TSUKUBA J. MATH.
Vol. 20 No. 2 (1996), 387-397

SELF-DUAL CONNECTIONS OF HOMOGENEOUS PRINCIPAL BUNDLES OVER QUATERNIONIC KAEHLER SYMMETRIC SPACES

By

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Abstract. A characterization of self-dual connections of homogeneous principal bundles over quaternionic Kaehler symmetric spaces in terms of the holonomy homomorphism, is given using the group theoretic approach.

Introduction.

Recently Mamone Capria and Salamon [C.S], and Nitta [N] extended the notion of (anti-)self-dual connections in 4-manifolds to the higher dimensional quaternionic Kaehler manifold and showed they are Yang-Mills connections. On the other hand, compact quaternionic Kaehler symmetric manifolds were classified by Wolf [W], and they are quotients $M = U/K$ of a compact simple Lie group U by a closed subgroup K with the spliting $K = A_1 \cdot L_1$ where A_1 is isomorphic to $Sp(1) = SU(2) = S^3$. Therefore it would be interesting to determine the invariant (anti-)self-dual connections of a homogeneous principal bundle over a compact quaternionic Kaehler symmetric manifold.

In this paper, using the group theoretic approach, we show:

THEOREM 1.2. *Let $M = U/K$ be a Wolf space, i.e., a compact quaternionic Kaehler symmetric manifold. Let P be a U-homogeneous principal bundle over M with the structure group G, and let $\lambda = \lambda(P)$ be the corresponding holonomy homomorphism of K into G. For a U-invariant connection ω on P, which is always a Yang-Mills connection, we have*

(i) *ω is self-dual $\Leftrightarrow \lambda|L_1 \equiv$ trivial, and*

(ii) *ω is anti-self-dual $\Leftrightarrow \lambda|A_1 \equiv$ trivial.*

1980 Mathematics Subject Classification (1990 Revision) Primary 81 T 13, 53 C 35, 53 C 07
Received August 8, 1994
Revised December 26, 1994

This theorem can be regarded as a natural generalization of Theorem 2 in [I], which determined the invariant (anti-)self-dual connections of invariant bundles over S^4.

§1. Preliminary.

In this section, we prepare the notion of (anti-)self-dual connections over a quaternionic Kaehler manifold, following Capria and Salamon [C.S], and Nitta [N], and explain Wolf's results about compact quaternionic Kaehler symmetric spaces.

1.1. Quaternionic Kaehler Manifolds.

DEFINITION. A $4n$-dimensional Riemannian manifold (M,g) is said to be *quaternionic Kaehler* (cf.[B]) if its holonomy group is contained in the subgroup $Sp(n)\cdot Sp(1)=Sp(n)\times Sp(1)/\{(I,1),(-I,-1)\}$ of $SO(4n)$, which is equivalent to the following: There exist an open covering $\{U_i\}$ of M, and almost complex structures I, J, and K on U_i such that

(a) the Riemannian metric g is hermitian for I, J, K on U_i,

(b) $K=IJ=-JI$,

(c) the covariant derivatives of I, J, K with respect to g on U_i, are linear combinations of I, J, K, and

(d) for each $x\in U_i\cap U_j$, the vector subspaces of $\mathrm{End}(T_xM)$ generated by $\{I,J,K\}$ coincides with each other for each i,j.

1.2. (Anti-)Self-Dual Connections.

Following [C.S], [N], we define (anti-)self-dual connections of a principal bundle P having the structure group G with the Lie algebra $\boldsymbol{g}$ over a quaternionic Kaehler manifold M.

Let $\boldsymbol{H}$ be the field of quaternions, and $\boldsymbol{H}^n$ the right $\boldsymbol{H}$-module of n-tuples of quaternions. The group $Sp(n)\cdot Sp(1)$ acts on $\boldsymbol{H}^n$ as left multiplications of $Sp(n)$, and right multiplications by $Sp(1)$. The $Sp(n)\cdot Sp(1)$-module $\wedge^2\boldsymbol{H}^n$ has the following irreducible decomposition:

$$\wedge^2\boldsymbol{H}^n=A_2'\oplus A_2''\oplus B_2, \tag{1.1}$$

where A_2' is the submodule of $Sp(n)$-fixed vectors, and B_2 is the one of $Sp(1)$-fixed vectors. By the assumption that the holonomy group of M is included in $Sp(n)\cdot Sp(1)$, we get the global decomposition of $\wedge^2T^*M$:

$$(1.2) \qquad \wedge^2 T^*M = A_2' \oplus A_2'' \oplus B_2,$$

where A_2', A_2'' and B_2 mean also the subbundles of $\wedge^2 T^*M$ corresponding to the above submodules.

DEFINITION. A connection ω on P is said to be an A_2' (resp. A_2'', B_2)-*connection* if the curvature form Ω^ω, which is a $\boldsymbol{g}$-valued 2 form on P, is a section of $\varepsilon \otimes \pi^* A_2'$ (resp. $\varepsilon \otimes \pi^* A_2'', \varepsilon \otimes \pi^* B_2$), where $\varepsilon = P \times \boldsymbol{g}$ (the product bundle), and $\pi^* A_2', \pi^* A_2'', \pi^* B_2$ are the pullback subbundles of $\wedge^2 T^*M$ by the projection $\pi; P \to M$.

Nitta [N] and Mamone Capria – Salamon [C.S] showed all the A_2', A_2'', B_2-connections are Yang-Mills connections of P, and B_2 (resp. A_2')-connections are the natural extension of self-dual (resp. anti-self-dual) connections in the case $\dim(M) = 4$, to the quaternionic Kaehler manifolds.

For later use, we express locally these connections. Take a local orthonormal frame field $\{e_i; i = 1, \cdots, 4n\}$ such that

$$\nabla e_i = 0,$$

$$I e_{4k+1} = e_{4k+2}, J e_{4k+1} = e_{4k+3}, K e_{4k+1} = e_{4k+4}, k = 0, 1, \cdots, n-1,$$

and denote by $\{\omega_i; i = 1, \cdots, 4n\}$, the dual basis. The rank 3-bundle A_2', and the rank $n(2n+1)$-bundle B_2 have the following bases of local sections, I, J, K for A_2', which are given by

$$I := \sum_{k=0}^{n-1} (\omega_{4k+1} \wedge \omega_{4k+2} + \omega_{4k+3} \wedge \omega_{4k+4}),$$

$$J := \sum_{k=0}^{n-1} (\omega_{4k+1} \wedge \omega_{4k+3} + \omega_{4k+4} \wedge \omega_{4k+2}),$$

$$K := \sum_{k=0}^{n-1} (\omega_{4k+1} \wedge \omega_{4k+4} + \omega_{4k+2} \wedge \omega_{4k+3}),$$

and $I_s, J_s, K_s, 0 \leqq s \leqq n-1$, $D_{pq}, E_{pq}, F_{pq}, G_{pq}$, $0 \leqq p < q \leqq n-1$, for B_2 are given by

$$I_s = \omega_{4s+1} \wedge \omega_{4s+2} - \omega_{4s+3} \wedge \omega_{4s+4},$$

$$J_s = \omega_{4s+1} \wedge \omega_{4s+3} - \omega_{4s+4} \wedge \omega_{4s+2},$$

$$K_s = \omega_{4s+1} \wedge \omega_{4s+4} - \omega_{4s+2} \wedge \omega_{4s+3},$$

$$D_{pq} = \omega_{4p+1} \wedge \omega_{4q+1} + \omega_{4p+2} \wedge \omega_{4q+2} + \omega_{4p+3} \wedge \omega_{4q+3} + \omega_{4p+4} \wedge \omega_{4q+4},$$

$$E_{pq} = \omega_{4p+1} \wedge \omega_{4q+2} - \omega_{4p+2} \wedge \omega_{4q+1} - \omega_{4p+3} \wedge \omega_{4q+4} + \omega_{4p+4} \wedge \omega_{4q+3},$$

$$F_{pq}=\omega_{4p+1}\wedge\omega_{4q+3}+\omega_{4p+2}\wedge\omega_{4q+4}-\omega_{4p+3}\wedge\omega_{4q+1}-\omega_{4p+4}\wedge\omega_{4q+2},$$

$$G_{pq}=\omega_{4p+1}\wedge\omega_{4q+4}-\omega_{4p+2}\wedge\omega_{4q+3}+\omega_{4p+3}\wedge\omega_{4q+2}-\omega_{4p+4}\wedge\omega_{4q+1}.$$

Then a connection ω is A_2' if and only if the curvature form $\Omega=\Omega^{\omega}$ can be expressed locally as

$$\Omega=a\otimes\pi^*I+b\otimes\pi^*J+c\otimes\pi^*K,$$

where a, b, c are $\boldsymbol{g}$-valued functions locally defined on P, and so on for A_2'', B_2-connections. Summing up these, we get then the following criterion to be A_2', A_2'', B_2-connections:

PROPOSITION 1.1. *For a connection ω on a principal G-bundle P over a quaternionic Kaehler manifold M,*

(1) *it is A_2', i.e., anti-self-dual, if and only if the curvature form Ω satisfies locally,*

$$\Omega(e_{4k+1}^*,e_{4k+2}^*)=\Omega(e_{4k+3}^*,e_{4k+4}^*)=\Omega(e_1^*,e_2^*)=\Omega(e_3^*,e_4^*),$$

$$\Omega(e_{4k+1}^*,e_{4k+3}^*)=\Omega(e_{4k+4}^*,e_{4k+2}^*)=\Omega(e_1^*,e_3^*)=\Omega(e_4^*,e_2^*),$$

$$\Omega(e_{4k+1}^*,e_{4k+4}^*)=\Omega(e_{4k+2}^*,e_{4k+3}^*)=\Omega(e_1^*,e_4^*)=\Omega(e_2^*,e_3^*),$$

for all $k=0,1,\cdots,n-1$, and all the other components of Ω are zero.

(2) *ω is an A_2''-connection if and only if $\Omega(u^*)=0$ for all the horizontal lifts u^* of elements u in $\wedge^2 TM$ dual to the bases of A_2', B_2.*

(3) *ω is a B_2, i.e., self-dual, -connection if and only if*

$$\Omega(e_{4s+1}^*,e_{4s+2}^*)=\Omega(e_{4s+4}^*,e_{4s+3}^*),$$

$$\Omega(e_{4s+1}^*,e_{4s+3}^*)=\Omega(e_{4s+2}^*,e_{4s+4}^*),$$

$$\Omega(e_{4s+1}^*,e_{4s+4}^*)=\Omega(e_{4s+3}^*,e_{4s+2}^*),\quad 0\leqq s\leqq n-1,$$

$$\Omega(e_{4p+1}^*,e_{4q+1}^*)=\Omega(e_{4p+2}^*,e_{4q+2}^*)=\Omega(e_{4p+3}^*,e_{4q+3}^*)=\Omega(e_{4p+4}^*,e_{4q+4}^*),$$

$$\Omega(e_{4p+1}^*,e_{4q+2}^*)=-\Omega(e_{4p+2}^*,e_{4q+1}^*)=-\Omega(e_{4p+3}^*,e_{4q+4}^*)=\Omega(e_{4p+4}^*,e_{4q+3}^*),$$

$$\Omega(e_{4p+1}^*,e_{4q+3}^*)=\Omega(e_{4p+2}^*,e_{4q+4}^*)=-\Omega(e_{4p+3}^*,e_{4q+1}^*)=-\Omega(e_{4p+4}^*,e_{4q+2}^*),$$

$$\Omega(e_{4p+1}^*,e_{4q+4}^*)=-\Omega(e_{4p+2}^*,e_{4q+3}^*)=\Omega(e_{4p+3}^*,e_{4q+2}^*)=-\Omega(e_{4p+4}^*,e_{4q+1}^*),$$

$$0\leqq p<q\leqq n-1,$$

and all the other components are zero. Here, for a vector field X on M, we *denote by* X^*, *the horizontal lift to* P.

1.3. Wolf's quaternionic Kaehler symmetric spaces.

A compact simply connected quaternionic Kaehler Riemannian symmetric space M, classified by Wolf [W], is the coset space $M = U/K$ of a compact simple Lie group U with trivial center by a closed subgroup K with $K = A_1 \cdot L_1$, where A_1, L_1 are closed subgroups of K given as follows: Taking a maximal torus T of K, let Δ be the root system of the complexification $\boldsymbol{u}^C$ of the Lie algebra $\boldsymbol{u}$ of U with respect to the Lie algebra $\boldsymbol{t}$, i.e., for $\alpha \in \Delta$,

$$[H, E_\alpha] = \sqrt{-1}\alpha(H)E_\alpha, \text{ for all } H \in \boldsymbol{t}.$$

Here $E_\alpha \in \boldsymbol{u}^C$ satisfies $\tau E_\alpha = E_{-\alpha}, B(E_\alpha, E_{-\alpha}) = -1$, where τ is the conjugation of $\boldsymbol{u}^C$ with respect to $\boldsymbol{u}$ and B is the Killing form of $\boldsymbol{u}^C$. For $\lambda, \mu \in \Delta, (\lambda, \mu) = -B(H_\lambda, H_\mu)$, and $H_\lambda \in \boldsymbol{t}$ is given by $-B(H_\lambda, H) = \lambda(H)$, for all $H \in \boldsymbol{t}$. Put $U_\alpha = E_\alpha + E_{-\alpha}, V_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha}), \alpha \in \Delta$. Let β be the highest root of Δ under some order. Define the Lie subalgebras $\boldsymbol{a}_1, \boldsymbol{l}_1$ of $\boldsymbol{u}$ by

$$\boldsymbol{a}_1 := \{H_\beta, U_\beta, V_\beta\}_{\boldsymbol{R}},$$

$$\boldsymbol{l}_1 := \{H \in \boldsymbol{t}; \beta(H) = 0\} \oplus \sum_{\alpha>0, (\alpha,\beta)=0} \{U_\alpha, V_\alpha\}_{\boldsymbol{R}},$$

and $\boldsymbol{k} := \boldsymbol{a}_1 \oplus \boldsymbol{l}_1$, and let A_1, L_1, K be the corresponding analytic subgroups of U. Denoting

$$\boldsymbol{p} := \sum_{\alpha>0, \alpha\neq\beta, (\alpha,\beta)\neq 0} \{U_\alpha, V_\alpha\}_{\boldsymbol{R}},$$

then $\boldsymbol{u} = \boldsymbol{k} \oplus \boldsymbol{p}$ is the Cartan decomposition of $\boldsymbol{u}$ and $M = U/K$ is a symmetric space. Moreover, the endomorphisms

$$I := \mathrm{Ad}(\exp\frac{1}{2}\pi X),\ J := \mathrm{Ad}(\exp\frac{1}{2}\pi Y),\ K := \mathrm{Ad}(\exp\frac{1}{2}\pi Z),$$

of $\boldsymbol{u}$ give the almost complex structures of $M = U/K$ around the origin, where

$$X = \frac{2}{(\beta,\beta)} H_\beta,\ Y = \frac{\sqrt{2}}{|\beta|} U_\beta,\ Z = \frac{\sqrt{2}}{|\beta|} V_\beta.$$

Namely,

$$I^2 = -\mathrm{Id},\ J^2 = -\mathrm{Id},\ K^2 = -\mathrm{Id},\ K = IJ, \text{ on } \boldsymbol{p}. \tag{1.1}$$

Note that, giving the inner product $(\cdot,\cdot) = -B(\cdot,\cdot)$ on $\boldsymbol{u}$,

$$(1.2)\qquad (X,X)=(Y,Y)=(Z,Z)=4/(\beta,\beta),$$
$$[X,Y]=2Z,\ (Y,Z)=2X,\ (Z,X)=2Y.$$

All compact simply connected quaternionic Kaehler symmetric spaces are exhausted by these coset spaces $M = U/K$.

Let P be a U-homogeneous principal G-bundle over M, and $\lambda=\lambda(P)$, the holonomy homomorphism of K into G at a point u_0 in P with $\pi(u_0)=o=\{K\}\in U/K$, i.e., for all $k\in K$, $ku_0=u_0\lambda(k)$ for some $\lambda(k)\in G$. We denote by the same letter its Lie algebra homomorphism. Then we obtain:

THEOREM 1.2. *Let $M = U/K$ be a Wolf space, i.e., a compact simply connected quaternionic Kaehler symmetric space. Let P be a U-homogeneous principal G-bundle over M, and $\lambda=\lambda(P)$, the holonomy homomorphism of K into G. For a U-invariant connection ω on P,*

(1) *ω is an A_2'-connection $\Leftrightarrow \lambda\equiv 0$ on $\boldsymbol{l}_1$,*

(2) *ω is an A_2''-connection $\Leftrightarrow \lambda\equiv 0$, in this case, P is trivial and ω is flat,*

(3) *ω is a B_2-connection $\Leftrightarrow \lambda\equiv 0$ on $\boldsymbol{a}_1$.*

For the proof, let us recall the facts (cf. [K.N, Chapter X]) about the invariant connections on homogeneous principal bundles. Note that a U-invariant connection ω must be *canonical*, i.e., the linear map Λ of $\boldsymbol{u}$ into $\boldsymbol{g}$ corresponding to ω satisfies $\Lambda\equiv 0$, on $\boldsymbol{p}$, because $M = U/K$ is symmetric (c.f. [K.N,] Theorem 3.1, p.230]). Then the curvature form Ω_{u_0} at the fixed point u_0 is given by

$$(1.3)\qquad 2\Omega_{u_0}(X^{\sim},Y^{\sim})=-\lambda([X,Y]),X,Y\in\boldsymbol{p}.$$

Here $X^{\sim}, X\in\boldsymbol{u}$, is a vector field on P defined by

$$X_p^{\sim}:=\frac{d}{dt}\Big|_{t=0}\exp(tX)p,\ p\in P.$$

Note that $X^{\sim}{}_{u_0}, X\in\boldsymbol{p}$, is the horizontal lift of the tangent vector $X_o\in T_oM$ since $\omega_{u_0}(X^{\sim})=\Lambda(X)=0$.

Using these facts, we will prove Theorem 1.2 in the section three.

§2. Some Lemmas about Wolf Spaces.

Before going into the proof of Theorem 1.2, we prepare some lemmas about the fine structure of quaternionic Kaehler symmetric spaces. We retain the notations in §1, in particular 1.3. We obtain then immediately:

LEMMA 2.1. *The endomorphisms I, J, K of the Lie algebra* $\boldsymbol{u}$ *satisfy the following:*

(I) $$I(H_\beta)=J(H_\beta)=K(H_\beta)=-H_\beta,\ I(H)=J(H)=K(H)=H,$$

for all $H\in\boldsymbol{t}$ *with* $\beta(H)=0$, *in particular,* $I^2=J^2=K^2=\mathrm{Id}$, *on* $\boldsymbol{t}$.

(II) *We put, for* $\alpha\in\Delta,\alpha>0$ *with* $\alpha\neq\beta$, *and* $(\alpha,\beta)\neq 0$,

$$J(\alpha)(H)=\alpha(JH),\ K(\alpha)(H)=\alpha(KH), H\in\boldsymbol{t}.$$

Then (1) $J(\alpha),K(\alpha)\in\Delta$, (2) $-J(\alpha)>0$, (3) $\alpha=-J(\alpha)$, (4) $\{\alpha\in\Delta;\alpha>0, \alpha\neq\beta, (\alpha,\beta)\neq 0\}=\{\alpha_1,\cdots,\alpha_n\}\cup\{-J(\alpha_1),\cdots,-J(\alpha_n)\}$ *(a disjoint union) for some mutually distinct* $\alpha_1,\cdots,\alpha_n$ *with* $4n=\dim(M)$. (5) $IE_\alpha=iE_\alpha, IE_{-\alpha}=-iE_{-\alpha}$, *in particular,* $IU_\alpha=V_\alpha, IV_\alpha=-U_\alpha$. (6) $JE_\alpha=c_\alpha E_{J(\alpha)}, JE_{-\alpha}=c_{-\alpha}E_{-J(\alpha)}, KE_\alpha=d_\alpha E_{K(\alpha)}, KE_{-\alpha}=d_{-\alpha}E_{-K(\alpha)}$. *Here the complex numbers* c_α, d_α*'s satisfy the relations:*

$$c_\alpha c_{-\alpha}=1,\ c_\alpha=c_{-\alpha},\ c_{J(\alpha)}=-c_{-\alpha},\ d_\alpha=-\mathrm{i}c_\alpha,\ d_{-\alpha}=\mathrm{i}c_{-\alpha}.$$

Moreover, we get:

LEMMA 2.2. (1) *For* $W\in\boldsymbol{k}$,

$$IW=W\Leftrightarrow W\in\boldsymbol{R}X\oplus\boldsymbol{l}_1,$$

$$JW=W\Leftrightarrow W\in\boldsymbol{R}Y\oplus\boldsymbol{l}_1 \text{ and}$$

$$KW=W\Leftrightarrow W\in\boldsymbol{R}Z\oplus\boldsymbol{l}_1.$$

(2) $IY=-Y, IZ=-Z;\ JX=-X, JZ=-Z;\ KX=-X, KY=-Y.$

Due to [Theorem 4.2, W], we get:

LEMMA 2.3.
$$\begin{aligned}&\{H\in\boldsymbol{t};\beta(H)=0\}\\&=\{H_\alpha-H_{\alpha'};\alpha,\alpha'\in\Delta,\alpha,\alpha'>0,(\alpha,\beta)>0,(\alpha',\beta)>0\}_{\boldsymbol{R}}\\&=\{H_\gamma\in\boldsymbol{t};\alpha\in\Delta,(\alpha,\beta)>0,\alpha=\frac{1}{2}\beta+\gamma,(\beta,\gamma)=0\}_{\boldsymbol{R}}.\end{aligned}$$

LEMMA 2.4. *According to the decomposition* $\boldsymbol{k}=\boldsymbol{a}_1+\boldsymbol{l}_1$, *we denote by* $X_{\boldsymbol{a}_1}$, *the* $\boldsymbol{a}_1$*-component, for* $X\in\boldsymbol{k}$. *Then for* $\alpha\in\Delta$, $\alpha>0$ *with* $\alpha\neq\beta$ *and* $(\alpha,\beta)\neq 0$,

$$[U_\alpha,JU_\alpha]_{\boldsymbol{a}_1}=(\sqrt{2})^{-1}|\beta|U_\beta, \text{ and } [U_\alpha,KU_\alpha]_{\boldsymbol{a}_1}=(\sqrt{2})^{-1}|\beta|V_\beta.$$

PROOF OF LEMMA 2.4. Since $[U_\alpha,JU_\alpha]$ and $[U_\alpha,KU_\alpha]\in\boldsymbol{k}$, are fixed by J and K, we get that $[U_\alpha,JU_\alpha]_{\boldsymbol{a}_1}\in\boldsymbol{R}Y$, $[U_\alpha,KU_\alpha]_{\boldsymbol{a}_1}\in\boldsymbol{R}Z$, by Lemma 2.2. Then we only may show

(2.1) $$(JU_\alpha,[Y,U_\alpha])=2, \text{ and } (KU_\alpha,[Z,U_\alpha])=2.$$

By definition of J, and K, and $\mathrm{ad}(Y)$ is skew symmetric relative to (,), we have

$$(JU_\alpha,[Y,U_\alpha])=\sum_{n=0}^{\infty}\frac{(\pi/2)^{2n+1}}{(2n+1)!}(-1)^n\left|(\mathrm{ad}Y)^{n+1}(U_\alpha)\right|^2,$$

$$(KU_\alpha,[Z,U_\alpha])=\sum_{n=0}^{\infty}\frac{(\pi/2)^{2n+1}}{(2n+1)!}(-1)^n\left|(\mathrm{ad}Z)^{n+1}(U_\alpha)\right|^2.$$

Here we show:

(2.2) $$[Y,U_\alpha]=\pm U_{\beta-\alpha},\ [Y,U_{\beta-\alpha}]=\pm U_\alpha,$$
$$[Z,U_\alpha]=\pm V_{\beta-\alpha},\ [Z,V_{\beta-\alpha}]=\pm U_\alpha,$$

which imply (2.1) because of $|U_\alpha|^2=|U_{\beta-\alpha}|^2=|V_{\beta-\alpha}|^2=2$.

The proof of (2.2) goes as follows: Recall that we took the Weyl basis $\{E_\alpha\}_\alpha\in\Delta$, i.e; $[E_\alpha,E_{\alpha'}]=N_{\alpha,\alpha'}E_{\alpha+\alpha'}$, with a real number $N_{\alpha,\alpha'}$, satisfying $N_{\alpha,\alpha'}=N_{-\alpha,-\alpha'}$, and $-B(E_\alpha,E_{-\alpha})=1$. Then we get

$$N_{\beta,-\alpha}{}^2=N_{-\alpha,\beta}{}^2=N_{\alpha,-\beta}{}^2=|\beta|^2/2.$$

In fact, since β is highest, $\beta-(-\alpha)$ is not a root. Then we get $N_{-\alpha,\beta}{}^2=k(-\alpha,-\alpha)/2$, where k is the integer in such a way that $\beta+(-\alpha),\cdots,\beta+k(-\alpha)$ are roots, but $\beta+(k+1)(-\alpha)$ is not root. On the other hand, $-k(-\alpha,-\alpha)/2=(\beta,-\alpha)=-|\alpha|^2/2$ due to Theorem 4.2 in [W]. Thus $N_{-\alpha,\beta}{}^2=|\beta|^2/2$. We get therefore

$$[Y,U_\alpha]=\pm U_{\beta-\alpha},\ [Z,U_\alpha]=\pm V_{\beta-\alpha}.$$

In a similar way, since

$$N_{-\beta,\beta-\alpha}{}^2=N_{\beta,-(\beta-\alpha)}{}^2=N_{\beta,J(\alpha)}{}^2=N_{J(\alpha),\beta}{}^2=|\beta|^2/2,$$

we obtain the rest of the equalities of (2.2). Q.E.D.

§3. Proof of Theorem 1.2.

We take an orthonormal frame field $\{e_i;i=1,\cdots,4n\}$ on a neighborhood of the origin o in $M=U/K$ as

$$(e_{4k+1})_o=U_{\alpha_{k+1}},(e_{4k+2})_o=IU_{\alpha_{k+1}},(e_{4k+3})_o=JU_{\alpha_{k+1}},(e_{4k+4})_o=KU_{\alpha_{k+1}},$$

for $k=0,1,\cdots,n-1$, where $\{\alpha_1,\cdots,\alpha_n\}$ is as in Lemma 2.1, (II),(4). Then by Proposition 1.1, since our I, J, K preserve the Lie bracket [,] of $\boldsymbol{u}$, we have:

(1) ω is an A_2'-connection if and only if, for $k=1,\cdots,n$,

$$(3.1)\quad \begin{aligned} &\lambda([U_{\alpha_k},IU_{\alpha_k}])=-\lambda(J[U_{\alpha_k},IU_{\alpha_k}])=\lambda([U_{\alpha_1},IU_{\alpha_1}])=-\lambda(J[U_{\alpha_1},IU_{\alpha_1}]),\\ &\lambda([U_{\alpha_k},JU_{\alpha_k}])=-\lambda(K[U_{\alpha_k},JU_{\alpha_k}])=\lambda([U_{\alpha_1},JU_{\alpha_1}])=-\lambda(K[U_{\alpha_1},JU_{\alpha_1}]),\\ &\lambda([U_{\alpha_k},KU_{\alpha_k}])=-\lambda(I[U_{\alpha_k},KU_{\alpha_k}])=\lambda([U_{\alpha_1},KU_{\alpha_1}])=-\lambda(I[U_{\alpha_1},KU_{\alpha_1}]),\end{aligned}$$

(2) ω is an A_2''-connection if and only if

$$(3.2)\quad \begin{cases} \lambda(\sum_{k=1}^{n}\{[U_{\alpha_k},IU_{\alpha_k}]+[JU_{\alpha_k},KU_{\alpha_k}]\})=0,\\ \lambda(\sum_{k=1}^{n}\{[U_{\alpha_k},JU_{\alpha_k}]+[KU_{\alpha_k},IU_{\alpha_k}]\})=0,\\ \lambda(\sum_{k=1}^{n}\{[U_{\alpha_k},KU_{\alpha_k}]+[IU_{\alpha_k},JU_{\alpha_k}]\})=0,\\ \lambda([U_{\alpha_s},IU_{\alpha_s}]=\lambda([JU_{\alpha_s},KU_{\alpha_s}]),\\ \lambda([U_{\alpha_s},JU_{\alpha_s}]=\lambda([KU_{\alpha_s},IU_{\alpha_s}]),\\ \lambda([U_{\alpha_s},KU_{\alpha_s}]=\lambda([IU_{\alpha_s},JU_{\alpha_s}]),\quad s=1,\cdots,n,\\ \lambda([U_{\alpha_p},U_{\alpha_q}]+[IU_{\alpha_p},IU_{\alpha_q}]+[JU_{\alpha_p},JU_{\alpha_q}]+[KU_{\alpha_p},KU_{\alpha_q}])=0,\\ \lambda([U_{\alpha_p},IU_{\alpha_q}]-[IU_{\alpha_p},U_{\alpha_q}]-[JU_{\alpha_p},KU_{\alpha_q}]+[KU_{\alpha_p},JU_{\alpha_q}])=0,\\ \lambda([U_{\alpha_p},JU_{\alpha_q}]+[IU_{\alpha_p},KU_{\alpha_q}]-[JU_{\alpha_p},U_{\alpha_q}]-[KU_{\alpha_p},IU_{\alpha_q}])=0,\\ \lambda([U_{\alpha_p},KU_{\alpha_q}]-[IU_{\alpha_p},JU_{\alpha_q}]+[JU_{\alpha_p},IU_{\alpha_q}]-[KU_{\alpha_p},U_{\alpha_q}])=0,\\ \qquad\qquad 1\leqq p<q\leqq n. \end{cases}$$

(3) ω is a B_2-connection if and only if

$$(3.3)\quad \begin{cases} \lambda([U_{\alpha_s},IU_{\alpha_s}])=\lambda(K[U_{\alpha_s},IU_{\alpha_s}]),\\ \lambda([U_{\alpha_s},JU_{\alpha_s}])=\lambda(I[U_{\alpha_s},JU_{\alpha_s}]),\\ \lambda([U_{\alpha_s},KU_{\alpha_s}])=\lambda(J[U_{\alpha_s},KU_{\alpha_s}]),\quad s=1,\cdots,n,\\ \lambda([U_{\alpha_p},KU_{\alpha_q}])=\lambda(I[U_{\alpha_p},KU_{\alpha_q}])=\lambda(J[U_{\alpha_p},U_{\alpha_q}])=\lambda(K[U_{\alpha_p},U_{\alpha_q}]),\\ \lambda([U_{\alpha_p},IU_{\alpha_q}])=\lambda(I[U_{\alpha_p},IU_{\alpha_q}])=\lambda(J[U_{\alpha_p},IU_{\alpha_q}])=\lambda(K[U_{\alpha_p},IU_{\alpha_q}]),\\ \lambda([U_{\alpha_p},JU_{\alpha_q}])=\lambda(I[U_{\alpha_p},JU_{\alpha_q}])=\lambda(J[U_{\alpha_p},JU_{\alpha_q}])=\lambda(K[U_{\alpha_p},JU_{\alpha_q}]),\\ \lambda([U_{\alpha_p},KU_{\alpha_q}])=\lambda(I[U_{\alpha_p},KU_{\alpha_q}])=\lambda(J[U_{\alpha_p},KU_{\alpha_q}])=\lambda(K[U_{\alpha_p},KU_{\alpha_q}]),\\ \qquad\qquad 1\leqq p<q\leqq n. \end{cases}$$

Now we will prove Theorem 1.2.

Case: A_2'-connection. If ω is an A_2'-connection, then by (3.1),

$$\lambda([U_{\alpha_k}, IU_{\alpha_k}]) = \lambda([U_{\alpha_1}, IU_{\alpha_1}]),\ k=1,\cdots,n.$$

By Lemma 2.1, it follows that $\lambda(H_{\alpha_k} - H_{\alpha_1}) = 0,\ k=1,\cdots,n$, which implies $\lambda \equiv 0$ on $\{H \in \boldsymbol{t}; \beta(H)=0\}$ by Lemma 2.3. Since λ is a homomorphism, $\lambda \equiv 0$ on $\boldsymbol{l}_1$. Conversely, assume that $\lambda \equiv 0$ on $\boldsymbol{l}_1$. Then by Lemma 2.4, we get,

$$\lambda([U_{\alpha_k}, IU_{\alpha_k}]) = -\lambda(J[U_{\alpha_k}, IU_{\alpha_k}]),$$

$$\lambda([U_{\alpha_k}, JU_{\alpha_k}]) = -\lambda(K[U_{\alpha_k}, JU_{\alpha_k}]),$$

$$\lambda([U_{\alpha_k}, IU_{\alpha_k}]) = -\lambda(J[U_{\alpha_k}, IU_{\alpha_k}]),$$

$$\lambda([U_{\alpha_k}, KU_{\alpha_k}]) = -\lambda(I[U_{\alpha_k}, KU_{\alpha_k}]),\ \text{and}$$

$$\lambda([U_{\alpha_k}, JU_{\alpha_k}]) = \lambda([U_{\alpha_1}, JU_{\alpha_1}]), \lambda([U_{\alpha_k}, KU_{\alpha_k}]) = -\lambda([U_{\alpha_1}, KU_{\alpha_1}]),$$

for $k=1,\cdots,n$. By Lemma 2.3, we get

$$\lambda([U_{\alpha_k}, IU_{\alpha_k}]) = \lambda([U_{\alpha_1}, IU_{\alpha_1}]),$$

thus ω is an A_2'-connection.

Case: A_2''-connection. By Lemma 2.4,

$$\sum_{k=1}^{n} \{[U_{\alpha_k}, JU_{\alpha_k}] - K[U_{\alpha_k}, JU_{\alpha_k}]\} \equiv |\beta|^2\, nY, (\text{mod } \boldsymbol{a}_1),\ \text{and}$$

$$\sum_{k=1}^{n} \{[U_{\alpha_k}, KU_{\alpha_k}] - I[U_{\alpha_k}, KU_{\alpha_k}]\} \equiv |\beta|^2\, nZ, (\text{mod } \boldsymbol{a}_1).$$

However, by Lemma 2.2, these $\boldsymbol{l}_1$-components are zero. We get, therefore, that the first three equalities of (3.2) are equivalent to saying that $\lambda \equiv 0$ on $\boldsymbol{a}_1 = \{X, Y, Z\}_{\boldsymbol{R}}$. Note that the elements appearing in the rest belong to $\boldsymbol{l}_1$, and $\{[U_{\alpha_k}, IU_{\alpha_k}] + J[U_{\alpha_k}, IU_{\alpha_k}];\ k=1,\cdots,n\}_{\boldsymbol{R}} = \{H \in \boldsymbol{t}; \beta(H)=0\}$. Then we get that the rest of the equations hold if and only if $\lambda \equiv 0$ on $\boldsymbol{l}_1$. Thus we obtain the desired result (2).

Case: B_2-connection. Assume that $\lambda \equiv 0$ on $\boldsymbol{a}_1$. Then, by Lemma 2.2, all the elements $W - IW, W - JW, W - KW$, with $W \in \boldsymbol{k}$, belong to $\boldsymbol{a}_1$. Thus (3.1) holds. Conversely, since $IU_{\alpha_k} = V_{\alpha_k}$, the first equality of (3.1) implies that $\lambda(H_{\alpha_k}) = -\lambda(H_{-J(\alpha_k)})$, i.e., $\lambda(H_{\alpha_k} + H_{-J(\alpha_k)}) = 0,\ k=1,\cdots,n$, and then we get

$\lambda(H_\beta)=0$ i.e., $\lambda\equiv 0$ on $\boldsymbol{a}_1$, since λ is a homomorphism. Theorem 1.2 is proved.

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