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PROPER n-HOMOTOPY EQUIVALENCES OF LOCALLY COMPACT POLYHEDRA

By

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Abstract. We prove the following theorem which is a locally compact analogue of results of S. Ferry and the author.

Theorem. Let $f: X \to Y$ be a proper map between finite dimensional locally compact polyhedra X and Y. Suppose that

(1) $\pi_i(f): \pi_i(X) \to \pi_i(Y)$ is an isomorphism for each $i \leq n$,

(2) f induces a surjection between the ends of X and Y, and

(3) f induces an isomorphism between the i-th homotopy groups of ends of X and Y for each $i \leq n$.

Then there exist a locally compact polyhedron Z and proper UV^n- maps $\alpha: Z \to X$ and $\beta: Z \to Y$ such that

(4) dim $Z \leq 2$ max(dim X,n)+3,

(5) $f \circ \alpha$ and β is properly n-homotopic, and

(6) α has at most countably many non-contractible fibre all of which have the homotopy type of S^{n+1}

1. Introduction.

The purpose of this note is to prove the following result which is a locally compact analogue of [F_2, Proposition 1.7] and [K].

MAIN THEOREM. *Let $f: X \to Y$ be a proper map between finite dimensional locally compact polyhedra X and Y. Suppose that*

(1) *$\pi_i(f): \pi_i(X) \to \pi_i(Y)$ is an isomorphism for each $i \leq n$,*

(2) *f induces a surjection between the ends of X and Y, and*

(3) *f induces an isomorphism between the i-th homotopy groups of ends of X and Y for each $i \leq n$.*

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(4) dim $Z \leq 2\ \max(\dim X, n)+3$,

(5) *$f \circ \alpha$ and β is properly n-homotopic, and*

(6) *α has at most countably many non-contractible fibre all of which have the homotopy type of S^{n+1}.*

A continuous map $f: X \rightarrow Y$ is said to be *proper* if it is closed and $f^{-1}(K)$ is compact for any compact subset K of Y. We do not assume that f is a surjection. A proper map $f: X \rightarrow Y$ is said to induce an *epimorphism between the i-th homotopy groups of the ends* if, for each compact subset K of Y, there is a compact subset L of Y containing K such that for each map $\beta: S^i \rightarrow Y-L$, there exists a map $\alpha: S^i \rightarrow X-f^{-1}(K)$ such that $f \circ \alpha \simeq \beta$ in $Y-K$. The map f is said to induce a *monomorphism between i-th homotopy groups of the ends* if for each compact subset K of Y, there exists another compact subset L of Y containing K such that, if a map $\alpha: S^i \rightarrow X-f^{-1}(L)$ satisfies that $f \circ \alpha \simeq 0$ in $Y-L$, then we have that $\alpha \simeq 0$ in $X-f^{-1}(K)$ (see [B, Chap. 6]). Two maps $f, g: X \rightarrow Y$ between locally compact separable metric spaces are said to be *properly n-homotopic* if, for each map proper $\alpha: K \rightarrow X$ of a locally compact separable metric space K with dim $K \leq n$, $f \circ \alpha$ is properly homotopic to $g \circ \alpha$.

It is known that there is a strong similarity between Menger (or μ^k-) manifold theory and Hilbert cube (or Q-) manifold theory. In μ^k-manifold theory, the proper $(k-1)$-homotopy theory plays the role similar to the usual homotopy theory in Q-manifold theory. In particular, the topological types of Q-manifolds are determined by their simple homotopy types, whereas the topological types of μ^k-manifolds are determined by their proper $(k-1)$-homotopy types. The above theorem provides an underlying reason for this correspondence.

For the proof of Main Theorem, we need locally compact analogues of results in [B, Appendix] and $[F_{3-4}]$. Once we obtain these analogues, the proof proceeds as in [K]. Throughout this paper, the reader is assumed to be familiar with the paper [B, Appendix], $[F_{3-4}]$ and $[K]$.

The possibility of obtaining the locally compact analogue as above was first asked in a discussion with A. Chigogidze. The author would like to express his sincere thanks to Professor A. Chigogidze for helpful discussion on this problem. The author would also like to express his thanks to the referee for helpful comments to make this paper readable.

2. Preliminaries.

Throughout this paper, spaces are assumed to be separable and metrizable.

DEFINITION 2.1. For a proper map $f: X \to Y$ between locally compact separable metric spaces, *M(f)* denotes the mapping cylinder of *f*. The standard *CE* retraction is denoted by $c(f): M(f) \to Y$.

(1) The map *f* is said to be *n-connected* if $\pi_i(M(f), X) = 0$ for each $i \leq n$.

(2) The map f is said to be *n-connected at infinity* if it induces an isomorphism between the i-th homotopy groups of ends for each $i \leq n-1$ and an epimorphism for i=n.

A pair of space (*P, Q*) is said to be *n-connected at infinity* if, for each compact subset *E* of *P*, there exists a compact subset *F* of *P* containing *E* such that;

For each $i \leq n$, each map $\alpha: (D^i, \partial D^i) \to (P - F, Q - F)$ is homotopic to a map $\beta: D^i \to Q - E$ in $P - E$ rel ∂D^i.

PROPOSITION 2.2. *Let $f: X \to Y$ be a proper map which is n-connected at infinity. Then (M(f),X) is n-connected at infinity.*

PROOF. For a given compact subset *E* of *M(f)*, let $E' = c(f)(E)$ and take a compact subset *C* of *Y* containing E' which satisfies the monomorphism condition at infinity with respect to E' and for each $i \leq n-1$. Next take a compact subset *D* of *Y* containing *C* which satisfies the epimorphism condition at infinity with respect to *C* and for each $i \leq n$. Let $F = c(f)^{-1}(D) \supset E$. Since *c(f)* is a proper map, *F* is compact and a standard argument shows that *F* is the desired compact set.

LEMMA 2.3. *Let $f: X \to Y$ be a proper cellular map between CW complexes which induces a surjection between ends of X and Y. There exist a CW complex M^* and a proper CE map $c: M(f) \to M^*$ such that $M^{*(0)} \subset X \subset M^*$ and $c|X = id$.*

PROOF. Since f induces a surjection between ends of X and Y, we have that:

For each compact subset K of Y and for any unbounded component N of $Y - K$, we have that $f^{-1}(N) \neq \phi$.

Using this, we can take an increasing sequence $K_1 \subset K_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} K_i = Y$ of compact subsets of *Y* satisfying the following condition.

(1) For each vertex $v \in c\ell(K_{i+1} - K_i)$, there exists an arc J_v in $c\ell(K_{i+1} - K_{i-1})$ joining v with a vertex $f(X_v)$, where X_v is a vertex of X.

Recall that the CW complex structure of $M(f)$ consists of the cells of X and Y and $\{e\times I \mid e$ is a cell of $X\}$. Thus $M(f)^{(0)} = X^{(0)} \cup Y^{(0)}$. Then $J_v \cup \{X_v\}\times I$ defines an arc connecting v with X_v in $M(f)$. Consider the union J of these arcs. By the condition (1), we can choose a countable collection $\{T_i\}$ of compact trees such that $\bigcup_{i=1}^{\infty} T_i \subset J$ and

(2) $\{T_i\}$ is a discrete collection and $\bigcup_{i=1}^{\infty} T_i \supset Y^{(0)}$.

Shrinking each T_i into a point, we obtain a CW complex M^* and a proper CE map $c: M(f) \to M^*$. From the construction, M^* contains X and X contains all vertices of M^*.

This completes the proof.

The following is an analogue of Whitehead Cell Trading Lemma (See for example, [Co, 7.3]) for locally compact CW complexes. Recall that two compact CW complexes X and Y are simple homotopy equivalent if and only if there exist a compact CW complex Z and CE maps of Z onto X and Z onto Y ([Chap]). Having this fact in mind, the proof is a simple modification of the one of [Co, 7.3].

PROPOSITION 2.4. *Let (K,L) be a pair of finite dimensional locally compact CW complexes with dim $K = N$ such that*

(1) *(K,L) is r-connected and r-connected at infinity,*

(2) $K = L \cup \bigcup_{j=1}^{\infty} e_j^r \cup \bigcup_{j=1}^{\infty} e_j^{r+1} \cup \cdots \bigcup_{j=1}^{\infty} e_j^N$.

Then there exists a CW complex Q containing L such that

(3) $Q = L \cup \bigcup_{j=1}^{\infty} E_j^{r+1} \cup \bigcup_{j=1}^{\infty} E_j^{r+2} \cup \cdots \bigcup_{j=1}^{\infty} E_j^N$, and

(4) *K is proper CE equivalent to Q relative to L, that is, there exist a CW complex Z which contains L and proper CE maps $\alpha: Z \to Q$ and $\beta: Z \to K$ such that $\alpha|L = id$ and $\beta|L = id$.*

OUTLINE OF PROOF. Let $I = [0,1]$ and let I^r be the r-cell. The r-cell I^r is naturally regarded as the face $I^r \times 0$ of I^{r+1}. Let $J^r = cl(\partial I^{r+1} - I^r \times 0)$. One can use the assumption (1) to obtain an increasing sequence $\phi = K_0 \subset K_1 \subset K_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} K_i = K$ of compact subcomplexes of K such that, for each $t \leq r$,

(5) each map $\alpha: (I^t, \partial I^t) \to (K_{i+1} - K_i, L - K_i)$ is homotopic to a map β: $I^t \to L - K_{i-1}$ rel ∂I^t in $K_{i+2} - K_{i-1}$, for each $i \geq 1$.

Using the condition (5), Proposition can be proved in the same way as the one in [Co, 7.3]. Take any r-cell $e_j^r \subset K$ and let $\varphi_j^r: I_j^r \to K - K_i$ be the characteristic map of e_j^r such that $\varphi_j^r(\partial I_j^r) \subset K^{(r-1)} \subset L$ (the r-cell is indexed as I_j^r). When $e_j^r \subset K_{i+1} - K_i$, the condition (5) guarantees that there exists a map F_j : $I_j^{r+1} \to K_{i+1}^{(r+1)} - K_{i-1}$ such that

$$(6)\quad \begin{aligned} F_j|I_j^r \times 0 &= \varphi_j^r,\ F_j(I_j^r \times 1) \subset L - K_{i-1}, \\ F_j|\partial I_j^r \times t &= \varphi_j^r|\partial I_j^r \ \textit{for each } t, \textit{and}\ F_j(\partial I_j^{r+1}) \subset K^{(r)}. \end{aligned}$$

Let $P = K \cup \bigcup_{F_j} I_j^{r+2}$, then one can define a proper CE map $\varphi : P \to K$ induced by the natural collapse $I_j^{r+2} \to I_j^{r+1} \times 0$.

Let $\psi : K \oplus \oplus_j I_j^{r+2} \to P$ be the quotient map and let $E_j^{r+1} = \psi(J_j^{r+1})$. Define P_0 by

$$P_0 = L \cup \bigcup_j e_j^r \cup E_j^{r+1}$$

It is easy to construct a CE retraction $g : P_0 \to L$. Let $Q = P \bigcup_g L$. The condition (6) guarantees that the collection of $(r+1)$-cells involved in the above construction is locally finite, so the same proof as the one of [Co, 1.9] works to produce a locally compact CW complex Z_1 which admits proper CE maps onto both of P and Q. Then Z_1 admits a proper CE map onto K as well.

Since dim $K < \infty$, repeating the above process finitely many times, one obtains the desired complex Z and proper CE maps. This completes the outline of the proof of Proposition.

3. Proof of Main Theorem.

For any $PL\,n$-manifold M^n, for any $\ell \geq 1$ and for any k with $2k+3 \leq n$, there exists a proper UV^k map $f : M \to M \times D^\ell$ such that $\mathrm{proj} \circ f$ is properly homotopic to id_M, where $\mathrm{proj} : M \times D^\ell \to M$ is the projection ([Ce], [F_3]). The same proof as [F_3] then can be adapted to prove the following result with a minor change.

PROPOSITION 3.1. (cf. [B, Appendix] and [F_3, Theorem 2]). *Let $f : M^n \to B$ be a proper map of a PL n-manifold M to a locally compact polyhedron B. Suppose that f induces a surjection between ends of M and B and f is (k+1)-connected and (k+1)-connected at infinity. If* $2k+3 \leq n$, there exists a proper *UV^k-map* $g : M \to B$ which is properly homotopic to *f*.

PROOF. Since *f* induces a surjection between ends, by Lemma 2.3, there exist a CW complex M^* and a proper CE map $c : M(f) \to M^*$ such that M^* contains M, $M \supset (M^*)^{(0)}$, and $c|M = \mathrm{id}$. Clearly, (M^*, M) is $(k+1)$-connected and $(k+1)$-connected at infinity. Applying Proposition 2.4 to (M^*, M), there exists a locally compact CW complex Q such that

(1) Q is obtained from M by attaching cells of dimension $\geq k+2$, and

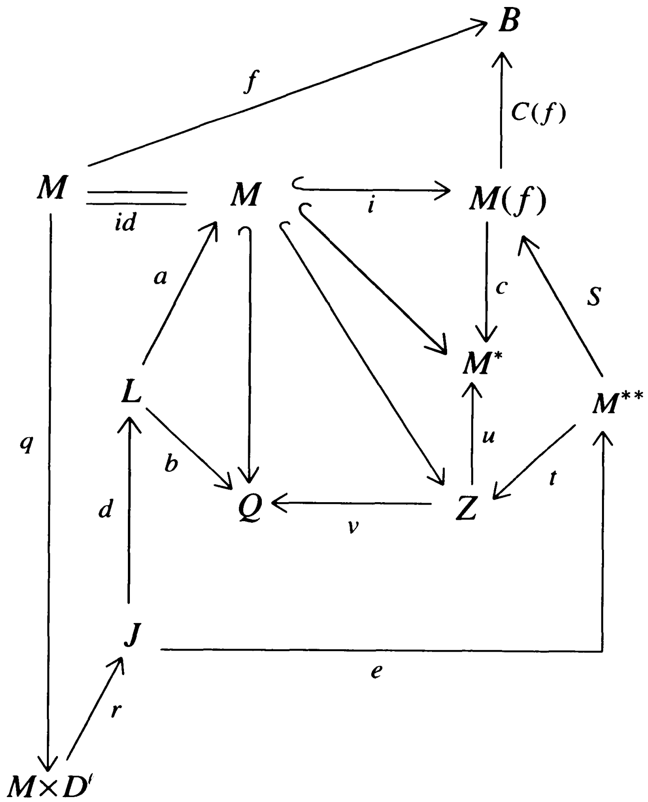
(2) there exist a locally compact CW complex Z containing M and proper CE

maps $u: Z \to M^*$ and $v: Z \to Q$ such that $u|M = \mathrm{id}$ and $v|M = \mathrm{id}$.

Take the pullback M^{**} of u and c and let $S: M^{**} \to M(f)$ and $t: M^{**} \to Z$ be the projections. Notice that both of s and t are proper CE maps. Let $\overline{Q}^{(j)} = Q^{(j)} \cup M$ $(k+2 \leq j \leq \dim Q = \dim M^*)$. As in [F_3, Theorem 2], one can construct a finite dimensional locally compact CW complexes L and a proper CE map $a: L \to M$ and a UV^k*-map* $b: L \to Q$ (use the induction on j) such that $i_{M,Q} \circ a$ is properly homotopic to b, where $i_{M,Q}$ is the inclusion $M \to Q$.

Again take the pullback J of $v \circ t: M^* \to Q$ and $b: L \to Q$ and let $d: J \to L$ and $e: J \to M^{**}$ be the projections. The map d is a proper CE map and e is a proper UV^k-map. As in [F_3, Theorem 2], for sufficiently large ℓ, there exists a proper CE map $r: M \times D^\ell \to J$ such that $a \circ d \circ r$ is properly homotopic to the projection $\mathrm{proj}: M \times D^\ell \to M$. Applying Cernavskii's Theorem mentioned at the beginning of this section, we can construct a proper UV^k-map $q: M \to M \times D^\ell$ such that $\mathrm{proj} \circ q$ is properly homotopic to id. Let $\varphi: M \to B$ be a UV^k-map defined by $\varphi = c(f) \circ s \circ e \circ r \circ q$. Let $i: M \to M(f)$ be the inclusion, then we can see that i is properly homotopic to $s \circ e \circ r \circ q$. Therefore, $f = c(f) \circ i$ is properly homotopic to φ.

This completes the proof.



Let $f:X \to Y$ be a map between locally compact separable metric spaces and let $\varepsilon: Y \to (0,1]$ be a continuous function. The map f is called an $AL^k(\varepsilon)$-*map* if for any locally compact polyhedral pair (P,Q) with $\dim P \le k$ and for any pair of maps $\alpha_0 : Q \to X$ and $\alpha : P \to Y$ such that $\alpha|Q = f \circ \alpha_0$, there exists an extension $\overline{\alpha} : P \to X$ of α_0 such that $d(f \circ \overline{\alpha}(x), \alpha(x)) < \varepsilon(\alpha(x))$ for each $x \in P$.

The proof of [F_4, Theorem 8.1] directly generalizes to prove the following result.

PROPOSITION 3.2. *Let B be a locally compact polyhedron. For each continuous function* $\varepsilon : B \to (0,1]$, *there exists a continuous function* $\delta : B \to (0,1]$ *such that*

for each k with $2k+3 \le n$ *and for each* $AL^{k+1}(\delta)$ *map* $f: M^n \to B$ *of a PL n-manifold M to B, there exists a proper* UV^k*-map* $\varphi : M \to B$ *which is properly* ε*-homotopic to f.*

Using Proposition 3.1 and 3.2, the proof of Main Theorem proceeds in the same way as in [K]. We briefly sketch the proof.

SKETCH OF THE PROOF OF MAIN THEOREM. Let $f: X \to Y$ be a proper map between finite dimensional locally compact polyhedra which induces surjection between the ends, an isomorphism between the i-th homotopy groups and the i-th homotopy groups of the ends for each $i \le n$.

Embed X into an Euclidean space of high dimension and take a regular neighbourhood M. We may assume t hat M is a PL manifold with $\dim M = 2\max(n, \dim X) + 3$ which admits a proper CE retraction onto X. In the sequel, we assume that $X = M$ for simplicity. Notice that f is n-connected and n-connected at infinity. Apply Proposition 3.1 to replace f by a UV^{n-1}-map which is denoted by the same symbol f. Take a continuous function $\delta : Y \to (0,1]$ such that

(1) any $AL^{n+1}(\delta)$-map $g: L \to Y$, where L is a PL manifold of $\dim \ge 2n+3$, is properly homotopic to a UV^n-map (Use Proposition 3.2 and the ANR property of Y).

As in [K], we can attach at most countably many (n+1)-cells to M to obtain a PL manifold $\underline{M}$ and an extension $\underline{f} : \underline{M} \to Y$ which is an $AL^{n+1}(\delta)$-map. By the choice of δ ,(1), there exists a UV^n- map $\varphi : M \to Y$ which is properly homotopic to $\underline{f}$.

Next we attach at most countably many (n+2)-cells to $\underline{M}$ to obtain a PL manifold M^* which admits a proper CE retraction $r: M^* \to M$ onto M. Attach (n+2) cells to Y using φ to obtain a polyhedron Y^*, so that φ naturally extends to

$\varphi^* : M^* \to Y^*$. Applying [F_1, Lemma 2.1], we can construct a proper CE map $c: Y^\wedge \to Y$ and a UV^n- map $u: Y^\wedge \to Y^*$ of a locally compact polyhedron $Y^\wedge$ such that $i_{YY}{}^* \circ c$ is properly homotopic to u and u has at most countably many non-contractible fibres all of which are homeomorphic to S^{n+1}($i_{YY}{}^*$is the inclusion $Y \to Y^*$). Take the pullback Z of φ^* and u and let $v: Z \to M^*$ and $w: Z \to Y^*$ be the projections. Then Z, $\alpha = r \circ v$ and $\beta = c \circ w$ are the required maps.

This completes the proof.

Since proper UV^n-maps between $(n+1)$-dimensional locally compact ANR's are proper n-homotopy equivalences ([Ch]), we have the following corollary.

COROLLARY. Let $f: M \to N$ be a proper map between at most $(n+1)$-dimensional locally compact ANR's. Then f is a proper n-homotopy equivalence if and only if f is n-homotopic to a proper UV^n-map.

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