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META-ABELIANIZATIONS OF SL$(2, Z[\frac{1}{p}])$ AND DENNIS-STEIN SYMBOLS

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Abstract. Using a Dennis-Stein symbol, we will study $K_2(2, \mathbf{Z}[\frac{1}{p}])$ and the meta-abelianization of $\mathrm{SL}(2, \mathbf{Z}[\frac{1}{p}])$.

1. Introduction.

Let Z be the ring of rational integers. For a given group G, we denote by G' the commutator subgroup $[G, G]$ of G, and by G'' the second commutator subgroup $[G', G']$ of G. Then we put $G^{ab} = G/G'$, the abelianization of G, and $G^{mab} = G/G''$, the meta-abelianization of G. The cyclic group of order m is denoted by Z_m, and the cyclic group of infinite order is denoted by Z instead of Z_∞. And, the semi-direct product $H = K \ltimes L$ of groups means $H = \langle K, L \rangle$, $K \cap L = 1$, and $H \rhd L$. Then we will obtain the following results.

THEOREM 1. *Let p be a prime number. Then*

$$\mathrm{SL}(2, \mathbf{Z}[\frac{1}{p}])^{mab} \simeq \begin{cases} Z_3 \ltimes (Z_2 \times Z_2) & p = 2; \\ Z_4 \ltimes Z_3 & p = 3; \\ Z_{12} \ltimes (Z_2 \times Z_6) & p \geq 5. \end{cases}$$

COROLLARY. *Suppose* $p \geq 5$. Then

$$\mathrm{SL}(2, \mathbf{Z}[\frac{1}{p}])^{mab} \simeq \mathrm{SL}(2, \mathbf{Z}[\frac{1}{2}])^{mab} \times \mathrm{SL}(2, \mathbf{Z}[\frac{1}{3}])^{mab}.$$

THEOREM 2.
(1) Suppose $p = 2, 3$. Then $K_2(2, \mathbf{Z}[\frac{1}{p}]) \simeq Z \times Z_{p-1}$, and $K_2(2, \mathbf{Z}[\frac{1}{p}])$ is central.
(2) Suppose $p \geq 5$. Then $K_2(2, \mathbf{Z}[\frac{1}{p}]) \supset Z \times Z$, and $K_2(2, \mathbf{Z}[\frac{1}{p}])$ is not central.

There is an algorithm to get a finite presentation of $\mathrm{SL}(2, \mathbf{Z}[\frac{1}{p}])$. Therefore, it might be possible to calculate the meta-abelianization of $\mathrm{SL}(2, \mathbf{Z}[\frac{1}{p}])$ when p is

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given. However, the main difficulty is that one cannot expect a uniform presentation of $\mathrm{SL}(2,\mathbf{Z}[\frac{1}{p}])$ for all p (cf. [4]). Here we will find some element $d(a,b)$, called a Dennis-Stein symbol, in $K_2(2,\mathbf{Z}[\frac{1}{p}])$ which leads to Theorem 1 as well as Theorem 2. Corollary can be also obtained from the result in [9].

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1. $K_2(2,A)$ and symbols.

For a commutative ring, A, with 1, we define the Steinberg group of rank one, called $\mathrm{St}(2, A)$, by generators: $x_{12}(t)$, $x_{21}(t)$ for $t \in A$ and defining relations:

$$x_{ij}(s)x_{ij}(t) = x_{ij}(s+t)$$

and

$$x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(t)x_{ji}(u^{-1})x_{ij}(-u) = x_{ji}(-u^{-2}t)$$

for $s,t \in A$, $u \in A^{\times}$ and $\{i,j\} = \{1,2\}$, where $A^{\times}$ is the unit group of A (cf. [2], [5]). Then, there is a natural homomorphism, π, of $\mathrm{St}(2, A)$ into $\mathrm{SL}(2, A)$ with

$$\pi\, x_{12}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \pi\, x_{21}(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} .$$

Put

$$K_2(2,A) = \ker[\pi : \mathrm{St}(2,A) \to \mathrm{SL}(2,A)] .$$

Now we define several elements in $\mathrm{St}(2, A)$. Set

$$\begin{aligned} w_{ij}(u) &= x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u), \\ h_{ij}(u) &= w_{ij}(u)w_{ij}(-1), \\ c(u,v) &= h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}, \\ d(a,b) &= x_{21}(-bu^{-1})x_{12}(-a)x_{21}(b)x_{12}(au^{-1})h_{12}(u)^{-1} \end{aligned}$$

for $a,b \in A$, $u, v \in A^{\times}, \{i,j\} = \{1,2\}$ with $1 - ab = u$. Then

$$\pi\, w_{12}(u) = \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}, \quad \pi\, w_{21}(u) = \begin{pmatrix} 0 & -u^{-1} \\ u & 0 \end{pmatrix},$$

$$\pi\, h_{12}(u) = \begin{pmatrix} u & o \\ 0 & u^{-1} \end{pmatrix}, \quad \pi\, h_{21}(u) = \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix},$$

and $c(u,v), d(a,b) \in K_2(2,A)$. The symbols $c(u,v)$ and $d(a,b)$ are called Steinberg symbols and Dennis-Stein symbols respectively.

2. The case of $\boldsymbol{Z}[\frac{1}{p}]$.

First, we shall recall that St(2,$\boldsymbol{Z}$) is isomorphic to the 3-braid group, $B_3 = \langle x, y | xyx = yxy \rangle$. Hence, we see $\mathrm{St}(2,\boldsymbol{Z})^{mab} \simeq Z \ltimes (Z \times Z)$. Since $K_2(2,\boldsymbol{Z}) \simeq \langle c(-1,-1) \rangle \simeq Z$ and $c(-1,-1)$ is corresponding to $x^{12} \equiv 1 \bmod B_3''$ (cf. [5], [8]), we obtain $\mathrm{SL}(2,\boldsymbol{Z})^{mab} \simeq Z_{12} \ltimes (Z \times Z)$.

Now we take a prime number p and consider $A = \boldsymbol{Z}[\frac{1}{p}]$. For each p, we define the group G_p by the generators x_1, x_2, y_1, y_2 and the defining relations

$$x_i y_i x_i = y_i x_i y_i (i = 1,2),\ x_1 = y_2^p,\ x_2 = y_1^p,$$
$$[(x_1 y_1 x_1)^2, y_2] = [(x_2 y_2 x_2)^2, y_1] = 1.$$

Then $\mathrm{St}(2,\boldsymbol{Z}[\frac{1}{p}]) \simeq G_p$ (cf. [7]). In [8], we already confirmed that

$$\mathrm{St}(2,\boldsymbol{Z}[\frac{1}{p}])^{mab} \simeq \begin{cases} Z_3 \ltimes (Z_2 \times Z_2) & p = 2; \\ Z_8 \ltimes Z_3 & p = 3; \\ Z_{p^2-1} \ltimes (Z \times Z) & p \geq 5. \end{cases}$$

To get this, we constructed M_p as follows:

$$M_2 = \langle \sigma, \tau_1, \tau_2 | \sigma^3 = \tau_1^2 = \tau_2^2 = [\tau_1, \tau_2] = 1, \sigma\tau_1\sigma^{-1} = \tau_1\tau_2, \sigma\tau_2\sigma^{-1} = \tau_1 \rangle,$$
$$M_3 = \langle \sigma, \tau | \sigma^8 = \tau^3 = 1, \sigma\tau\sigma^{-1} = \tau^2 \rangle,$$
$$M_p = \langle \sigma, \tau_1, \tau_2 | \sigma^{p^2-1} = [\tau_1, \tau_2] = 1, \sigma\tau_1\sigma^{-1} = \tau_1\tau_2^{-1}, \sigma\tau_2\sigma_1^{-1} = \tau_1 \rangle$$

with $p \geq 5$. Then we obtain $G_p^{mab} \simeq M_p$ for every p, which gives the explicite group structure of $\mathrm{St}(2,\boldsymbol{Z}[\frac{1}{p}])^{mab}$ as above. In fact, we can easily see that there is a group homomorphism α_p of G_p onto M_p satisfying

$$\begin{cases} \alpha_2(x_1) = \sigma\tau_1, \alpha_2(y_1) = \sigma, \alpha_2(x_2) = \sigma^2, \alpha_2(y_2) = \sigma^2\tau_1\tau_2 & (p = 2); \\ \alpha_3(x_1) = \sigma\tau, \alpha_3(y_1) = \sigma, \alpha_3(x_2) = \sigma^3, \alpha_3(y_2) = \sigma^3\tau & (p = 3); \\ \alpha_p(x_1) = \sigma\tau_1, \alpha_p(y_1) = \sigma, \alpha_p(x_2) = \sigma^p, \alpha_p(y_2) = \sigma^p\tau_1 & (p \equiv 1 \bmod 6); \\ \alpha_p(x_1) = \sigma\tau_1, \alpha_p(y_1) = \sigma, \alpha_p(x_2) = \sigma^p, \alpha_p(y_2) = \sigma^p\tau_1^{-1}\tau_2 & (p \equiv 5 \bmod 6). \end{cases}$$

This map α_p induces an isomorphism of G_p^{mab} onto M_p.

If $p = 2$, then $K_2(2,\boldsymbol{Z}[\frac{1}{2}])$ is generated by $c(-1,-1)$ (cf. [1]), and $c(-1,-1)$ is corresponding to $1 \in M_2$. Therefore, $\mathrm{SL}(2,\boldsymbol{Z}[\frac{1}{2}])^{mab} \simeq Z_3 \ltimes (Z_2 \times Z_2)$. If $p = 3$, then $K_2(2,\boldsymbol{Z}[\frac{1}{3}])$ is generated by $c(-1,-1)$ and $c(3,-1)$ (cf. [1]), which are corresponding to $1 \in M_3$ and $\sigma^4 \in M_3$ respectively. Therefore, $\mathrm{SL}(2,\boldsymbol{Z}[\frac{1}{3}])^{mab} \simeq (Z_4 \ltimes Z_3)$.

Next suppose $p \geq 5$. Then we will choose some Dennis-Stein symbols, and consider their images in M_p. Note that if $1-ab=\pm p$, then $d(a,b)= w_{12}(1)x_{12}(\pm bp^{-1})w_{12}(-1)x_{12}(-a)x_{21}(b)x_{12}(\pm ap^{-1})w_{21}(\pm p)w_{12}(1) \in \mathrm{St}(2,\boldsymbol{Z}[\frac{1}{p}])$, which is corresponding to

$$e(a,b)=(x_1y_1x_1)y_2^{\pm b}(x_1y_1x_1)^{-1}x_1^{-a}y_1^{-b}y_2^{\pm a}(x_2y_2x_2)^{\mp 1}(x_1y_1x_1) \in G_p.$$

Since $y_i^{p^2-1} \equiv 1 \bmod G_p''$ (cf. [8]), then $y_1 \equiv x_2^p \bmod G_p''$ and $y_2 \equiv x_1^p \bmod G_p''$. Hence, $e(a,b) \equiv x_2^{\pm b}x_1^{-a}y_1^{-b}y_2^{\pm a}(x_2y_2x_2)^{\mp 1}(x_1y_1x_1) \bmod G_p''$.

If $p=6k+1, k=2^l m, (2,m)=1$, then

$$\begin{aligned}
&\alpha_p e(-2^{l+1},3m)\\
&=(\sigma^p)^{3m}(\sigma\tau_1)^{2^{l+1}}\sigma^{-3m}(\sigma^p\tau_1)^{-2^{l+1}}(\sigma^p\sigma^p\tau_1\sigma^p)^{-1}(\sigma\tau_1\sigma\sigma\tau_1)\\
&=\sigma^{3pm}\sigma^{2^{l+1}}\rho\sigma^{-3m}\rho^{-1}\sigma^{-2^{l+1}p}\sigma^{3p+3}\\
&=\sigma^{3(p-1)m}\sigma^{2^{l+1}}\rho^{-2}\sigma^{-2^{l+1}p}\sigma^{-3(p-1)}\\
&=\sigma^{(p-1)\{3(m-1)+2^{l+1}\}}\rho',
\end{aligned}$$

where $(\sigma\tau_1)^{2^{l+1}}=\sigma^{2^{l+1}}\rho, \rho=\tau_1\tau_2$ or $\tau_1^{-1}\tau_2^2$, and $\rho' \in \{\tau_1^{-2}\tau_2^4, \tau_1^{\pm 4}\tau_2^{\mp 2}, \tau_1^2\tau_2^2\}$. In particular, the order of $d(-2^{l+1},3m)$ is infinite.

If $p=6k-1, k=2^l m, (2,m)=1$, then

$$\begin{aligned}
&\alpha_p e(2^{l+1},3m)\\
&=(\sigma^p)^{-3m}(\sigma\tau_1)^{-2^{l+1}}\sigma^{-3m}(\sigma^p\tau_1^{-1}\tau_2)^{-2^{l+1}}(\sigma^p\sigma^p\tau_1^{-1}\tau_2\sigma^p)(\sigma\tau_1\sigma\sigma\tau_1)\\
&=\sigma^{-3pm}\sigma^{-2^{l+1}}\rho\sigma^{-3m}\rho^{-1}\sigma^{-2^{l+1}p}\sigma^{3p+3}\\
&=\sigma^{-3(p+1)m}\sigma^{-2^{l+1}}\rho^{-2}\sigma^{-2^{l+1}p}\sigma^{3(p+1)}\\
&=\sigma^{-(p+1)\{3(m-1)+2^{l+1}\}}\rho',
\end{aligned}$$

where $(\sigma\tau_1)^{-2^{l+1}}=\sigma^{-2^{l+1}}\rho, \rho=\tau_1\tau_2$ or $\tau_1\tau_2^{-2}$, and $\rho' \in \{\tau_1^{-2}\tau_2^4, \tau_1^4\tau_2^{-2}, \tau_1^{-2}\tau_2^{-2}\}$. In particular, also in this case, the order of $d(2^{l+1},3m)$ is infinite.

PROOF OF THEOREM 1. For p = 2, 3, we already discussed completely. Suppose $p \geq 5$. Then the homomorphism π of $\mathrm{St}(2,\boldsymbol{Z}[\frac{1}{p}])$ onto $\mathrm{SL}(2,\boldsymbol{Z}[\frac{1}{p}])$ induces the homomorphism, called $\bar{\pi}$, of M_p onto $\mathrm{SL}(2,\boldsymbol{Z}[\frac{1}{p}])^{mab}$. Since $\sigma^{12}, \tau_1^6, \tau_1^2\tau_2^2 \in \ker\bar{\pi}$ as above, we obtain a homomorphism of $Z_{12} \ltimes (Z_2 \times Z_6)$ onto $\mathrm{SL}(2,\boldsymbol{Z}[\frac{1}{p}])^{mab}$. On the other hand, we see that $\mathrm{PSL}(2,\boldsymbol{Z}/3\boldsymbol{Z}) \simeq \mathfrak{A}_4 \simeq Z_3 \ltimes (Z_2 \times Z_2)$ (cf.[3]) and $\mathrm{SL}(2,\boldsymbol{Z}/4\boldsymbol{Z}) \simeq Z_4 \ltimes \mathfrak{A}_4$ (cf. Section 3). Hence, $\mathrm{SL}(2,\boldsymbol{Z}[\frac{1}{p}])^{mab} \simeq Z_{12} \ltimes (Z_2 \times Z_6)$. □

Considering the action of σ in M_p, one reaches Corollary easily. And, the result in Theorem 2(1) is already known (cf. [1], [5], [6], [7]).

PROOF OF THEOREM 2(2). In $K_2(2,\mathbf{Z}[\frac{1}{p}])$, we have found three elements

$$c_1 = c(-1,-1),\ c_2 = c(p,-1),\ d = d(\mp 2^{l+1}, 3m)$$

as before, where $p = 6k \pm 1$. Let L be the subgroup of $K_2(2,\mathbf{Z}[\frac{1}{p}])$ generated by c_1, c_2, d. Then, L is abelian, and d is not central and of infinite order in $\mathrm{St}(2,\mathbf{Z}[\frac{1}{p}])$ $\mathrm{St}(2,\mathbf{Z}[\frac{1}{p}])$ by the structure of M_p. Therefore, $c_1^{n_1} c_2^{n_2} d^{n_3} = 1$ with $n_1, n_2, n_3 \in \mathbf{Z}$ implies $n_3 = 0$ and $c_1^{n_1} c_2^{n_2} = 1$. Then, since the image of c_1 (resp. c_2) in the stable K_2 over the field of real numbers is of infinite order (resp. trivial), n_1 must be 0 (cf. [5]). Hence, $L = \langle c_1, c_2, d \rangle \simeq Z \times Z_n \times Z$, where n is the order of c_2 and ≥ 2. □

In particular, for every $p \geq 5$, we get $K_2(2,\mathbf{Z}[\frac{1}{p}]) \not\simeq Z \times Z_{p-1}$ (cf. [8; Theorem 9]).

3. Some remarks around $\mathbf{SL}(2,\mathbf{Z}/4\mathbf{Z})$.

The group $\mathbf{SL}(2,\mathbf{Z}/4\mathbf{Z})$ is generated by

$$r_1 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},\ r_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},\ s = \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix},$$

and the subgroup generated by r_1, r_2 (resp. s) is isomorphic to $\mathfrak{A}_4$ (resp. Z_4). Hence, we see

(1) $\mathrm{SL}(2,\mathbf{Z}/4\mathbf{Z}) \simeq Z_4 \ltimes \mathfrak{A}_4$.

In particular, $\mathrm{SL}(2,\mathbf{Z}/4\mathbf{Z})^{mab} \simeq Z_4 \ltimes Z_3$. Furthermore, by some easy and routine calculation, we obtain the following as an appendix:

(2) $\mathrm{GL}(2,\mathbf{Z}/4\mathbf{Z}) \simeq \mathrm{GL}(2,\mathbf{F}_2[\xi]/(\xi^2)) \simeq \mathfrak{S}_3 \ltimes (Z_2)^4 \simeq Z_2 \ltimes (\mathfrak{S}_4 \times Z_2)$,

(3) $\mathrm{PGL}(2,\mathbf{Z}/4\mathbf{Z}) \simeq \mathrm{PGL}(2,\mathbf{F}_2[\xi]/(\xi^2)) \simeq \mathrm{SL}(2,\mathbf{F}_2[\xi]/(\xi^2)) \simeq \mathfrak{S}_4 \times Z_2$,

(4) $\mathrm{PSL}(2,\mathbf{Z}/4\mathbf{Z}) \simeq \mathrm{PSL}(2,\mathbf{F}_2[\xi]/(\xi^2)) \simeq \mathfrak{S}_4$.

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