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THE MODULI SPACE OF ANTI-SELF-DUAL CONNECTIONS OVER HERMITIAN SURFACES

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1 Introduction

Let (M, g) be a compact Hermitian surface with an orintation induced by the complex structure of M, and P a principal bundle over M with structure group $SU(n)$. Then a canonical representation ρ of $SU(n)$ induces a smooth complex vector bundle $E=P\times_{\rho}\mathbf{C}^{n}$. A necessary and sufficient condition for a $SU(n)$-connection D on E to be an anti-self-dual connection is that the curvature of D is a differential 2-form of type (1,1), and is orthogonal to the fundamental form Φ of (M, g). Hence, a holomorphic structure is induced on E and hence on $\mathrm{End}^{0}E$ (the subbundle of $\mathrm{End}E$ consisting of endomorphisms with trace 0) by an anti-self-dual connection D. Itoh ([4]) showed that the moduli space of anti-self-dual connections over Kähler surfaces is a complex manifold. We will extend this result over Kähler surfaces to over Hermitian surfaces, which are not necessarily Kählerian.

Let K_{M} be a canonical line bundle over M. We define $\tilde{H}_{D}=H_{D}^{0}(M;\mathcal{O}(\mathrm{End}^{0}E\otimes K_{M})$ as the space of holomorphic sections, where $\mathrm{End}^{0}E$ is endowed with the holomorphic structure induced from the irreducible anti-self-dual connection D. We denote by $\mathcal{M}$ the moduli space of irreducible anti-self-dual connections (the quotient space of irreducible anti-self-dual connections by the gauge transformation group $SU(E)$), and set $\mathcal{M}_{0}$ as follows: $\mathcal{M}_{0}=\{[D]\in\mathcal{M}\,\tilde{H}_{D}=(0)\}$. Then we obtain the following

THEOREM 1. *Let M be a compact Hermitian surface. If $\mathcal{M}_{0}$ is not empty, then $\mathcal{M}_{0}$ is a complex manifold.*

We can make H_{D} vanish under a certain condition. On a Hermitian manifold

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(M, g), $\mathrm{Scal}(g)$ denotes the scalar curvature of the Hermitian connection with respect to g. Then we have the following vanishing theorem.

PROPOSITION 1. Let (M, g) be a compact Hermitian surface with fundamental form Φ which satisfies $\partial\bar{\partial}\Phi = 0$. If $\int_M \mathrm{Scal}(g)\,dv \geq 0$, then $\tilde{H}_D = (0)$.

With this proposition, Theorem 1 implies the following.

THEOREM 2. *Let (M, g) be a compact Hermitian surface which satisfies the same condition as proposition 1. If $\mathcal{M}$ is not empty, then $\mathcal{M}$ is a complex manifold.*

2. Two moduli spaces

In this section we will recall the moduli spaces of anti-self-dual connections and holomorphic semi-connections following [1], [4], and [5].

Let (M, g) be a compact oriented Hermitian surface with fundamental form $\Phi = \sqrt{-1}\sum g_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$. We will denote by A^p(resp. $A^{p,q}$) the space of real valued smooth p-forms (resp. (p,q)-forms) on M. Then we have the decomposition of the space of 2-forms,

$$A^2 \otimes \mathbf{C} = A^{2,0} \oplus A^{1,1} \oplus A^{0,2}. \tag{2.1}$$

The fundamental form Φ decomposes $A^{1,1}$ further:

$$A^{1,1} = A_\Phi^{1,1} \oplus (A_\Phi^{1,1})^\perp, \tag{2.2}$$

where

$$A_\Phi^{1,1} = \{f\Phi : f \in C^\infty(M;\mathbf{C})\}, \tag{2.3}$$

and

$$(A_\Phi^{1,1})^\perp = \{\psi = \sum \psi_{\alpha\bar{\beta}} dz^\alpha \wedge \bar{z}^\beta : \sum g^{\alpha\bar{\beta}} \psi_{\alpha\bar{\beta}} = 0\}. \tag{2.4}$$

$(A_\Phi^{1,1})^\perp$ is the space of all primitive (1,1)-forms in (M, g). We put

$$A_+^2 = (A^{2,0} + A_\Phi^{1,1} + A^{0,2}) \cap A^2, \tag{2.5}$$

and

$$A_-^2 = (A_\Phi^{1,1})^\perp \cap A^2. \tag{2.6}$$

Then A_+^2(resp. $/A_-^2$) is the self-dual part (resp. the anti-self-dual part) of A^2 ([1]). Then projection from A^2 onto A_-^2 is denoted by p_+.

Let P be a principal bundle over M with structure group $SU(n)$. Then the canonical representation ρ of $SU(n)$ induces a smooth complex vector bundle $E = P \times_\rho \mathbf{C}^n$. We denote by h and ω, the Hermitian structure and the n-form on E defined by the $SU(n)$-structure of P, respectively. Let $GL(E)$ denote the group of C^∞-bundle automorphisms of E (inducing the identity transformations on the base manifold M). Let $SL(E)$ (resp. $SU(E)$) denote the subgroup of $GL(E)$ consisting of bundle automorphisms of E (resp. unitary automorphisms of (E, h)) with determinant 1. They are called the gauge transformation groups of E. Let End^0 (resp. End^0 (E, h)) the subbundle of the endomorphism bundle $\mathrm{End}E$ consisting of endomorphisms (resp. skew-Hermitian endomorphisms) with trace 0. End^0 (E, h) is the real subbundle of $\mathrm{End}^0 E$ and we have

$$\mathrm{End}^0 E = \mathrm{End}^0(E,h) \oplus \sqrt{-1}\mathrm{End}^0(E,h). \tag{2.7}$$

For $\psi = \psi_0 + \sqrt{-1}\psi_1, \psi_0 - \sqrt{-1}\psi_1$, we denote the complex conjugate by $\overline{\psi}$, which is defined by $\overline{\psi} = \psi_0 - \sqrt{-1}\psi_1$.

An $SU(n)$-connection D in (E, h) is a connection in E preserving h and ω, i.e., a homomorphism $D: A^0(E) \to A^1(E)$ over $\mathbf{C}$ such that

$$\begin{aligned} &D(f\sigma) = \sigma df + f.D\sigma \quad \text{for } f \in A^0_{\mathbf{C}}, \sigma \in A^0_{\mathbf{C}}(E), \\ &Dh = 0, \\ &D\omega = 0. \end{aligned} \tag{2.8}$$

The set of $SU(n)$-connections has an affine structure. Namely, it is given by $\{D + v : v \in A^1(\mathrm{End}^0(E,h))\}$ for a fixed $SU(n)$-connection D. We can extend an $SU(n)$-connection D to a connection in End^0 (E, h). We call D irreducible when the kernel of $D: A^0(\mathrm{End}^0(E,h)) \to A^1(\mathrm{End}^0(E,h))$ is trivial. An $SU(n)$-connection D is called anti-self-dual, if the curvature form $R(D)$ belongs to $A^2_-(\mathrm{End}^0(E,h)$, namely $p_+ R(D) = 0$. Let Asd be the set of all anti-self-dual $SU(n)$-connections in (E, h). The gauge transformation group $SU(E)$ acts on the space of $SU(n)$-connections and leaves Asd invariant. Thus we obtain the moduli space Asd/SU(E) of anti-self-dual $SU(n)$-connection in (E, h).

A semi-connection D'' in E is a linear map $D'' : A^0(E) \to A^{0,1}(E)$ satisfying $D''(f\sigma) = D''f\sigma + f\,D''\sigma$ for $\sigma \in A^0(E), f \in C^\infty(M;\mathbf{C})$. Moreover we assume that D'' preserves the n-form ω, i.e.; $D''\omega = 0$. The set of semi-connections has an (complex) affine space. Namely, it is given by $\{D'' + v : v \in A^{0,1}(\mathrm{End}^0 E)\}$ for a fixed semi-connection D''. We can extend D'' to a semi-connection in $\mathrm{End}^0 E$. We call D'' simple when the kernel of $D'' : A^0(\mathrm{End}^0 E) \to A^{0,1}(\mathrm{End}^0 E)$ is trivial. A semi-connection D'' which satisfies $D'' \circ D'' = 0$ defines a unique holomorphic structure on E. We call such a semi-connection holomorphic. Let Hol be the set of all holomorphic semi-connections in E. The gauge transformation group $SL(E)$ acts on the space of semi-connections and leaves Hol invariant. Thus we obtain the moduli space Hol/SL(E) of holomorphic semi-connections in E.

Let D be an $SU(n)$-connection in (E, h). Set $D = D' + D''$ where $D' : A^0(E) \to A^{0,1}(E)$. Then D'' is a semi-connection in E. This natural map $D \mapsto D''$ is a bijective map of the set of $SU(n)$-connections onto the set of semi-connections. If D is anti-self-dual, D'' is holomorphic. In fact the (0,2)-component of $R(D) = D'' \circ D''$. Thus we obtain a natural map $f : \mathrm{Asd}/SU(E) \to \mathrm{Hol}/SL(E)$. It is known that f is an injective map (cf. [5, p.243]). Moreover we have

LEMMA 1. *If an anti-self-dual connection D is irreducible, then D'' is simple.*

Proof) Suppose $\phi \in A^0(\mathrm{End}^0 E)$ be a holomorphic section of $\mathrm{End}^0 E$. Then $D''\phi = 0$. By the vanishing theorem of the holomorphic sections ([5]), we obtain $D\phi = 0$. By the assumption that D is an irreducible connection, we conclude $\phi \equiv 0$.

In order to consider infinitesimal deformations, we introduce two complexes (2.9), (2.10), and their cohomology groups. For $D \in \mathrm{Asd}$ set

$$0 \to A^0(\mathrm{End}^0(E,h)) \xrightarrow{D} A^1(\mathrm{End}^0(E,h)) \xrightarrow{D_+} A^2_+(\mathrm{End}^0(E,h)) \to 0 \qquad (2.9)$$

where $D_+ = p_+ \circ D$. Their cohomology groups are denoted by $H^p_D (p = 0,1,2)$. For $D'' \in \mathrm{Hol}$, we consider the Dolbeault complex

$$0 \to A^{0,0}(\mathrm{End}^0 E) \xrightarrow{D''} A^{0,1}(\mathrm{End}^0 E) \xrightarrow{D''} A^{0,2}(\mathrm{End}^0 E) \to 0 \qquad (2.10)$$

and their cohomology groups are denoted by $H^{o,p}_{D''} (p = 0,1,2)$.. We set

$$\mathcal{M}_0 = \{[D] \in \mathrm{Asd}/SU(E) : D \text{ is irreducible and } H^2_D \text{ vanishes}\} \qquad (2.11)$$

Then it is known that $\mathcal{M}_0$ is a smooth manifold, and its tangent space at $[D]$ is naturally isomorphic to H^1_D. We set

$$\mathcal{H}_0 = \{[D''] \in \mathrm{Hol}/SU(E) : D'' \text{ is simple and } H^{0,2}_{D''} \text{ vanishes}\} \qquad (2.12)$$

Similarly it is known that $\mathcal{H}_0$ is a complex manifold and its tangent space at $[D'']$ is naturally isomorphic to $H^{0,1}_{D''}$.

Now we consider the following natural homomorphism between two complexes (2.9), (2.10) for an irreducible anti-self-dual $SU(n)$-connection D and its corresponding holomorphic semi-connection D:

$$
\begin{array}{ccccccccc}
0 & \to & A^0(\mathrm{End}^0(E,h)) & \xrightarrow{D} & A^1(\mathrm{End}^0(E,h)) & \xrightarrow{D_+} & A^2_+(\mathrm{End}^0(E,h)) & \to & 0 \\
 & & \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 & & \\
0 & \to & A^{0,0}(\mathrm{End}^0 E) & \xrightarrow{D''} & A^{0,1}(\mathrm{End}^0 E) & \xrightarrow{D''} & A^{0,2}(\mathrm{End}^0 E) & \to & 0
\end{array}
\tag{2.13}
$$

where

$$h_0 : \text{inclusion}$$
$$h_1 : \alpha \to \alpha^{0,1}$$
$$h_2 : \alpha \to \alpha^{0,2}$$

and $\alpha^{0,p}$ represents the $(0, p)$-component of α. Itoh showed that h_p induces an isomorphism of H^p_D onto $H^{0,p}_{D''}(p = 0,1,2)$ when (M, g) is a Kähler surface. We can extend this result to the case of a Hermitian surface. Its proof will be given in section 3. Therefore we have $f(\mathcal{M}_0) \subset \mathcal{H}_0$ for the natural map f. Moreover it is known that f is a differentiable map. Since we can regard the differential f_* of f at $[D]$ as h_1, f is a diffeomorphisms of $\mathcal{M}_0$ into $\mathcal{H}_0$. Thus it has been shown that $\mathcal{M}_0$ is a complex manifold. We note that $H^{0,2}_{D''}$ is isomorphic to $\tilde{H}_D = H^0(M, \mathcal{O}(\mathrm{End}^0 E \otimes K_M))$ by the Serre duality. Hence our Theorem 2 has been proved.

3. Isomorphisms between cohomology groups H^p_D and $H^{0,p}_{D''}$

In this section, we prove that for an irreducible anti-self-dual connection the cohomology groups H^p_D are isomorphic to $H^{0,p}_{D''}(p = 0,1,2)$ in the diagram (2.13).

We first begin with the preparation for the proof. On a Hermitian surface (M, g), we define differential 1-forms $\theta = -d^*\Phi\eta = \theta \circ J$, and (1,0)-form $\varphi = \eta + \sqrt{-1}\theta$. Here J is the complex structure of (M, g). Then we obtain following formulas by direct calculation.

LEMMA 2. *For the operators acting on $A^p(\mathrm{End}^0 E)$, the following formulas hold:*

$$D'^* = -\sqrt{-1}(D''\Lambda - \Lambda D'') + \frac{1}{2}(p-2)i(\overline{\varphi}) - \frac{\sqrt{-1}}{2}\varepsilon(\overline{\varphi})\Lambda \tag{3.1}$$

$$D''^* = \sqrt{-1}(D'\Lambda - \Lambda D') + \frac{1}{2}(p-2)i(\varphi) - \frac{\sqrt{-1}}{2}\varepsilon(\varphi)\Lambda \tag{3.2}$$

It is known that there is a unique Hermitian metric up to the homothety such that

$d^*\eta = 0$ in the conformal class of the given Hermitian metric ([3]). Moreover the anti-self-duality is preserved by a conformal change of the metric. Therefore we may assume that $d^*\eta = 0$ on the given Hermitian surface. Define a mapping $\mathcal{L} : A^0(\mathrm{End}^0 E) \to A^0(\mathrm{End}^0 E)$ by $\mathcal{L} = -\sqrt{-1}\Lambda D'D''$. Then we have

LEMMA 3. *On* $A^0(\mathrm{End}^0 E)$

$$\mathcal{L} = \frac{1}{2}(\Delta_D + i(\eta)D), \tag{3.3}$$

where $\Delta_D = D^* D$.

PROOF) In fact

$$\begin{aligned} \Delta_D = D^* D &= (D'^* + D''^*)(D' + D'') \\ &= D'^* D' + D''^* D''. \end{aligned} \tag{3.4}$$

Using equations (3.1) and (3.2), we see that

$$\begin{aligned} D'^* D' + D''^* D'' &= \sqrt{-1}\Lambda D''D' - \frac{1}{2}i(\bar{\varphi})D' - \sqrt{-1}\Lambda D'D'' - \frac{1}{2}i(\varphi)D'' \\ &= \sqrt{-1}\Lambda(D'D'' - D''D') - i(\eta)D. \end{aligned} \tag{3.5}$$

Since D is an anti-self-dual connection, for $\psi \in A^0(\mathrm{End}^0 E)$, we have

$$\begin{aligned} \Lambda(D'D'' + D''D')\psi &= \Lambda R(D)(\psi) \\ &= \Lambda(R(D) \circ \psi - \psi \circ R(D)) \\ &= (\Lambda R(D))\psi - \psi(\Lambda R(D)) \\ &= 0. \end{aligned} \tag{3.6}$$

It follows that

$$\Delta_D = -2\sqrt{-1}\Lambda D'D'' - i(\eta)D. \tag{3.7}$$

Then we obtain (3.3).

From Lemma 3 we see that $\mathcal{L}(A^0(\mathrm{End}^0 E, h)) \subset A^0(\mathrm{End}^0 E, h))$. Let $\mathcal{L}^*$ be the formal adjoint operator of $\mathcal{L}$. For $\phi, \psi \in A^0(\mathrm{End}^0(E, h))$,

$$\begin{aligned} (\mathcal{L}\phi, \psi)_M &= \left(\frac{1}{2}\Delta_D\phi + \frac{1}{2}i(\eta)D\phi, \psi\right)_M \\ &= \left(\phi, \frac{1}{2}\Delta_D\psi + \frac{1}{2}D^*\varepsilon(\eta)\psi\right)_M. \end{aligned} \tag{3.8}$$

Consequently we have

$$\mathscr{S}^* = \frac{1}{2}(\Delta_D + D^*\varepsilon(\eta)). \tag{3.9}$$

By the direct calculation on $A^0(\mathrm{End}^0(E,h))$, we have

$$\begin{aligned} D^*\varepsilon(\eta) &= \varepsilon(d^*\eta) - i(\eta)D \\ &= -i(\eta)D. \end{aligned} \tag{3.10}$$

Consequently, we obtain

$$\mathscr{S}^* = \frac{1}{2}(\Delta_D - i(\eta)D). \tag{3.11}$$

LEMMA 4. *On* $A^0(\mathrm{End}^0(E,h))$, *we have*

$$\ker\mathscr{S} = \ker\mathscr{S}^* = \ker D \tag{3.12}$$

Proof) It is clear that $\ker D \subset \ker\mathscr{S}$, and $\ker D \subset \ker\mathscr{S}^*$ by (3.3), (3.11). Conversely suppose that $\mathscr{S}\phi = 0$, for $\phi \in A^0(\mathrm{End}^0(E,h))$. Then

$$\begin{aligned} 0 &= (\mathscr{S}\phi, \phi)_M \\ &= \left(\frac{1}{2}\Delta_D\phi + \frac{1}{2}i(\eta)D\phi, \phi\right)_M \\ &= \frac{1}{2}(D\phi, D\phi)_M + \frac{1}{2}(i(\eta)D\phi, \phi)_M \end{aligned} \tag{3.13}$$

Using (3.10), we see that

$$\begin{aligned} (i(\eta)D\phi, \phi)_M &= (\phi, D^*\varepsilon(\eta)\phi)_M \\ &= -(\phi, i(\eta)D\phi)_M \\ &= -(i(\eta)D\phi, \phi)_M. \end{aligned} \tag{3.14}$$

Then

$$(i(\eta)D\phi, \phi)_M = 0, \tag{3.15}$$

From (3.13) it follows that $D\phi = 0$. Noting that $\mathscr{S}(A^0(\mathrm{End}^0(E,h))) \subset A^0(\mathrm{End}^0(E,h))$, we obtain

$$\ker\mathscr{S} \subset \ker D \tag{3.16}$$

Owing to (3.11), we obtain $\ker\mathscr{S}^* \subset \ker D$ similarly.

THEOREM 3. *Let D be an irreducible anti-self-dual* $SU(n)$-*connection. Then the homomorphisms of the cohomology groups* $h_p : H_D^p \to H_{D''}^{0,p}$ $(p = 0,1,2)$ *induced from the diagram (2.13) are isomorphisms.*

PROOF)

$\underline{h_0}$:

By Lemma 1, we have $H^0 \to H^{0,0} = 0$. Therefore it is trivial that h_0 is isomorphic.

$\underline{h_1}$:

First we show the injectivity of h_1. Suppose $[\alpha] \in H^1$ and $h_1([\alpha]) = 0$. That is $\alpha \in A^1(\mathrm{End}^0(E,h))$ satisfies $D_+\alpha = 0$ and there exists $\phi \in A^0(\mathrm{End}^0 E)$ such that $h_1(\alpha) = \alpha^{0,1} = D''\phi$. Since $D_+\alpha = 0, \Lambda(D''D'\overline{\phi} + D'D''\phi) = 0$. We set $\phi = \phi_0 + \sqrt{-1}\phi_1$ and $\overline{\phi} = \phi_0 \sqrt{-1}\phi_1$ for $\phi_0, \phi_1 \in A^0(\mathrm{End}^0(E,h))$. Then

$$\begin{aligned} 0 &= \Lambda(D''D'\phi_0 - \sqrt{-1}D''D'\phi_1 + D'D''\phi_0 + \sqrt{-1}D'D''\phi_1) \\ &= \Lambda(D''D'\phi_0 + D'D''\phi_0) - \sqrt{-1}\Lambda(D''D'\phi_1 - D'D''\phi_1) \end{aligned} \tag{3.17}$$

Since D is an anti-self-dual connection,

$$\Lambda(D''D' + D''D')\phi_0 = (\Lambda R(D))\phi_0 = 0, \tag{3.18}$$

and

$$-\sqrt{-1}\Lambda(D'D'' - D''D')\phi_1 = 2\mathcal{I}\phi_1 . \tag{3.19}$$

Therefore we have $2\mathcal{I}\phi_1 = 0$. Together with Lemma 4, the irreducibility of D implies $\phi_1 \equiv 0$. Consequently

$$\alpha = \alpha^{1,0} + \alpha^{0,1} = D'\phi_0 + D''\phi_0 = D\phi_0 \tag{3.20}$$

and then $[\alpha] = 0$ in H^1_D. It is shown that h_1 is injective.

Next, in order to prove the surjectivity of h_1, given $\beta \in A^{0,1}(\mathrm{End}^0 E)$ satisfying $D''\beta = 0$, we will find $[\alpha] \in H^1_D$ such that $h_1([\alpha]) = [\beta]$ in $H^{0,1}_{D''}$. To do so, we put $\alpha = \overline{\beta} + D'\overline{\psi} + \beta + D''\psi \in A^1(\mathrm{End}^0(E,h))$. The equation $D_+\alpha = 0$ means

$$D''\alpha^{0,1} = D''(\beta + D''\psi) = 0 \tag{3.21}$$

and

$$\begin{aligned} \Lambda(D''\alpha^{1,0} + D'\alpha^{0,1}) &= \Lambda(D''\overline{\beta} + D''D'\overline{\psi} + D'\beta + D'D''\psi) \\ &= \Lambda(D''\overline{\beta} + D'\beta + 2\sqrt{-1}\Lambda D'D''\psi_1 = 0, \end{aligned} \tag{3.22}$$

where $\psi = \psi_0 + \sqrt{-1}\psi_1$. Therefore we have

$$2\mathcal{I}\psi_1 = \Lambda(D''\overline{\beta} + D'\beta) \tag{3.23}$$

By Lemma 4 and the irreducibility of D, the kernel of $\mathcal{I}^*$ is trivial. Then we can find ψ_1 which satisfies the equation (3.23). Taking ψ_0 suitably, we obtain

$\alpha \in A^1(\mathrm{End}^0 E, h))$ satisfying $h_1([\alpha]) = [\beta]$.

$\underline{h_2}$:

It is clear that h_2 is surjective. So we show the injectivity. Let ψ be an element of $A_+^2(\mathrm{End}^0(E,h))$. We decompose ψ as follows: $\psi = \psi^{2,0} + (1/2)\Phi \wedge \phi + \psi^{0,2}$ for $\phi \in A^0(\mathrm{End}^0(E,h))$. Suppose $h_2([\psi]) = 0$. That is, there exists a $\beta \in A^{0,1}(\mathrm{End}^0 E,)$ such that $h_2(\psi) = \psi^{0,2} = D''\beta$. We will find $\alpha \in A^1(\mathrm{End}^0(E,h))$ such that $\psi = D_+\alpha$. To do so, we put $\alpha = \overline{\beta} + D'\overline{\gamma} + \beta + D''\gamma$ for some $\gamma \in A^0(\mathrm{End}^0 E)$. Then we have

$$\begin{aligned}\psi &= D_+\alpha \\ &= D'(\overline{\beta} + D'\overline{\gamma}) + \frac{1}{2}\Phi \wedge \Lambda\{D''(\overline{\beta} + D'\overline{\gamma}) + D'(\beta + D''\gamma)\} + D''(\beta + D''\gamma)\end{aligned} \tag{3.24}$$

We set $\gamma = \gamma_0 + \sqrt{-1}\gamma_1$ for $\gamma_0, \gamma_1 \in A^0(\mathrm{End}^0 E, h))$. Then

$$\begin{aligned}\phi &= \Lambda(D''\overline{\beta} + D''D'\gamma + D'\beta + D'D''\gamma) \\ &= \Lambda(D''\overline{\beta} + D'\beta) + 2\Lambda D'D''\gamma_1.\end{aligned} \tag{3.25}$$

Therefore we have

$$2\mathscr{S}\gamma_1 = \Lambda(D''\overline{\beta} + D'\beta) - \phi. \tag{3.26}$$

The solution γ_1 of (3.26) exists since D is irreducible and ker $\mathscr{S}^* = \{0\}$. We have found α satisfying $\psi = D_+\alpha$.

4. Vanishing of $\tilde{H}_D$

In this section, we will prove Proposition 1 in the introduction. First we recall the results obtained by Gauduchon in [2]. Let (M, g) be an m-dimensional compact Hermitian manifold with $\partial\overline{\partial}\Phi^{m-1} = 0$. Let L be a holomorphic line bundle over (M, g), and h be its Hermitian structure. We denote by k the mean curvature of (L, h). We use the notation "mean curvature" following Kobayashi [5, p. 51] and it is called the Ricci-scalar in Gauduchon [2]. Then the following holds ([2]):

1. $\int_M k\,dv$ is independent of the Hermitian structure h.
2. There exists a unique Hermitian structure h_0 on L (up to the homothety) such that its mean curvature k_0 is constant.

In particular, applying the above results to the canonical line bundle K_M, we obtain the Hermitian structure with constant mean curvature k_0. We note that

$k_0 \mathrm{Vol}(M,g) = -\int_M \mathrm{Scal}(g)\,dv$, where $\mathrm{Scal}(g)$ denotes the scalar curvature of the Hermitian connection with respect to g.

Now we return to the proof of Proposition 1. The C^∞-Hermitian vector bundle (E, h) has a holomorphic structure defined by the anti-self-dual $SU(n)$-connection D. D is the Hermitian connection of (E, h) with respect to this holomorphic structure and it has mean curvature 0 and so for $\mathrm{End}^0 E$. Together with the former, it implies that the tensor product $F = \mathrm{End}^0 E \otimes K_M$ admits a Hermitian structure with mean curvature $k_0 I_L$. If $k_0 < 0$, by the vanishing theorem of the holomorphic sections ([5, pp. 49–53]), $\mathrm{End}^0 E \otimes K_M$ admits no nonzero holomorphic sections. Further, if $k_0 = 0$, then every holomorphic section is parallel. Let f be a nonzero holomorphic section section of $\mathrm{End}^0 E \otimes K_M$. For each point x on M, consider the eigenspace of the homomorphism f_x. These eigenspaces define a parallel subbundle of E. This contracts that D is an irreducible connection. Consequently, even if $k_0 = 0, \mathrm{End}^0 E \otimes K_M$ has no nonzero holomorphic sections.

REMARK: Let (M, g) be a compact anti-self-dual Hermitian surface (i.e., its Weyl conformal curvature tensor W belongs to A_-^2) with $\partial\bar{\partial}\Phi = 0$. Then we have $\int_M \mathrm{Scal}(g)\,dv \geq 0$ and the equality holds if and only if (M, g) is Kählerian (cf. Boyer [6]).

References

[1] Atiyah, M. F., Hitchin, N. J., and Singer, I. M.: Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London. A, **362** (1978) 425–461.

[2] Gauduchon, P.: Fibrès Hermitian à endomorphisme de Ricci non négatif, Bull. Soc. Math. France., **105** (1977) 113–140.

[3] Gauduchon, P.: Le théorème de l'excentricité nulle, C. R. Acad. Sc. Paris. t. **285** (1977) 387–390.

[4] Itoh, M.: The moduli space of Yang-Mills connections over a Kähler surface is a complex manifold, Osaka J. Math., **22** (1985) 845–862.

[5] Kobayashi, S.: "Differential geometry of complex vector bundles", Iwanami Shoten (1987).

[6] Boyer, C. P.: Conformal duality and compact complex surfaces, Math. Ann., **274** (1986) 517–526.