alpha$. Then from this we obtain

$$\frac{1+x}{1-x}+\frac{x-1}{x+1}+(n-2)\left(x-\frac{1}{x}\right)=(2n-3)\frac{x^2-1}{x}$$

or

$$(n-1)x^4-2(n-3)x^2+n-1=0.$$

But this is impossible because the discriminant of this equation is negative.

Let $M$ be of type $D$ (which is a tube of radius $r$). Then $M$ has five distinct constant principal curvatures $\mu_1=(1+x)/(1-x)$ with $m(\mu_1)=4$, $\mu_2=(x-1)/(x+1)$ with $m(\mu_2)=4$, $\mu_3=x$ with $m(\mu_3)=4$, $\mu_4=-1/x$ with $m(\mu_4)=4$ and $\alpha=(x-1/x)$ with $m(\alpha)=1$, where $x$ = *cotr* and dim$M$ = 17 (cf. Remark 2). We have again as in case of type $C$, that $\varphi V_{\mu_1}=V_{\mu_2}$, $\varphi V_{\mu_3}=V_{\mu_3}$ and $\varphi V_{\mu_4}=V_{\mu_4}$. Thus the condition $\nabla_\xi C=0$ is equivalent to $h=\mu_1+\mu_2+(2n-3)\alpha$. This becomes $(n-4)x^4-2(n-7)x^2+n-4=0$. From this we get $n\le 5$ or equivalently $M\le 9$, a contradiction.

Finally, let $M$ be of type $E$ (which is a tube of radius $r$). Then as above $M$ has the same five distinct constant principal curvatures $\mu_1, \mu_2, \mu_3, \mu_4$ and $\alpha$ but with multiplicity $m(\mu_1) = m(\mu_2) = 6$, $m(\mu_3) = m(\mu_4) = 8$ and $m(\alpha) = 1$ (cf. Remark 2). By virtue of the discussion in cases of type $C$ or $D$ we have only to solve the equation $h - \mu_1 - \mu_2 - (2n-3)\alpha = 0$. Namely we have the equation $(n-6)x^4 - 2(n-11)x^2 + (n-6) = 0$. But in our case dim$M$ = 29, or equivalently $n$ = 15. Thus we have $9x^4 - 8x^2 + 9 = 0$, which is impossible. This completes the proof of the assertion that $m < 2$.

We will examine now the cases $m = 0$ and $m = 1$ separately. Let $m = 0$. Our real hypersurface $M$ has three distinct principal curvatures and it is of case (1) with $0 < r(\neq \pi/4) < \pi/2$ in the Theorem. Now let $m = 1$. Our real hypersurface $M$ has at most five distinct principal curvatures $2cot2r$ with $m(2cot2r)=1$, $cotr$ with $m(cotr) = 2n-2k-2$, $-tanr$ with $m(-tanr) = 2k-2$, $\lambda_1$ with $m(\lambda_1) = 1$ and $\lambda_2$ with $m(\lambda_2) = 1$. Since the multiplicities of the principal curvatures of $M$ do not match with the multiplicities of any homogeneous real hypersurface (cf. Remark 2), the manifold $M$ is not homogeneous. Hence both $\lambda_1$ and $\lambda_2$ are not constant (cf. Theorem B). Moreover, Theorem D shows that $\lambda_1$ and $\lambda_2$ can be expressed as: $\lambda_1 = cot(r-\theta)$ and $\lambda_2 = cot(r+\theta)$, where $cot\theta$ is a principal curvature of the Kaehler submanifold $\tilde{N}$. In addition equation (3.6) yields that $cot^2 r = (n-k-1)/(k-1)$. Hence the manifold $M$ is of case (3) in the Theorem.

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