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COVERS AND ENVELOPES OVER GORENSTEIN RINGS

By

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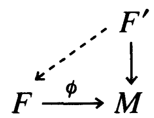
Abstract. A module over a Gorenstein ring is said to be Gorenstein injective if it splits under all modules of finite projective dimension. We show that over a Gorenstein ring every module has a Gorenstein injective envelope. We apply this result to the group algebra $\hat{Z}_pG$ (with G a finite group and $\hat{Z}_p$ the ring of p-adic integers for some prime p) and show that ever finitely generated $\hat{Z}_pG$-module has a cover by a lattice. This gives a way of lifting finite dimensional representations of G over $\boldsymbol{Z}/(p)$ to modular representations of G over $\hat{Z}_p$.

1. Introduction

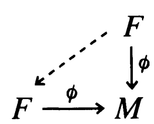
In this paper we will use the terminology of Enochs [6]. We recall that if $\mathscr{F}$ is a class of left R-modules for some ring R, then a linear map $\phi: F \to M$ with $F \in \mathscr{F}$ is called an $\mathscr{F}$-cover of M if

a) any diagram



can be completed to a commutative diagram if $F' \in \mathscr{F}$

b) The diagram



can be completed to a commutative diagram only by automorphisms of F.

If $\phi: F \to M$ satisfies a) and perhaps not b), then it called an $\mathscr{F}$-precover of M.

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If an $\mathscr{F}$-cover of M exists it is unique to isomorphism. If $\mathscr{F}$ is for example, the class of flat modules, then an $\mathscr{F}$-cover is called a flat cover.

$\mathscr{F}$-preenvelopes and $\mathscr{F}$-envelopes are defined dually. We follow a convention like the one above when dealing with preenvelopes and envelopes. So our terminology agrees with the customary terminology. For example, injective envelopes and projective covers are as usual.

We note that Auslander and Reiten [2] use the terms right $\mathscr{F}$- approximation, minimal right $\mathscr{F}$-approximation, left $\mathscr{F}$-approximation and minimal left $\mathscr{F}$-approximation for $\mathscr{F}$-precover, $\mathscr{F}$-cover, $\mathscr{F}$-preenvelope and $\mathscr{F}$-envelope respectively. Also see Auslander and Smalφ [3].

DEFINITION 1.1. If $\mathscr{F}$ is a class of left (right) R-modules, we let $\mathscr{F}^{\perp}$ be the class of left (right) R-modules K such that $\mathrm{Ext}^1(F,K)=0$ for all $F\in\mathscr{F}$. We let ${}^{\perp}\mathscr{F}$ be those K such that $\mathrm{Ext}^1(K,F)=0$ for all $F\in\mathscr{F}$.

We note that if $0\to K\to F\to M\to 0$ is an exact sequence of left R-modules with $F\in\mathscr{F}$ for some class of left R-modules and with $K\in\mathscr{F}^{\perp}$, then for any $F'\in\mathscr{F}$,

$$\mathrm{Hom}(F',X)\to\mathrm{Hom}(F',M)\to\mathrm{Ext}^1(F',K)=0$$

is exact. This shows that $F\to M$ is an $\mathscr{F}$-precover of M. Wakamatsu's lemma (see [2]) says that conversely, if

$$0\to K\to F\to M\to 0$$

is exact and if $F\to M$ is an $\mathscr{F}$-cover of M where $\mathscr{F}$ is a class of modules closed under extensions, then $K\in\mathscr{F}^{\perp}$. There is a dual result concerning envelopes. These ideas are considered in section 3. In section 5 we use these ideas to prove the existence of covers and envelopes.

If R is a Gorenstein ring, the class $\mathscr{L}$ of left R-modules of finite projective dimension is an important class of modules. We call the modules in $\mathscr{L}^{\perp}$ the Gorenstein injective modules. We consider the Gorenstein injective modules in section 4 and then in section 6 show that over a Gorenstein ring every left R-module has a Gorenstein injective envelope and an $\mathscr{L}$-cover.

We note that Auslander and Reiten consider similar questions in [2] but concerning finitely generated modules over artin rings. We remark that with these chain conditions it is easier to prove the existence of covers (envelopes) once precover (preenvelopes) are known to exist.

In the final sections we consider the Gorenstein ring $\hat{Z}_pG$ (with $\hat{Z}_p$ the ring of p-adic integers and G any finite group). In this case we note that Gorenstein injective envelopes are just divisible envelopes. We use these envelopes to show that every finitely generated left $\hat{Z}_p G$-module has a cover by a lattice and then that the category of lattices is a stably reflective subcategory of the category of finitely generated left $\hat{Z}_p G$-modules. We note that our results give a canonical way of lifting representations of G over $Z/(p)$ to modular representations of G over $\hat{Z}_p$.

For a ring R we have

THEOREM 1.2 ([6], Proposition 2.1 and Theorem 2.1). *Every left R-module has an injective cover if and only if R is left noetherian.*

We say that a class $\mathcal{F}$ of left R-modules is resolving if it contains the projective modules and is closed under extensions and taking kernels of surjective maps. If $\mathcal{F}$ is resolving, it is easy to argue that for $K \in \mathcal{F}^{\perp}$, $\mathrm{Ext}^i(F,K)=0$ for all $i \geq 1$ and all $F \in \mathcal{F}$.

If R is a ring, the symbol ${}_R M$ will indicate that M is a left R-module.

2. Gorenstein Rings

In this short section we define the Gorenstein rings, note how they can be constructed and give some properties of their modules. Most of the content of this section is due to Iwanaga.

DEFINITION 2.1 (Iwanaga [12]. A ring is said to be Gorenstein if it is left and right noetherian and if it has finited injective dimension as a module over itself both on the left and on the right. If R is Gorenstein and $n \geq 0$ is an upper bound for these two injective dimensions, then R is said to be n-Gorenstein.

REMARK 2.2. If R is n-Gorenstein, it can be argued that the ring of lower triangular matrices (of any given size) over R is $n+1$-Gorenstein.

The path algebra of any finite quiver over a Gorenstein ring $R \neq 0$ is Gorenstein if and only if the quiver is the disjoint union of cycles with no multiple edges and quivers not having cycles (Bronstein, Enochs, Herzog [4]).

If R is n-Gorenstein and G is a finite group then RG is n-Gorenstein ([4], Eilenberg and Nakayama).

The 0-Gorenstein rings are just the quasi-Frobenius rings. Over a quasi-Frobenius ring a module is injective if and only if it is projective. The following can be interpreted as a higher dimensional analogue of this fact.

THEOREM 2.3 ([12], Iwanaga). *If R is n-Gorenstein and if M is a left R-module the following are equivalent:*

a) *proj.* dim $M < \infty$

b) *proj.* dim $M \leq n$

c) *inj.* dim $M < \infty$

d) *inj.* dim $M \leq n$

DEFINITION 2.4 (Iwanaga[12]). If R is ring, a ring extension $R \subset A$ is said to be left quasi-Frobenius extension if ${}_RA$ is finitely generated and projective and if ${}_AA_R$ is a direct summand of the direct sum of a finite number of copies of $\mathrm{Hom}_R({}_RA_A, {}_RR_R)$. A right quasi-Frobenius extension is defined similarly. If $R \subset A$ is both a left and a right quasi-Frobenius extension, then it is called a quasi-Frobenius extension.

REMARK. If G is a finite group and R is any ring, $R \subset RG$ is a quasi-Frobenius extension.

THEOREM 2.5 [12], Iwanaga). If $R \subset A$ is a quasi-Frobenius extension then R is n-Gorenstein if and only if A is.

3. Fibrations and Wakamatsu's lemma

Let R be a ring and $\mathscr{X}$ a class of modules closed under extensions. We show that if $0 \to L \to X \to M \to 0$ is an exact sequences of left R-modules with $X \to M$ an $\mathscr{X}$-cover, then in fact $X \to M$ is an $\mathscr{X}$-fibration. We take this to mean that if $A \subset B$ are left R-modules with $B/A \in \mathscr{X}$, then a linear map $A \to X$ has an extension to B if and only if $A \to X \to M$ has an extension to B. Furthermore any extension $B \to M$ can be lifted to an extension $B \to X$.

With the above conditions as hypotheses this result can be stated as:

PROPOSITION 3.1. Any commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \swarrow & \downarrow \\ X & \longrightarrow & M \end{array}$$

can be completed to a commutative diagram.

PROOF. We form the pushout

$$\begin{array}{ccccccccc} 0 & \rightarrow & A & \rightarrow & B & \rightarrow & B/A & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \rightarrow & X & \rightarrow & P & \rightarrow & B/A & \rightarrow & 0 \end{array}$$

By our hypothesis on $\mathscr{X}, P \in \mathscr{X}$.

This gives rise to a commutative

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & P \swarrow & \downarrow \\ X \nearrow & \searrow & \\ X & \longrightarrow & M \end{array}$$

Since $X \to M$ is an $\mathscr{X}$-cover and $P \in \mathscr{X}$, there is a map $P \to X$ making

$$\begin{array}{ccc} P & & \\ \downarrow & \searrow & \\ X & \longrightarrow & M \end{array}$$

commutative. Since $X \to M$ is a cover the composition $X \to P \to X$ is an automorphism if X. Composing $P \to X$ with the inverse of this automorphism if necessary, we see we can assume $X \to P \to X$ is the identity on X. Then

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & P \swarrow & \downarrow \\ \swarrow & & \\ X & \longrightarrow & M \end{array}$$

is seen to be the desired commutative diagram.

COROLLARY 3.2 (Wakamatsu's lemma [2] and [13]). *If* $0 \to L \to X \to M \to 0$ *is exact and* $X \to M$ *is an* $\mathscr{X}$*-cover then* $L \in \mathscr{X}^{\perp}$.

PROOF. If $A \to L$ is a linear map then $A \to L \to X \to M$ can be extended to the constant map $B \to M$. Then the map $B \to X$ guaranteed by the proposition has its image in L and gives and extension $B \to L$ of $A \to L$ i.e. $\mathrm{Hom}(B, L) \to \mathrm{Hom}(A, L) \to 0$ is exact.

If $0 \to L \to N \to Y \to 0$ is then any exact sequence with $Y \in \mathscr{X}$, let $A = N$ and $B = N$. Since $\mathrm{Hom}(N, L) \to \mathrm{Hom}(L, L) \to 0$ is exact, this sequence splits and so $\mathrm{Ext}^1(Y, L) = 0$.

REMARK 3.3. See (lemma 2.2 of [8] and Proposition 2.2 and Corollary 1 of [6]) for special cases of the above Proposition and its Corollary.

Dual proofs give

PROPOSITION 3.4 *If $\mathscr{X}$ is a class of left R-modules closed under extensions and if*

$$\begin{array}{ccccccccc} & & 0 & \to & M & \to & X & \to & L & \to & 0 \\ & & & & \downarrow & \swarrow & \downarrow & & & & \\ 0 & \to & Y & \to & B & \to & C & \to & 0 & & \end{array}$$

is any commutative diagram with exact rows such that $M \to X$ is an $\mathscr{X}$-envelope and such that $Y \in \mathscr{X}$, then the diagram can be completed to a commutative diagram.

COROLLARY 3.5. $L \in {}^{\perp}\mathscr{X}$.

4. Gorenstein Injective Modules

In this sections we define Gorenstein injective modules over Gorenstein rings. We consider their behavior under restrictions of scalars when we have a quasi-Frobenius extension.

DEFINITION 4.1. For any ring R we will let $\mathscr{L}$ denote the class of left R-modules of finite projective dimension.

DEFINITION 4.2. For a Gorenstein ring R, $\mathscr{L}^{\perp}$ will be called the class of Gorenstein injective left R-modules. Hence K is Gorenstein injective if and only if $\mathrm{Ext}^1(L,K)=0$ whenever proj. dim $L<\infty$. Since $\mathscr{L}$ is a resolving category we know that in fact $\mathrm{Ext}^i(L,K)=0$ for all $i \geq 1$.

The Gorenstein injective right R-modules are defined analogously.

PROPOSITION 4.3. *If R is Gorenstein, K is a Gorenstein injective left R-module and $E \to K$ is an injective cover of K then $E \to K$ is surjective and is also an $\mathscr{L}$-cover of K. Furthermore, ker($E \to K$) is Gorenstein injective.*

PROOF. Let $0 \to R \to \overline{E} \to C \to 0$ be exact with $\overline{E}$ injective. Then by Theorem 2.3, $C \in \mathscr{L}$, so $\mathrm{Ext}^1(C,K)=0$. Hence $\mathrm{Hom}(\overline{E},K) \to \mathrm{Hom}(R,K) \to 0$ is exact. But any linear map $\overline{E} \to K$ can be factored $\overline{E} \to E \to K$. Since every $x \in E$ is in the image of a linear of a linear map $R \to K$, it is in the image of an extension $\overline{E} \to K$. Then any factorization $\overline{E} \to E \to K$ shows that x is in the image of $E \to K$. Let $\mathscr{E}$ denote the class of injective left R-modules. Then by Corollary 3.2, $N = \ker(E \to K) \in \mathscr{E}^{\perp}$ i.e. $\mathrm{Ext}^1(\overline{E},N)=0$ for all $\overline{E} \in \mathscr{E}$. But we know $\operatorname{Ext}^i(L,K)=0$ for $i\geq 1$ and $L\in\mathscr{L}$. Since $\mathscr{E}\subset\mathscr{L}$ we have $\operatorname{Ext}^i(\overline{E},K)=0$ for $\overline{E}\in\mathscr{E}$ and $i\geq 1$.

Then by the exactness of

$$0\to N\to E\to K\to 0$$

with E injective, we get that $\operatorname{Ext}^i(\overline{E},N)=0$ for all $\overline{E}\in\mathscr{E}$. Since any $L\in\mathscr{L}$ has finite injective dimension, we get that $\operatorname{Ext}^1(L,N)=0$ for all $L\in\mathscr{L}$ i.e. $N\in\mathscr{L}^\perp$. This implies that $E\to K$ is an $\mathscr{L}$-precover since $E\in\mathscr{L}$. Then since $E\to K$ is an injective cover, we get that $E\to K$ is an $\mathscr{L}$-cover. But then by Corollary 3.2, ker($E\to K$) is in $\mathscr{L}^\perp$, i.e. ker($E\to K$) is Gorenstein injective.

PROPOSITION 4.4. *If R is an n-Gorenstein ring then a left R-module K is Gorenstein injective if and only if K is an n-th cosyzygy, i.e. there is an exact sequence*

$$0\to M\to E^0\to E^1\to\cdots\to E^{n-1}\to K\to 0 \text{ with } E^0,\cdots,E^{n-1}\text{injective.}$$

PROOF. Given such an exact sequence of modules over the n-Gorenstein ring R, let proj. $\dim_R L<\infty$. Then by Theorem 2.3, proj. $\dim L\leq n$. Hence $\operatorname{Ext}^1(L,K)=\operatorname{Ext}^{n+1}(L,M)=0$. Hence $K\in\mathscr{L}^\perp$.

Conversely, suppose $K\in\mathscr{L}^\perp$. let $E\to K$ be an injective cover of K. This exists by Theorem 1.2 above. If $\overline{K}=\ker(E\to K)$ then $K\in\mathscr{L}^\perp$ by Proposition 4.3. Then if $\overline{E}\to\overline{K}$ is an injective cover of $\overline{K}$, we have the exact sequence $\overline{E}\to E\to K\to 0$. It is clear this procedure can be continued and so gives the result.

THEOREM 4.5 (Auslander, Buchweitz [1]. *If R is Gorenstein, then for any left R-module M there is an exact sequence*

$$0\to M\to K\to L\to 0$$

with $K\in\mathscr{L}^\perp$ and $L\in\mathscr{L}$.

PROOF. The argument is dual to the proof of Theorem 1.1 in [1]. We let $\mathscr{X}\in\mathscr{L}^\perp$ be the class of Gorenstein injective modules and let $\mathscr{E}\subset\mathscr{X}$ be the injective modules. Then in the language of [1], Proposition 4.4 says that $\mathscr{E}$ is a generator for $\mathscr{X}$. Then an appeal to Proposition 4.4 says for every module M, there is an exact sequence $0\to M\to X^0\to\cdots\to X^n\to 0$ (taking R to be n-Gorenstein) with $X^0,\cdots,X^n\in\mathscr{X}$. Then the dual of Theorem 1.1 of [1] gives the result.

We note that there is a different proof of this claim in [9].

COROLLARY 4.6. *$\mathscr{L} \cap \mathscr{L}^{\perp}$ is the class of injective left R-modules.*

PROOF. If $M \in \mathscr{L} \cap \mathscr{L}^{\perp}$, then by Theorem 2.3 inj. $\dim M \leq n$ if we take R to be n-Gorenstein.

Let $0 \to M \to E^0(M) \to \cdots \to E^k(M) \to 0$ be a minimal injective resolutions of M with $E^k(M) \neq 0$ (so $k \leq n$). By Theorem 2.3, $E^k(M) \in \mathscr{L}$. If $k \geq 1$, then

$$\mathrm{Ext}^k(E^k(M), M) = 0 \text{ so } E^{k-1}(M) \to E^k(M)$$

has a section. By minimality, this is impossible so $k = 0$. Hence M is injective.

PROPOSITION 4.7. *If R is Gorenstein then $^{\perp}(\mathscr{L}^{\perp}) = \mathscr{L}$.*

PROOF. This follows from Proposition 1.11 and Proposition 4.1 of [10].

COROLLARY 4.8. *A Gorenstein ring has finite left global dimension if and only if the class of Gorenstein injective left R-modules coincides with the class of injective left R-modules.*

PROOF. Letting $\mathscr{E}$ be the class of injective left R-modules, then if $\mathscr{L}^{\perp} = \mathscr{E}$, we get $\mathscr{L} = {}^{\perp}(\mathscr{L}^{\perp}) = {}^{\perp}\mathscr{E}$ is the class of all modules and so R has finite left global dimensions. Conversely if R has finite left global dimension then $\mathscr{L}$ is the class of all modules, so $\mathscr{L}^{\perp} = \mathscr{E}$.

REMARK 4.9. Other properties of Gorenstein injective modules are given in [9].

REMARK 4.10. If $R \subset A$ is a quasi-Frobenius extension, then for any injective left R-module E, $A \otimes_R E$ is an injective left A-module. For ${}_A A_R$ is a summand of copies of $\mathrm{Hom}_R({}_R A_A, {}_R R_R)$. But $\mathrm{Hom}_R(A, R) \otimes_R E \cong \mathrm{Hom}_R(A, E)$ as left R-modules since ${}_R A$ is finitely generated and projective. But $\mathrm{Hom}_R(A, E)$ is an injective left A-module.

THEOREM 4.11. If $R \subset A$ is a quasi-Frobenius extension with R a Gorenstein ring, then a left A-module K is Gorenstein injective if and only if if ${}_R K$ is Gorenstein injective.

PROOF. Let ${}_A K$ be Gorenstein injective. Assume R is n-Gorenstein. Then by Theorem 2.5, so is A, By Proposition 4.4, there is an exact sequence

$$E^0 \to E^1 \to \cdots \to E^{n-1} \to K \to 0$$

of left A-modules with the E^i injective left A-modules. Since ${}_RA$ is flat, the E^i are also injective as left R-modules, and so by Proposition 4.4, K is Gorenstein injective as an R-module.

Conversely, suppose ${}_AK$ is such that ${}_RK$ is Gorenstein injective. Let $E^0 \to \cdots \to E^{n-1} \to K \to 0$ be an exact sequence of left R-modules with the E^i injective left R-modules. Tensoring we get the exact sequence

$$A \otimes_R E^0 \to \cdots \to A \otimes_R E^{n-1} \to A \otimes_R K \to 0$$

exact, and by the remark above, the A-modules $A \otimes_R E^i$ are injective. Hence $A \otimes_R K$ is a Gorenstein injective left A-module. We have the obvious A-linear surjection $A \otimes_R K \to K$ which has an R-linear section. Hence if

$$0 \to N \to A \otimes_R K \to K \to 0$$

is exact, N is a summand of $A \otimes_R K$ as an R-module. By the first part of the proof, $A \otimes_R K$ is a Gorenstein injective R-module, and hence so is N. We can now repeat the procedure with N.

In this manner we can construct an exact sequence

$$0 \to M \to K^0 \to \cdots \to K^{n-1} \to K \to 0$$

of left A-modules with $K^0, \cdots, K^{n-1}$ all Gorenstein injective left A-modules. Hence if ${}_AL$ has finite projective dimension, $\mathrm{Ext}_A^j(L, K^i) = 0$ for $j \geq 1$ and $0 \leq i \leq n-1$.

But then $\mathrm{Ext}_A^1(L,K) \cong \mathrm{Ext}_A^{n+1}(L,M)$. Since A is n-Gorenstein, proj. $\dim L \leq n$ and so $\mathrm{Ext}_A^{n+1}(L,M) = 0$. So $\mathrm{Ext}_A^1(L,K) = 0$ for all such L and hence K is a Gorenstein injective left A-module.

5. Generators of Extensions

Wakamatsu's lemma suggests that when trying to find an $\mathscr{X}$-envelope of a left R-module M we consider exact sequences

$$0 \to M \to N \to L \to 0$$

with $L \in {}^{\perp}\mathscr{X}$. This suggests we make the definition below (with th $\mathscr{L}$ in the definition thought of as ${}^{\perp}\mathscr{X}$ for some $\mathscr{X}$)

DEFINITION 5.1. *Let $\mathscr{L}$ be a class of left R-modules and let M be a left R-module. Then an element $\xi \in \mathrm{Ext}^1(L,M)$ where $L \in \mathscr{L}$ is said to generate $\mathrm{Ext}^1(\mathscr{L},M)$ if for any $\bar{L} \in \mathscr{L}$ and $\bar{\xi} \in \mathrm{Ext}^1(\bar{L},M)$ there is a linear $f : \bar{L} \to L$ such* *that* $\mathrm{Ext}^1(f,M)(\xi)=\bar{\xi}$.

Diagrammatically this says we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & \bar{G} & \to & \bar{L} & \to & 0 \\ & & \| & & \downarrow & & \downarrow f & & \\ 0 & \to & M & \to & G & \to & L & \to & 0 \end{array}$$

where the rows represent the extensions $\bar{\xi}$ and ξ. Then $\xi \in \mathrm{Ext}^1(L,M)$ is said to be a minimal generator if it is a generator and if for any $f \in \mathrm{Hom}(L,L)$ such that and $\mathrm{Ext}^1(f,M)(\xi)=\xi$, f is necessarily an automorphism of L. Then we see that if $\xi \in \mathrm{Ext}^1(L,M)$ and $\bar{\xi} \in \mathrm{Ext}^1(\bar{L},M)$ are both minimal generators of $\mathrm{Ext}^1(\mathscr{L},M)$, then any $f \in \mathrm{Hom}(\bar{L},L)$ such that $\mathrm{Ext}^1(f,M)(\xi)=\bar{\xi}$ is an isomorphism.

If $\xi \in \mathrm{Ext}^1(L,M)$ is a generator of $\mathrm{Ext}^1(\mathscr{L},M)$ and $\mathrm{Ext}^1(f,M)(\bar{\xi})=\xi$ for $\bar{\xi} \in \mathrm{Ext}^1(\bar{L},M)$ and $\bar{L} \in \mathscr{L}$ then $\bar{\xi}$ is also a generator.

PROPOSITION 5.2. *If $\mathscr{L}$ is a class of left R-modules closed under extensions and if $\xi: 0 \to M \to K \to L \to 0$ is a minimal generator for $\mathrm{Ext}^1(\mathscr{L},M)$ then $K \in \mathscr{L}^{\perp}$.*

PROOF. Given a short exact sequence $0 \to K \to N \to \bar{L} \to 0$ with $\bar{L} \in \mathscr{L}$, using a pushout we can construct a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & K & \to & L & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & N & \to & P & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \bar{L} & = & \bar{L} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

with exact rows and columns. Since $\mathscr{L}$ is closed under extensions, $P \in \mathscr{L}$. Since $\xi: 0 \to M \to K \to L \to 0$ generates $\mathrm{Ext}^1(\mathscr{L},M)$, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & N & \to & P & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & K & \to & L & \to & 0 \end{array}$$

Since $\xi: 0 \to M \to K \to L \to 0$ is minimal, the composition $L \to P \to L$ is an automorphism of L. Hence $K \to N \to K$ is an automorphism of K and so

$0 \to K \to N \to \bar{L} \to 0$ splits.

We now get out main result.

THEOREM 5.3. *If R is a ring, $\mathcal{L}$ is a class of left R-modules closed under direct limits and direct summands and if for some left R-module M $\mathrm{Ext}^1(\mathcal{L},M)$ has a generator, then there is a minimal generator for $\mathrm{Ext}^1(\mathcal{L},M)$.*

PROOF. We will let $\mathcal{D}$ be the category of exact sequences $0 \to M \to N \to L \to 0$ where $L \in \bar{L}$. A morphism of $\bar{\xi}: 0 \to M \to \bar{N} \to \bar{L} \to 0$ to $\xi: 0 \to M \to N \to L \to 0$ will be given by a commutative diagram

$$\begin{array}{ccccccccccc} \bar{\xi} & : & 0 & \to & M & \to & \bar{N} & \to & \bar{L} & \to & 0 \\ & & & & \| & & \downarrow & & \downarrow & & \\ \xi & : & 0 & \to & M & \to & N & \to & L & \to & 0 \end{array}$$

We note that ξ is a generator for $\mathrm{Ext}^1(\bar{L},M)$ and only if $\mathrm{Hom}_D(\bar{\xi},\xi) \neq \varnothing$ of all objects $\bar{\xi}$ of $\mathcal{D}$. Also note that $\mathcal{D}$ has direct limits since $\bar{L}$ is closed under limits and since the direct limit functor is exact.

Now let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ whose objects are the ξ which are generators of $\mathrm{Ext}^1(\bar{L},M)$ (or equivalently such that $\mathrm{Hom}_D(\bar{\xi},\xi) \neq \varnothing$ for all objects $\bar{\xi}$ of $\mathcal{D}$). Then $\mathcal{C}$ may not have direct limits, but every directed system in $\mathcal{C}$ certainly admits a map into some object of $\mathcal{C}$ (gotten by mapping the limit of the system, which is in $\mathcal{D}$, to some object of $\mathcal{C}$).

Also suppose $\xi: 0 \to M \to N \to L \to 0$ is in $\mathcal{C}$ and that we have a morphism

$$\begin{array}{ccccccccccc} \xi & : & 0 & \to & M & \to & N & \to & L & \to & 0 \\ & & & & \| & & \downarrow g & & \downarrow h & & \\ \xi & : & 0 & \to & M & \to & N & \to & L & \to & 0 \end{array}$$

such that $\ker(g \circ g) = \ker(g)$, $\ker(h \circ h) = \ker(\mathrm{h})$, in $(g \circ g) = \mathrm{im}(g)$ and $(h \circ h) = \mathrm{im}(h)$. Then $\mathrm{im}(h)$ is a direct summand of L and so in $\bar{L}$ by our hypothesis on $\bar{L}$. Also the sequence $0 \to M \to \mathrm{im}\,(g) \to \mathrm{im}\,(h) \to 0$ is exact and so is an object in $\mathcal{C}$.

We now appeal to Theorem 1 of [11] and see that $\mathrm{Ext}^1(\bar{L},M)$ has a minimal generator.

REMARK. In [11] the symbol ker(f) has a different meaning than if does here. However, we see that the morphism $\xi \to \xi$ above is (in the terminology of [11]) a quasi-retraction.

6. Covers and Envelopes

We apply the results of the previous section to argue that over Gorenstein rings every module has Gorenstein injective envelope and a cover by a module of finite projective dimension.

THEOREM 6.1. *If R is a Gorenstein ring, the every left R-module has a Gorenstein injective envelope.*

PROOF. If M is a left R-module, Theorem 4.5 shows that there is an exact sequence

$$0 \to M \to K \to L \to 0$$

with K Gorenstein injective and $L \in \mathscr{L}$. Since $K \in \mathscr{L}^{\perp}$, $\mathrm{Ext}^1(\bar{L}, M) = 0$ for all $\bar{L} \in \mathscr{L}$.

But then if $0 \to M \to \bar{K} \to \bar{L} \to 0$ is exact with $\bar{L} \in \mathscr{L}$ then

$$\mathrm{Hom}(\bar{K}, K) \to \mathrm{Hom}(M, K) \to \mathrm{Ext}^1(\bar{L}, K) = 0$$

is exact, so

$$\begin{array}{ccc} M & \to & \bar{K} \\ \| & & \vdots\!\downarrow \\ M & \to & K \end{array}$$

can be completed to a commutative diagram. This shows that $0 \to M \to K \to L \to 0$ generated $\mathrm{Ext}^1(\mathscr{L}, M)$. By Theorem 2.3, $\mathscr{L}$ is closed under inductive limits, so there is a minimal generator

$$0 \to M \to K \to L \to 0$$

of $\mathrm{Ext}^1(\mathscr{L}, M)$. By Proposition 5.2, $K \in \mathscr{L}^{\perp}$ i.e. K is Gorenstein injective. But then if $\bar{K} \in \mathscr{L}^{\perp}$

$$\mathrm{Hom}(K \to \bar{K}) \to \mathrm{Hom}(M, \bar{K}) \to \mathrm{Ext}^1(L, \bar{K}) = 0$$

is exact showing that $M \to K$ is a $\mathscr{L}^{\perp}$-preenvelope. Since $0 \to M \to K \to L \to 0$ is a minimal generator, $M \to K$ is also a $\mathscr{L}^{\perp}$-envelope, i.e. is a Gorenstein injective envelope.

THEOREM 6.2. *If R is a Gorenstein ring, then every left R-module has an $\mathscr{L}$-cover.*

Proof. By Theorem 6.1 we know M has a Gorenstein injective envelope, i.e. a $\mathscr{L}^{\perp}$-envelope, say $M \to K$. Easily $M \to K$ is an injection since $M \subset E$ for some injective module E and $E \in \mathscr{L}^{\perp}$.

Let $0 \to M \to K \to L \to 0$ be exact. Then by Corollary 3.5, $L \in {}^{\perp}(\mathcal{L}^{1})$. By Proposition 4.7, ${}^{\perp}(\mathcal{L}^{\perp}) = \mathcal{L}$ so $L \in \mathcal{L}$.

Let $E \to K$ be an injective cover of K. Then using a pull back we can form a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & U & \to & P & \to & M & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & U & \to & E & \to & K & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & L & & L & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

with exact rows and columns.

Since E, $L \in \mathcal{L}$, we have $P \in \mathcal{L}$. $U \in \mathcal{L}^{\perp}$ by Proposition 4.3 and Corollary 3.2. This gives that $P \to M$ is an $\mathcal{L}$-precover. It remains to argue that $P \to M$ is a cover. To facilitate the argument we will make the obvious identifications (e.g. U with a submodule of P, P/U with M etc.).

Let $P \to P$ is a map over M. We need to show this map is an isomorphism.

The map $P \to P$ can be extended to a map $E \to E$ since E is injective and $P \subset E$. since $P \to P$ maps U into U, the map $E \to E$ induces a map $K = E/U \to E/U = K$. The map $P \to P$ induces the identity map an $M = P/U$, so $K \to K$ is such that

$$\begin{array}{ccc} M & \longrightarrow & K \\ & \searrow & \downarrow \\ & & K \end{array}$$

is commutative. Since $M \to K$ is an envelope, $K \to K$ is an automorphism of K. Then since $E \to K$ is an injective cover and $E \to E$ is such that

$$\begin{array}{ccc} E & \to & K \\ \downarrow & & \downarrow \\ E & \to & K \end{array}$$

is commutative, $E \to E$ is an automorphism of E.

Then our maps $E \to E$, $K \to K$, $P \to P$ and id: $M \to M$ give a map of the pull back

$$\begin{array}{ccc} P & \rightarrow & M \\ \downarrow & & \downarrow \\ E & \rightarrow & K \end{array}$$

into itself.

Since this map is an isomorphism on each of E,K and M, it is also an isomorphism on P.

7. Divisible Envelopes and Lattice Covers

In this section we let D be a discrete valuation ring and let G be a finite group. Let $E=E_D(k)$ where k is the residue field of D and for any DG-module (left or right), let $N^{\upsilon}=\mathrm{Hom}_D(N,E)$ (i.e. N^{υ} is the Matlis dual of N). So if N is a left DG-module, N^{υ} is a right DG-module. We will use the familiar properties of the Matlis dual.

A left DG-module will be said to be divisible if it is divisible if it is divisible as a D-module.

THEOREM 7.1. *Every left DG-module has a divisible envelope.*

PROOF. Since $D\subset DG$ is a quasi-Frobenius extension, by Theorem 2.5, DG is 1-Gorenstein. By Theorem 4.11 and Corollary 4.8, a left DG-module K is Gorenstein injective if and only if K is divisible. Now we only need appeal to Theorem 6.

PROPOSITION 7.2. *If M is a left DG-module which is artinian as a D-module and if $M\rightarrow K$ is a Gorenstein injective envelope of M (as DG-modules) then K is also an artinian D-module.*

PROOF. By Theorem 4.11 and Corollary 4.8, K is a divisible D-module, so $E_D(M)\subset K$ (as D-modules). But $E_D(M)$ is an artinian D-module.

Let $\overline{K}=\sum gE_D(M)$ (the sum over $g\in G$). Then $\overline{K}$ is a DG-module which is divisible as a D-module. Hence $\overline{K}$ is Gorenstein injective DG-module. Then $M\subset\overline{K}$ can be factored $M\rightarrow K\rightarrow\overline{K}$ since $M\rightarrow K$ is an envelope. But then $K\rightarrow\overline{K}\rightarrow K$ (with $\overline{K}\rightarrow K$ the canonical injective) is an automorphism of K, so $\overline{K}\rightarrow K$. Hence ${}_DK$ is artinian.

We recall that by a lattice we mean a left DG-module U which is free and finitely generated as a D-module.

Note that if U is a lattice, then the right DG-module U^{υ} is divisible as a D-

module and so is Gorenstein injective.

Conversely if D is complete and if K is a Gorenstein injective right DG-module with ${}_DK$ artinian, then K^{υ} is a lattice.

This gives a duality which will be used in the proof below.

THEOREM 7.2. *If D is complete, every finitely generated left DG-module has a cover by a lattice.*

PROOF. Let M be a finitely generated left DG-module. Then M^{υ} is artinian as a D-module and so if $M^{\upsilon} \to K$ is a Gorenstein injective envelope, K is also artinian, and so reflexive. Since K is also a divisible D-module, K^{υ} is a lattice. We argue that $K^{\upsilon} \to M^{\upsilon\upsilon} = M$ is the desired cover.

If U is a lattice then U is reflexive and U^{υ} is Gorenstein injective. Hence to complete

$$\begin{array}{ccc} U & & \\ \vdots & \searrow & \\ \downarrow & & \\ K^{\upsilon} & \longrightarrow & M \end{array}$$

to a commutative diagram it suffices to complete

$$\begin{array}{ccc} M^{\upsilon} & \longrightarrow & K = K^{\upsilon\upsilon} \\ & \searrow & \vdots \\ & & \downarrow \\ & & U^{\upsilon} \end{array}$$

But we can complete the latter to a commutative diagram. If

$$\begin{array}{ccc} K^{\upsilon} & & \\ & \searrow & \\ \downarrow & & M \\ & \nearrow & \\ K^{\upsilon} & & \end{array}$$

is a commutative diagram, then so is

$$\begin{array}{ccc} & & K = K^{\upsilon\upsilon} \\ & \nearrow & \\ M & & \downarrow \\ & \searrow & \\ & & K \end{array}$$

Since $M^{\upsilon} \to K$ is an envelope, $K \to K$ is an isomorphism, and hence so is $K^{\upsilon} \to K^{\upsilon}$.

PROPOSITION 7.3. *Let D be complete. If M is a finitely generated left DG-module and $U \to M$ is a cover by a lattice, then $P = \ker(U \to M)$ is a (finitely* *generated) projective DG-module.*

PROOF. Clearly P is a free DG-module, so P^{υ} is a Gorenstein injective right DG-module. Since

$$0 \to M^{\upsilon} \to U^{\upsilon} \to P^{\upsilon} \to 0$$

is exact and since by the duality mentioned above we have that $M^{\upsilon} \to U^{\upsilon}$ is an Gorenstein injective envelope, the by Corollary 3.5, $P^{\upsilon} \in {}^{\perp}(\mathscr{L}^{\perp}) = \mathscr{L}$. Hence $P^{\upsilon} \in \mathscr{L} \cap \mathscr{L}^{\perp}$ and so is an injective DG-module by Corollary 4.6. Then by duality again we easily see that $P^{\upsilon\upsilon} = P$ is a projective DG-module.

We note that for any finitely generated projective left DG-module P, id: $P \to P$ is its cover by a lattice.

Now assume D is a complete discrete valuation domain and that G is a finite group. Let $\mathscr{C}$ be the category of finitely generated left DG-modules. By the stable category $\overline{\mathscr{C}}$ we mean the category whose objects are the objects of $\mathscr{C}$ and whose morphisms are equivalence classes of linear maps with $f,g\colon M_1 \to M_2$ equivalent if and only if $f-g$ can be factored through a projective left DG-module.

let $\mathscr{U} \subset \mathscr{C}$ be the full subcategory of lattices and let $\overline{\mathscr{U}}$ be its associated stable category.

For each finitely generated left DG-module, we pick a lattice cover $U \to M$.

If $f: M_1 \to M_2$ is linear between two such finitely generated modules and if $f_1: U_1 \to M_1$, $f_2: L_2 \to M_2$ are their lattice covers then there is a $g: U_1 \to U_2$

$$\begin{array}{ccc} U_1 & \to & M_1 \\ g\downarrow & & \downarrow f \\ U_2 & \to & M_2 \end{array}$$

commutative. If $\bar{g}: U_1 \to U_2$ is another map making the diagram commutative, then im $(g-\bar{g}) \subset \ker(U_2 \to M_2)$. Since $\ker(U_2 \to M_2)$ is projective, the equivalence class of g in $\overline{\mathscr{U}}$ is well defined.

With the obvious notation, we now note that a factorization $M_1 \to P \to M_2$ with P finitely generated and projective leads to a factorization $L_1 \to P \to L_2$. With these remarks we see we have a well defined additive functor $\overline{\mathscr{C}} \to \overline{\mathscr{U}}$ which is the identity on $\overline{\mathscr{U}} \subset \overline{\mathscr{C}}$. so we say $\overline{\mathscr{U}}$ is stably reflective in $\overline{\mathscr{C}}$.

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