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A NOTE ON FREE DIFFERENTIAL GRADED ALGEBRA RESOLUTIONS

By

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Introduction

We work ove a field k. A differential graded algebra (dga for short) in this paper is a graded k-algebra $U=\oplus_{n\geq 0}U_n$ with differential d of degree -1. Given a k-algebra R, it is well-known that there exists a free dga resolution $\varepsilon: U\rightarrow R$ (Baues [2]). That is, U is a dga which is free as a graded algebra, ε is a dga map, and the sequence

$$\cdots\xrightarrow{d}U_n\xrightarrow{d}\cdots\xrightarrow{d}U_0\xrightarrow{\varepsilon}R\rightarrow 0$$

is exact. Such a resolution is thought of as a prolongation of a presentation of R by generators and relations, and expected to contain lots of information about homology of R. Although free dga's frequently appear in homotopical algebra such as [2], not much seems to be known about the structure of free dga resolutions of algebras.

We study here a relationship between a free dga resolution of R and a free bimodule resolution of the R-bimodule R. Let U be a dga which is free on a graded space E, and $\varepsilon: U\rightarrow R$ an augmentation map. We construct a complex $R\otimes E\otimes R$ of free R-bimodules with augmentation $\sigma: R\otimes E\otimes R\rightarrow\Omega_R$, where Ω_R is the kernel of the multiplication map $R\otimes R\rightarrow R$. If ε is a resolution, then so is σ (Proposition 1.2). The converse is true when R is a connected graded algebra and U,ε are taken to be compatible with the grading of R (Theorem 3). Therefore, the verification of the exactness of $\varepsilon: U\rightarrow R$ reduces to that of $\sigma: R\otimes E\otimes R\rightarrow\Omega_R$, which is much easier.

Using this criterion, we give explicit free dga resolutions of Koszul algebras and their generalizations.

NOTATION. For a graded module $M=\oplus_{n\geq 0}M_n$, we write $M_+=\oplus_{n>0}M_n$. For a

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k-module V, $T(V)$ is the tensor algebra on V. When V is a graded k-module, we give $T(V)$ the induced grading.

1. The bimodule resolution associated with a dga resolution

A dga is a graded algebra $U=\oplus_{n\geq 0}U_n$ equipped with a linear map $d:U\to U$ such that $d^2=0, d(U_n)\subset U_{n-1}$ and $d(xy)=d(x)y+(-1)^p xd(y)$ for $x\in U_p, y\in U_q$. A dga (U,d) is said to be free if the graded algebra U is free, that is, $U=T(E)$ for some graded subspace E of U.

Any algebra can be viewed as a dga concentrating in degree 0. Let U be a dga and R an algebra. A dga map $\varepsilon:U\to R$ is called a resolution if

$$\cdots\xrightarrow{d}U_n\xrightarrow{d}\cdots\xrightarrow{d}U_0\xrightarrow{\varepsilon}R\to 0$$

is exact.

It is well-known that given an algebra R, there exist a free dga U and a resolution $\varepsilon:U\to R$. For example, see [2, Lemma 7.21], where a more general statement is proved. Although our results are logically independent of this fact, we briefly review a construction of a free dga resolution.

First, take a surjective algebra map $U^{(0)}=T(E_0)\to R$ from a tensor algebra. Suppose we have constructed a dga $U^{(n)}$ which is free on a graded space $E_0\oplus\cdots\oplus E_n$, and a dga map $U^{(n)}\to R$ which induces isomorphism on homology in degree $< n$. Then take a linear map $\phi:E_{n+1}\to \mathrm{Ker}(d:U_n^{(n)}\to U_{n-1}^{(n)})$ so that $\mathrm{Im}\,\phi$ covers $H_n(U^{(n)})$. Put $U^{(n+1)}=T(E_0\oplus\cdots\oplus E_{n+1})$ and extend the differential of $U^{(n)}$ to the differential d of $U^{(n+1)}$ so that $d\,|\,E_{n+1}=\phi$. Then $H_n(U^{(n+1)})=0$. Thus we obtain an increasing sequence of free dga's $U^{(n)}, n\geq 0$. Then $U=\bigcup_n U^{(n)}$ together with the map $U_0=U^{(0)}\to R$ provides a free dga resolution of R.

In this section we give a construction of a free R-bimodule resolution of the R-bimodule R from a free dga resolution of R. This is based on an idea of Shukla in [5].

Let R be an algebra, $U=(T(E),d)$ a free dga and $\varepsilon:U\to R$ a dga map. Define an R-bimodule map $\rho:R\otimes U\otimes R\to R\otimes E\otimes R$ by

$$\rho(1\otimes x_1\cdots x_n\otimes 1)=\sum_{i=1}^{n}\varepsilon(x_1\cdots x_{i-1})\otimes x_i\otimes\varepsilon(x_{i+1}\cdots x_n)$$

for $x_1,\cdots,x_n\in E$ and

$$\rho(1\otimes 1\otimes 1)=0.$$

Then $\rho(R\otimes U_n\otimes R)\subset R\otimes E_n\otimes R$, because $\varepsilon(U_+)=0$.

Define an R-bimodule map $\partial:R\otimes E\otimes R\to R\otimes E\otimes R$ as the composite

$$R\otimes E\otimes R \hookrightarrow R\otimes U\otimes R \xrightarrow{1\otimes d\otimes 1} R\otimes U\otimes R \xrightarrow{\rho} R\otimes E\otimes R .$$

Then $\partial(R\otimes E_n\otimes R)\subset R\otimes E_{n-1}\otimes R$.

Define an R-bimodule map $\sigma: R\otimes E\otimes R\to R\otimes R$ by

$$\sigma(1\otimes x\otimes 1)=\varepsilon(x)\otimes 1-1\otimes\varepsilon(x) .$$

σ vanishes on $R\otimes E_+\otimes R$.

PROPOSITION 1.1. $\partial^2=0, \sigma\partial=0$.

Thus we obtain a complex of R-bimodules

$$\cdots\xrightarrow{\partial} R\otimes E_n\otimes R\xrightarrow{\partial}\cdots\xrightarrow{\partial} R\otimes E_0\otimes R\xrightarrow{\sigma} R\otimes R\xrightarrow{\text{mult}} R\to 0 .$$

PROPOSITION 1.2. *If $\varepsilon: U\to R$ is a resolution, then this complex is exact.*

2. Proof of Propositions 1.1 and 1.2

Viewing U as just an algebra, we form the standard free resolution of the U-bimodule U ([3]):

$$\cdots\xrightarrow{\delta} U^{\otimes(n+2)}\xrightarrow{\delta}\cdots\xrightarrow{\delta} U\otimes U\xrightarrow{\text{mult}} U\to 0$$

where

$$\delta(u_0\otimes\cdots\otimes u_{n+1})=\sum_{i=0}^{n}(-1)^i u_0\otimes\cdots\otimes u_i u_{i+1}\otimes\cdots\otimes u_{n+1} .$$

Each term of the resolution is a complex as a tensor product of the complex U, and each δ is a chain map, because the multiplication $U\otimes U\to U$ is so.

Now regard R as a U-bimodule through the map $\varepsilon: U\to R$. Applying the functor $R\otimes_U(\;)\otimes_U R$ to the standard resolution, we obtain a complex

$$\cdots\xrightarrow{\gamma} R\otimes U^{\otimes n}\otimes R\xrightarrow{\gamma}\cdots\xrightarrow{\gamma} R\otimes R\xrightarrow{\text{mult}} R\longrightarrow 0 ,$$

whose terms are complexes and differentials γ are chain maps. So we have a double complex B having terms $B_{pq}=(R\otimes U^{\otimes p}\otimes R)_q$ for $p,q\geq 0$. This was considered by Shukla [5]. The propositions are proved by relating $H^{\mathrm{I}}_p H^{\mathrm{II}}_q(B)$ and $H^{\mathrm{II}}_q H^{\mathrm{I}}_p(B)$ with $H_{p+q}(\operatorname{tot} B)$. Here $H^{\mathrm{I}}, H^{\mathrm{II}}$ mean the homology with respect to the first, second index respectively, and tot B is the total complex of B.

We first treat $H^{\mathrm{I}}H^{\mathrm{II}}(B)$.

(i) We have a diagram

$$\begin{array}{ccccccc} R\otimes U\otimes U\otimes R & \xrightarrow{\gamma} & R\otimes U\otimes R & \xrightarrow{\rho} & R\otimes E\otimes R & \longrightarrow & 0 \\ & & \gamma\downarrow & \swarrow\sigma & & & \\ & & R\otimes R & & & & \end{array}$$

where the first row is exact and the triangle is commutative.

PROOF. Forget the differential graded structure of U for a moment. As U is the tensor algebra on E, a minimal free resolution of the U-bimodule U is given by

$$0\to U\otimes E\otimes U\xrightarrow{\tau}U\otimes U\xrightarrow{\text{mult}}U\to 0$$

where $\tau(1\otimes x\otimes 1)=x\otimes 1-1\otimes x$ for $x\in E$([2, p. 181, Ex2]). Define a U-bimodule map $\theta: U\otimes U\otimes U\to U\otimes E\otimes U$ by

$$\theta(1\otimes x_1\cdots x_n\otimes 1)=\sum_{i=1}^{n} x_1\cdots x_{i-1}\otimes x_i\otimes x_{i+1}\cdots x_n$$

$$\theta(1\otimes 1\otimes 1)=0$$

for $x_1,\cdots,x_n\in E$. θ is the identity on $U\otimes E\otimes U$ and the diagram

$$\begin{array}{ccc} U\otimes U\otimes U & \xrightarrow{\delta} & U\otimes U \\ \theta\downarrow & & \| \\ U\otimes E\otimes U & \xrightarrow{\gamma} & U\otimes U \end{array}$$

is commutative. By the exactness of the standard and the minimal resolutions, it follows that the sequence

$$U\otimes U\otimes U\otimes U\xrightarrow{\delta}U\otimes U\otimes U\xrightarrow{\theta}U\otimes E\otimes U\to 0$$

is exact. Now apply $R\otimes_U(\,)\otimes_U R$ to the above diagram and the sequence. As $\rho=R\otimes_U\theta\otimes_U R$, $\sigma=R\otimes_U\tau\otimes_U R$, the assertion follows.

PROOF OF PROPOSITION 1.1. Since γ is a chain map, it follows from the exact sequence of (i) that $R\otimes E\otimes R$ becomes a complex with differential ∂' and ρ becomes a chain map. By the definition of ∂ and the fact that ρ is the identity on $R\otimes E\otimes R$, we know $\partial'=\partial$. Thus $(R\otimes E\otimes R,\partial)$ is a complex and ρ,σ are chain maps.

(ii) The homology of the complex

$$\cdots\xrightarrow{\partial}R\otimes E_n\otimes R\xrightarrow{\partial}\cdots\xrightarrow{\partial}R\otimes E_0\otimes R\xrightarrow{\sigma}R\otimes R$$

at $R\otimes E_n\otimes R$ is isomorphic to $H_{n+1}(\text{tot}\,B)$.

PROOF. As noted in (i), the U-bimodule U has projective dimension 1. So $H_p^{\mathrm{I}}(B)=\mathrm{Tor}_p^{U\otimes U^{op}}(R\otimes R,U)=0$ for $p>1$. By (i), we have $H_1^{\mathrm{I}}(B)\cong \mathrm{Ker}\,\sigma$, $H_0^{\mathrm{I}}(B)\cong \mathrm{Cok}\,\sigma$. So

$$\begin{aligned} H_1^{\mathrm{I}}(B_{\cdot,q}) &\cong R\otimes E_q\otimes R && \text{if } q>0\\ &\cong \mathrm{Ker}(R\otimes E_0\otimes R \xrightarrow{\ \sigma\ } R\otimes R) && \text{if } q=0\\ H_0^{\mathrm{I}}(B_{\cdot,q}) &= 0 && \text{if } q>0. \end{aligned}$$

Hence $H_q^{\mathrm{II}}H_1^{\mathrm{I}}(B)$ is the homology at $R\otimes E_q\otimes R$ of the complex in the statement and the spectral sequence degenerates to give $H_q^{\mathrm{II}}H_1^{\mathrm{I}}(B)\cong H_{q+1}(\mathrm{tot}\,B)$ for q ≥ 0.

(iii) If $\varepsilon:U\to R$ is a resolution, then $H_n(\mathrm{tot}\,B)=0$ for $n>0$.

PROOF. By Künneth we have

$$H_q^{\mathrm{II}}(B_{p,\cdot})=H_q(R\otimes U^{\otimes p}\otimes R)\cong\begin{cases} R^{\otimes(p+2)} & \text{if } q=0\\ 0 & \text{if } q>0\end{cases}$$

and the complex

$$\cdots\to H_0^{\mathrm{II}}(B_{p,\cdot})\to\cdots\to H_0^{\mathrm{II}}(B_{0,\cdot})$$

is isomorphic to the standard free resolution of the R-bimodule R, which is acyclic. Hence $H_p^{\mathrm{I}}H_q^{\mathrm{II}}(B)=0$ unless $p=q=0$. Then $H_n(\mathrm{tot}\,B)=0$ for $n>0$.

Now Proposition 1.2 follows from (ii) and (iii).

3. Case where *R* is graded

In this section we state a converse of Proposition 1.2 under certain assumptions. As before, let R be a k-algebra, $U=(T(E),d)$ a free differential graded algebra with augmentation $\varepsilon:U\to R$. Here we further assume that

- R is a connected graded algebra, that is, $R=\oplus_{m\geq 0}R^m$ with $R^0=k$.
- E has another grading $E=\oplus_{m\geq 0}E^m$ compatible with the original one, that is, $E_n=\oplus_{m\geq 0}E_n^m$ with $E_n^m=E^m\cap E_n$.
- $E_0^0=0$.

Then the upper grading of E induces the grading $U=\oplus_{m\geq 0}U^m$ so that U is a doubly graded algebra. The third condition means that the graded algebra U_0 is connected. We finally assume

- $d(U^m)\subset U^m, \varepsilon(U^m)\subset R^m$.

REMARK. For any connected graded algebra R, one can find a resolution $\varepsilon: U \to R$ satisfying the above conditions. This is easily seen from the construction reviewed in Section 1.

THEOREM 3. *The following are equivalent.*

(1) $\varepsilon: U \to R$ *is a resolution.*

(2) *The complex*

$$\cdots \xrightarrow{\partial} R \otimes E_n \otimes R \xrightarrow{\partial} \cdots \xrightarrow{\partial} R \otimes E_0 \otimes R \xrightarrow{\sigma} R \otimes R \xrightarrow{\text{mult}} R \to 0$$

is exact.

(3) *The complex*

$$\cdots \xrightarrow{\bar{\partial}} E_n \otimes R \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} E_0 \otimes R \xrightarrow{\bar{\sigma}} R \xrightarrow{\eta} k \to 0$$

is exact, where $\bar{\partial} = k \otimes_R \partial$, $\bar{\sigma} = k \otimes_R \sigma$ *and* η *is the projection.*

(2) $\Rightarrow$ (3) is obvious, and (3) $\Rightarrow$ (2) follows from a version of Nakayama's lemma. We shall prove (3) $\Rightarrow$ (1) in the next section.

The map $\bar{\partial}$ on E is given also as $E \xrightarrow{d} E \otimes U \xrightarrow{1 \otimes \varepsilon} E \otimes R$, and $\bar{\sigma} = -\varepsilon$ on E (see (v) of the next section).

4. Proof of Theorem 3

Let $\mathfrak{N}$ be the set of finite sequences $\nu = (\nu_1, \cdots, \nu_r)$ of non-negative integers. Write $|\nu| = \nu_1 + \cdots + \nu_r$, $l(\nu) = r$. For $\mu = (\mu_1, \cdots, \mu_q)$ and $\nu = (\nu_1, \cdots, \nu_r)$, define

$$\mu\nu = (\mu_1, \cdots, \mu_q, \nu_1, \cdots, \nu_r).$$

We write also $0^k = (0, \cdots, 0), 0^k 1 = (0, \cdots, 0, 1)$ (k is the number of 0).

Define a partial order $<$ on $\mathfrak{N}$ as follows. For μ and ν as above we set $\mu < \nu$ if $|\mu| = |\nu|$ and $\mu_1 = \nu_1, \cdots, \mu_{h-1} = \nu_{h-1}, \mu_h < \nu_h$ for some $h \leq q, r$. Note that $\mu 0^i < \nu 0^j$ if and only if $\mu < \nu$. Let $\mathfrak{N}' = \{(\nu_1, \cdots, \nu_r) \in \mathfrak{N} \mid r > 0, \nu_r > 0\}$. Then $<$ is a total order on the subset $\{\nu \in \mathfrak{N}' \mid |\nu| = n\}$ for each $n > 0$.

For $\nu = (\nu_1, \cdots, \nu_r) \in \mathfrak{N}'$, define $\nu_- = (\nu_1, \cdots, \nu_{r-1}, \nu_r - 1)$.

For $\nu \in \mathfrak{N}$, we set $E_\nu = E_{\nu_1} \otimes \cdots \otimes E_{\nu_r}$. Then

$$U = T(E) = \bigoplus_{\nu \in \mathfrak{N}} E_\nu, \ U_0 = T(E_0) = \bigoplus_{k \geq 0} E_{0^k} \ U_+ = \bigoplus_{\nu \in \mathfrak{N}'} E_\nu U_0.$$

For $\nu \in \mathfrak{N}$, let

$$\text{pr}_\nu : U \to E_\nu, \ \pi_\nu : U \to \bigoplus_{k \geq 0} E_{\nu 0^k} = E_\nu U_0$$

be the projections with respect to the above decomposition for U.

(i) $d(E) \subset EU$.

PROOF. Clearly $d(E_n) \subset U_{n-1} \subset EU$ if n > 1. And

$$d(E_1) \subset \mathrm{Ker}(\varepsilon : U_0 \to R) \subset U_0^+ = E_0 U_0$$

by the assumption $U_0^0 = k$.

We define a map $d_\star : U_+ \to U$ as follows. First, $d_\star$ on E_+ is the composite

$$E_+ \xrightarrow{d} E \otimes U \xrightarrow{1 \otimes \pi_\varnothing} E \otimes U_0,$$

where $\pi_\varnothing$ is the projection onto U_0. As $U_+ = U \otimes E_+ \otimes U_0$, we can then define $d_\star$ on U_+ by

$$d_\star(xyz) = (-1)^p x d_\star(y) z$$

for $x \in U_p, y \in E_+, z \in U_0$. Clearly $d_\star$ is right U_0-linear and left skew U-linear. Also $d_\star(E_\nu U_0) \subset E_{\nu\_} U_0$ for $\nu \in \mathfrak{A}'$.

(ii) For $x \in E_\nu U_0$ with $\nu \in \mathfrak{A}'$ we have

$$d(x) - d_\star(x) \in \bigoplus_{\substack{\mu \in \mathfrak{A}' \\ \mu < \nu_-}} E_\mu U_0$$

and in particular $d_\star(x) = \pi_{\nu_-} d(x)$.

PROOF. Let $\nu = (\nu_1, \cdots, \nu_r)$. As d and $d_\star$ are right U_0-linear, we may assume $x = x_1 \cdots x_r$ with $x_1 \in E_{\nu_1}, \cdots, x_r \in E_{\nu_r}$. Then

$$\begin{aligned} d(x) &= \sum_{i=1}^{r} \pm x_1 \cdots x_{i-1} d(x_i) x_{i+1} \cdots x_r \\ &= \sum_{i=1}^{r} \sum_{\lambda} \pm x_1 \cdots x_{i-1} \mathrm{pr}_\lambda d(x_i) x_{i+1} \cdots x_r \end{aligned}$$

where λ runs over elements of $\mathfrak{A}$ such that $|\lambda| = \nu_i - 1, l(\lambda) \geq 1$ (by (i)). If $i < r$ or if $i = r$ and $\lambda_1 < \nu_r - 1$, then

$$(\nu_1, \cdots, \nu_{i-1}) \lambda (\nu_{i+1}, \cdots, \nu_r) < \nu_-.$$

If $i = r$ and $\lambda_1 = \nu_r - 1$, then $\lambda = (\nu_r - 1, 0, \cdots, 0)$. The sum of the terms for such i, λ is

$$\pm x_1 \cdots x_{r-1} \sum_{k \geq 0} \mathrm{pr}_{(\nu_r - 1)0^k} d(x_r) = d_\star(x).$$

Thus $d(x) - d_\star(x) \in \oplus_{\mu < \nu_-} E_\mu$.

(iii) $\mathrm{pr}_{\lambda\mu} d(xy) = (-1)^{|\lambda|} x \, \mathrm{pr}_\mu d(y)$ for $\lambda, \mu \in \mathfrak{A}, x \in E_\lambda, y \in U$.

PROOF. Expanding $d(x)$ as above, we see $\mathrm{pr}_{\lambda\mu}(d(x)y)=0$.

(iv) Let $\mu\in\mathfrak{A}$, $\nu\in\mathfrak{A}'$, $\mu<\nu$. If $\pi_{\nu\_}d(E_\mu)\neq 0$, then $\mu=\nu\_0^i10^j$ for some $i,j\geq 0$.

PROOF. We have $\mathrm{pr}_\lambda d(E_\mu)\neq 0$ for some $\lambda=\nu\_0^k, k\geq 0$. Put $r=l(\nu), q=l(\mu)$, $p=l(\lambda)=r+k$. Then $q\leq p$ by (i), and there exists $h\leq q$ such that

$$\begin{aligned}&\mu_1=\lambda_1,\cdots,\mu_{h-1}=\lambda_{h-1},\\&\mu_h=\lambda_h+\cdots+\lambda_{h+p-q}+1,\\&\mu_{h+1}=\lambda_{h+p-q+1},\cdots,\mu_q=\lambda_p.\end{aligned}$$

If $h<r$, then $\mu>\nu$, a contradiction. If $h=r$, then $\mu=\nu 0^{q-r}$, which is also impossible. Hence $h>r$ and $\mu=\nu\_0^i10^j$ for some i,j.

Let $\eta:R\to k$ be the projection map. $E\otimes R\cong k\otimes_R(R\otimes E\otimes R)$ becomes a complex with differential $\bar\partial=k\otimes_R\partial$ and augmentation $\bar\sigma=k\otimes_R\sigma:E\otimes R\to R$.

(v) We have

$$\bar\partial|E:E\xrightarrow{d}E\otimes U\xrightarrow{1\otimes\varepsilon}E\otimes R,\quad \bar\sigma|E=-\varepsilon.$$

PROOF. By the definition of ρ and the fact $\eta\varepsilon(E)=0$, we have a commutative diagram

$$\begin{array}{ccc}U&\xrightarrow{p|U}&R\otimes E\otimes R\\ \cup&&\downarrow{\scriptstyle \eta\otimes1\otimes1}\\ E\otimes U&\xrightarrow{1\otimes\varepsilon}&E\otimes R.\end{array}$$

From this and (i), the first assertion follows. The second is clear.

(vi) The following diagram is commutative.

$$\begin{array}{ccc}E_n\otimes U_0&\xrightarrow{1\otimes\varepsilon}&E_n\otimes R\\ {\scriptstyle d_*}\downarrow&&\downarrow{\scriptstyle\bar\partial}\\ E_{n-1}\otimes U_0&\xrightarrow[1\otimes\varepsilon]{}&E_{n-1}\otimes R.\end{array}$$

PROOF. Follows from (v) and the definition of d_*.

From now on we assume (3) of Theorem 3.

(vii) $\varepsilon:U\to R$ is onto.

PROOF. The exactness of $E_0\otimes R\xrightarrow{\bar\sigma}R\xrightarrow{\eta}k$ implies $R^+=\varepsilon(E_0)R$. So R is generated by $\varepsilon(E_0)$ as a k-algebra.

(viii) $U_1\xrightarrow{d}U_0\xrightarrow{\varepsilon}R$ is exact.

PROOF. By (vi) we have a commutative diagram

$$\begin{array}{ccccc} U_1 & \supset & E_1 \otimes U_0 & \xrightarrow{1\otimes\varepsilon} & E_1 \otimes R \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d_\star} & & \downarrow{\scriptstyle \bar\partial} \\ U_0 & \supset & E_0 \otimes U_0 & \xrightarrow{1\otimes\varepsilon} & E_0 \otimes R \\ {\scriptstyle \varepsilon}\downarrow & & & & \downarrow{\scriptstyle -\bar\sigma} \\ R & & = & & R. \end{array}$$

Consider the ideal $I = \mathrm{Ker}(\varepsilon : U_0 \to R) \subset U_0^+ = E_0 \otimes U_0$. By the exactness of the right column of the diagram, we have $I \subset d_\star(E_1 \otimes U_0) + \mathrm{Ker}(1 \otimes \varepsilon)$, hence $I \subset d(U_1) + E_0 I$. Since $d(U_1)$ is an ideal of U_0, we have $I = d(U_1)$ by Nakayama's lemma.

(ix) $U_{n+1} \xrightarrow{d} U_n \xrightarrow{d} U_{n-1}$ is exact for $n > 0$.

PROOF. Fix $p \geq 0$. Let us show the exactness in upper degree p. Firstly we note that the set $\{\nu \in \mathfrak{A}' | \pi_\nu(U_n^p) \neq 0\}$ is finite. Indeed, such ν must satisfy $|\nu| = n$ and $\#\{i | \nu_i = 0\} \leq p$ because $E_0^0 = 0$. So $l(\nu) \leq n + p$.

For $0 \neq x \in U_n^p$, let $H(x)$ be the greatest element of $\{\nu \in \mathfrak{A}' | \pi_\nu(x) \neq 0\}$ with respect to the order <. We shall show that if $0 \neq x \in \mathrm{Ker}(d : U_n^p \to U_{n-1}^p)$, then there exists $y \in U_{n+1}^p$ such that $H(x - d(y)) < H(x)$ or $x - d(y) = 0$. Then the exactness will follow by induction.

Put $H(x) = \nu = (\nu_1, \cdots, \nu_r)$ and $\nu_r = m > 0, \nu' = (\nu_1, \cdots, \nu_{r-1})$. Also put $x_\mu = \pi_\mu(x)$ for $\mu \in \mathfrak{A}'$. Then $x = x_\nu + \sum_{\mu<\nu} x_\mu$. We have

$$0 = \pi_{\nu\_} d(x) = \pi_{\nu\_} d(x_\nu) + \sum_{\mu<\nu} \pi_{\nu\_} d(x_\mu).$$

By (ii), $\pi_{\nu\_} d(x_\nu) = d_\star(x_\nu)$. If $\mu < \nu$ and $\pi_{\nu\_} d(x_\mu) \neq 0$, then, by (iv), $\dot\mu 0^k = \nu\_0^i 1 0^j$ for some i, j, k. Hence $\mu = \nu\_0^i 1$ as $\mu \in \mathfrak{A}'$. So $x_\mu \in E_{\nu\_} \otimes U_1$. Then, by (iii), $\pi_{\nu\_} d(x_\mu) \in E_{\nu\_} \otimes d(U_1)$. Thus we know $d_\star(x_\nu) \in E_{\nu\_} \otimes d(U_1)$. Hence $(1 \otimes \varepsilon) d_\star(x_\nu) = 0$ $(*)$.

By (vi), the diagram

$$\begin{array}{ccc} E_{\nu'} \otimes E_{m+1} \otimes U_0 & \xrightarrow{1\otimes 1\otimes\varepsilon} & E_{\nu'} \otimes E_{m+1} \otimes R \\ {\scriptstyle d_\star}\downarrow & & \downarrow{\scriptstyle 1\otimes\bar\partial} \\ E_{\nu'} \otimes E_m \otimes U_0 & \xrightarrow{1\otimes 1\otimes\varepsilon} & E_{\nu'} \otimes E_m \otimes R \\ {\scriptstyle d_\star}\downarrow & & \downarrow{\scriptstyle 1\otimes\bar\partial} \\ E_{\nu'} \otimes E_{m-1} \otimes U_0 & \xrightarrow{1\otimes 1\otimes\varepsilon} & E_{\nu'} \otimes E_{m-1} \otimes R \end{array}$$

commutes up to sign. By $(*)$, (vii) and the exactness of the right column, there exists $z \in (E_{\nu'} \otimes E_{m+1} \otimes U_0)^p$ such that

$$x_\nu - d_*(z) \in \mathrm{Ker}(E_\nu \otimes U_0 \xrightarrow{1\otimes\varepsilon} E_\nu \otimes R) = E_\nu \otimes d(U_1).$$

Since d_* operates as $\pm 1 \otimes d$ on $E_\nu \otimes U_1$, we have $x_\nu - d_*(z) = d_*(u)$ for some $u \in (E_\nu \otimes U_1)^p$.

But by (ii),

$$d(z) - d_*(z) \in \bigoplus_{\mu<\nu} E_\mu U_0$$
$$d(u) - d_*(u) \in \bigoplus_{\mu<\nu} E_\mu U_0,$$

where $\mu \in \mathfrak{A}'$. Hence

$$x - d(z+u) = x_\nu - d(z+u) + \sum_{\mu<\nu} x_\mu \in \bigoplus_{\mu<\nu} E_\mu U_0$$

as required.

5. Examples of resolutions

We shall first give a free dga resolution of a Koszul algebra. Let R be a connected graded algebra generated by elements of degree 1 with defining relations of degree 2. So we can write as $R = T(V)/(I)$ where $I \subset V \otimes V$. Put

$$I^{(n)} = \bigcap_{i+j=n-2} V^{\otimes i} \otimes I \otimes V^{\otimes j} \subset V^{\otimes n}$$

for $n \geq 0$. We understand $I^{(0)} = k$, $I^{(1)} = V$. R is called a Koszul algebra if the following complex of right R-modules is exact.

$$\to I^{(n)} \otimes R \to I^{(n-1)} \otimes R \to \cdots \to R \to k \to 0.$$

Here the differential is induced by the inclusion maps

$$I^{(n)} \subset I^{(n-1)} \otimes V \subset I^{(n-1)} \otimes R.$$

For equivalent definitions of Koszul algebras, see [1], [4].

Let $\Delta_{p,q} : I^{(p+q)} \to I^{(p)} \otimes I^{(q)}$ be the inclusion map for $p, q > 0$. Let $E = \oplus_{n\geq 1} I^{(n)}$ with bigrading $E_n = E^{n+1} = I^{(n+1)}$. Put $U = T(E)$. Let $d : U \to U$ be the derivation such that

$$d(x) = \sum_{\substack{p+q=n+1 \\ p,q>0}} (-1)^p \Delta_{p,q}(x) \quad \text{for } x \in E_n.$$

Let $\varepsilon : U \to R$ be the algebra map such that $\varepsilon(x) = x$ for $x \in E_0 = V$ and $\varepsilon(x) = 0$ for $x \in E_+$.

PROPOSITION 5.1. *$\varepsilon:(U,d)\to R$ is a free dga resolution.*

PROOF. $d^2=0$ follows from the coassociativity of $\Delta_{p,q}$. $\varepsilon d=0$ is clear. Let $\bar{\partial}$ be as in Theorem 3. For $x\in E_n$ we have

$$\bar{\partial}(x)=(1\otimes\varepsilon)\sum_{p+q=n+1}(-1)^p\Delta_{p,q}(x)=(-1)^n\Delta_{n,1}(x).$$

Hence, up to sign, $\bar{\partial}$ coincides with the differential of the above free resolution of the R-module k. So $\varepsilon:U\to R$ is a resolution.

We next introduce a generalization of a Koszul algebra, for which we give a free dga resolution. Fix an integer $e\geq 2$. Let $R=T(V)/(I)$ with $I\subset V^{\otimes e}$. As before, define

$$I^{(n)}=\bigcap_{i+j=n-e}V^{\otimes i}\otimes I\otimes V^{\otimes j}\subset V^{\otimes n}.$$

Consider the complex of right R-modules

$$\to I^{(en+1)}\otimes R\to I^{(en)}\otimes R\to\cdots\to I^{(1)}\otimes R\to I^{(0)}\otimes R\to k\to 0$$

where the differential is induced by the inclusion maps

$$I^{(en+1)}\subset I^{(en)}\otimes V\subset I^{(en)}\otimes R$$
$$I^{(en)}\subset I^{(e(n-1)+1)}\otimes V^{\otimes(e-1)}\subset I^{(e(n-1)+1)}\otimes R.$$

We say R is an e-Koszul algebra if the above complex is exact.

REMARK. (i) 2-Koszul just means Koszul. (ii) $k[x]/(x^e)$ is e-Koszul. (iii) Let $J\subset V\otimes V$. If $T(V)/(J)$ is Koszul, then $T(V)/(J^{(e)})$ is e-Koszul. We omit the proof.

Let us give a free dga resolution of an e-Koszul algebra $R=T(V)/(I)$. Let

$$E=\bigoplus_{n\geq 0}I^{(en+1)}\oplus\bigoplus_{n\geq 1}I^{(en)}$$
$$E_{2n}=E^{en+1}=I^{(en+1)},\ E_{2n-1}=E^{en}=I^{(en)}.$$

Put $E_{\mathrm{ev}}=\oplus_{n\geq 0}E_{2n}$, $E_{\mathrm{od}}=\oplus_{n\geq 1}E_{2n-1}$. Let

$$\delta_{11}:E_{\mathrm{od}}\to E_{\mathrm{od}}\otimes E_{\mathrm{od}}$$
$$\delta_0:E_{\mathrm{od}}\to E_{\mathrm{ev}}^{\otimes e}$$
$$\delta_{10}:E_{\mathrm{ev}}\to E_{\mathrm{od}}\otimes E_{\mathrm{ev}}$$
$$\delta_{01}:E_{\mathrm{ev}}\to E_{\mathrm{ev}}\otimes E_{\mathrm{od}}$$

be the maps whose components are respectively the inclusion maps

$$I^{(e(i+j))} \longrightarrow I^{(ei)} \otimes I^{(ej)}$$
$$I^{(e(i_1+\cdots+i_c+1))} \longrightarrow I^{(ei_1+1)} \otimes \cdots \otimes I^{(ei_c+1)}$$
$$I^{(e(i+j)+1)} \longrightarrow I^{(ei)} \otimes I^{(ej+1)}$$
$$I^{(e(i+j)+1)} \longrightarrow I^{(ei+1)} \otimes I^{(ej)}.$$

Put $U = T(E)$. Let $d : U \to U$ be the derivation such that

$$E = \bigoplus_{n \geq 0} d = \begin{cases} \delta_0 - \delta_{11} & \text{on } E_{\text{od}} \\ \delta_{01} - \delta_{10} & \text{on } E_{\text{ev}}. \end{cases}$$

Let $\varepsilon : U \to R$ be the algebra map which is the identity on $E_0 = V$ and vanishes on E_+.

PROPOSITION 5.2. *$\varepsilon : (U,d) \to R$ is a free dga resolution.*

PROOF. Again $d^2 = 0$ is a consequence of the coassociativity of the maps $\delta_{11}, \delta_0, \delta_{10}, \delta_{01}$. Recall the description of $\bar{\partial}$ after Theorem 3. We have the equalities of maps

$$\begin{aligned}
&(E_{2n-1} \xrightarrow{\delta_0} (E \otimes U)_{2n-2} \xrightarrow{1 \otimes \varepsilon} E_{2n-2} \otimes R) \\
&= (E_{2n-1} \hookrightarrow E_{2n-2} \otimes E_0^{\otimes(e-1)} \hookrightarrow E_{2n-2} \otimes R), \\
&(E_{2n} \xrightarrow{\delta_{10}} (E \otimes E)_{2n-1} \xrightarrow{1 \otimes \varepsilon} E_{2n-1} \otimes R) \\
&= (E_{2n} \hookrightarrow E_{2n-1} \otimes E_0 \hookrightarrow E_{2n-1} \otimes R), \\
&(1 \otimes \varepsilon)\delta_{01} = 0, \\
&(1 \otimes \varepsilon)\delta_{11} = 0.
\end{aligned}$$

Hence $\bar{\partial}$ equals the differential of the free resolution of the R-module k up to sign. So by Theorem 3 $\varepsilon : U \to R$ is a resolution.

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