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PSEUDO-UMBILICAL SUBMANIFOLDS OF A SPACE FORM $N^{n+p}(C)$

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Abstract. Let M be an n-dimensional pseudo-umbilical submanifold in an $(n+p)$-dimensional space form $N^{n+p}(C)$. In this paper, we obtain some generalizations of B. Y. Chen in [1].

§1. Introduction.

Let $N^{n+p}(C)$ be an $(n+p)$-dimensional space form with constant sectional curvature C, and M an n-dimensional submanifold in $N^{n+p}(C)$. Let h be the second fundamental form of the immersion and ξ the mean curvature vector, we denote by $\langle\cdot,\cdot\rangle$ the scalar product in $N^{n+p}(C)$. If there exists a function λ on M such that

$$\langle h(X,Y),\zeta\rangle=\lambda\langle X,Y\rangle \tag{1.1}$$

for all tangent vectors X,Y on M, then M is called a pseudo-umbilical submanifold in $N^{n+p}(C)$ (cf. [1]). It is clear that $\lambda\geq 0$. B. Y. Chen [1] proved: (1) Let M be an n-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(C)$. Then

$$\int_M[nH\Delta H+n(C+H^2)S-\left(2-\frac{1}{p}\right)S^2-n^2H^2C]dv\leq 0,$$

where S, H and dV denote the square of the length of h, the mean curvature of M and the volume element of M, respectively. (2) Let M be an n-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(C)$. If

$$nH\Delta H+n(C+H^2)S-\left(2-\frac{1}{p}\right)S^2-n^2H^2C\leq 0,$$

then the second fundamental form is parallel and S is constant.
In this paper, we obtain the following generalizations of (1) and (2).

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THEOREM 1. *Let M be an n-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(C)$. Then*

$$\int_M \left[n(C+H^2)S - \frac{3}{2}S^2 - n^2H^2C \right] dv \leq 0, \quad \text{for } p>1$$

and

$$\int_M \left[n(C+4H^2)S - \frac{3}{2}S^2 - n^2H^2C - \frac{5}{2}n^2H^4 \right] dv \leq 0, \quad \text{for } p>2.$$

THEOREM 2. *Let M be an n-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(C)$. If*

$$nH\Delta H + n(C+H^2)S - \frac{3}{2}S^2 - n^2H^2C \geq 0, \quad \text{for } p>1 \tag{1.2}$$

or

$$nH\Delta H + n(C+4H^2)S - \frac{3}{2}S^2 - n^2H^2C - \frac{5}{2}n^2H^4 \geq 0, \quad \text{for } p>2, \tag{1.3}$$

then the second fundamental form is parallel and S is constant. In particular, if the equality of (1.2) holds and $C = 1$, then M is totally geodesic or $n=2$ and M is a veronese surface in $S^4(1)$ and if the equality of (1.3) holds and $C = 1$, then M is totally geodesic, where $S^4(1)$ denotes the 4-dimensional unit sphere.

If $H = 0$ and $C = 1$, then Theorem 2 was proved jointly by A. M. Li and J. M. Li in [2].

§2. Local formulas.

We shall make use of the following convention on the ranges of indices:

$$A, B, \cdots, = 1, \cdots, n, n+1, \cdots, n+p; i, j, \cdots, = 1, \cdots, n; \alpha, \beta, \cdots, = n+1, \cdots, n+p.$$

We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$ in $N^{n+p}(C)$. Such that, restricted to M the vectors $e_1, \cdots, e_n$ are tangent to M and $\{\omega_A\}$ is the field of dual frames. Then the structure equations of $N^{n+p}(C)$ are given by

$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{CD} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = C(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Restricting these forms to M, we have

$$\omega_\alpha = 0, \omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, h_{ji}^\alpha = h_{ij}^\alpha. \tag{2.4}$$

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \tag{2.5}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.6}$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{2.7}$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \frac{1}{2}\sum_{ij} R_{\alpha\beta ij}\omega_i \wedge \omega_j, \tag{2.8}$$

$$R_{\alpha\beta ij} = \sum_k (h_{ik}^\alpha h_{kj}^\alpha - h_{ik}^\alpha h_{kj}^\alpha). \tag{2.9}$$

h_{ijk}^α and h_{ijkl}^α are given by

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum_k h_{kj}^\alpha \omega_{ki} - \sum_k h_{ik}^\alpha \omega_{kj} - \sum_\beta h_{ij}^\beta \omega_{\beta\alpha} \tag{2.10}$$

and

$$\sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha - \sum_l h_{ljk}^\alpha \omega_{li} - \sum_l h_{ilk}^\alpha \omega_{lj} - \sum_l h_{ijl}^\alpha \omega_{lk} - \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}, \tag{2.11}$$

respectively, where

$$h_{ijk}^\alpha = h_{ikj}^\alpha, \tag{2.12}$$

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha, = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{mj}^\alpha R_{mikl} + \sum_m h_{ij}^\beta R_{\beta\alpha kl}. \tag{2.13}$$

We call $h = \sum_{ij\alpha} h_{ij}^\alpha \omega_i \omega_j e_\alpha$ the second fundamental form of the immersed manifold M. We denote the square of the length of h by $S = \sum_{ij\alpha} (h_{ij}^\alpha)^2$. $\zeta = \frac{1}{n}\sum_\alpha trH_\alpha e_\alpha$ and $H = \|\zeta\| = \sqrt{\frac{1}{n}\sum_\alpha (trH_\alpha)^2}$ denote the mean curvature vector and the mean curvature of M, respectively. Here tr is the trace of the matrix $H_\alpha = (h_{ij}^\alpha)$. Now, let e_{n+p} be parallel to ζ. Then we get

$$trH_{n+p} = nH, trH_\alpha = 0, \alpha \neq n+p. \tag{2.14}$$

The Laplacian Δh_{ij}^α of the second fundamental form h_{ij}^α is defined by $\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$. By a simple calculation we have (cf. [1])

$$\frac{1}{2}\Delta S = \sum_{ijk\alpha} (h_{ijk}^\alpha)^2 + \sum_{ij\alpha} (h_{ij}^\alpha)\Delta h_{ij}^\alpha$$

$$= \sum_{ijk\alpha}(h_{ijk}^{\alpha})^2 + \sum_{ijk\alpha} h_{ij}^{\alpha}h_{kkij}^{\alpha} + \sum_{ijkl\alpha} h_{ij}^{\alpha}h_{lk}^{\alpha}R_{ijkl} + \sum_{ijkl\alpha} h_{ij}^{\alpha}h_{li}^{\alpha}R_{lkjk} + \sum_{ijk\alpha\beta} h_{ij}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk}$$

$$= \sum_{ijk\alpha}(h_{ijk}^{\alpha})^2 + nH\Delta H + n(C+H^2)S - n^2H^2C - \sum_{\alpha\beta}(trH_\alpha H_\beta)^2$$

$$+\sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2. \tag{2.15}$$

§3. Proofs of Theorems.

From (1.1) and (2.14) we get $\sum_{\alpha} trH_\alpha h_{ij}^{\alpha} = n\lambda\delta_{ij}, H^2 = \lambda$ and

$$h_{ij}^{n+p} = H\delta_{ij}. \tag{3.1}$$

In order to prove our Theorems, we need the following Lemma 1 which can be proved by diagonalizing the matrix (trH_iH_j) and using the inequality tr $(H_iH_j - H_jH_i)^2 \geq -2trH_i^2trH_j^2$ ([1]), and Lemma 2.

LEMMA 1 [2]. *Let* $H_i (i \geq 2)$ *be symmetric* $(n\times n)$*-matrices,* $S_i = trH_i^2$ *and* $S = \sum_i S_i$. *Then*

$$\sum_{ij} tr(H_iH_j - H_jH_i)^2 - \sum_{ij}(tr\,H_iH_j)^2 \geq -\frac{3}{2}S^2 \tag{3.2}$$

and the equality holds if and only if all $H_i = 0$ *or there exist two of* H_i *different from zero. Moreover, if* $H_1 \neq 0, H_2 \neq 0, H_i = 0 (i \neq 1,2)$, *then* $S_1 = S_2$ *and there exists an orthogonal* $(n\times n)$*-matrix* T *such that*

$$TH_1^t T = \sqrt{\frac{S_1}{2}}\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right),\quad TH_2^t T = \sqrt{\frac{S_1}{2}}\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)$$

LEMMA 2. *When* $p > 2$,,

$$\sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta}(tr\,H_\alpha H_\beta)^2 \geq -\frac{3}{2}S^2 + 3nH^2S - \frac{5}{2}n^2H^4$$

PROOF. Using (2.14) and (3.1), when $p > 2$, we have

$$\sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta}(trH_\alpha H_\beta)^2$$

$$= \sum_{\alpha\beta\neq n+p} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta\neq n+p}(trH_\alpha H_\beta)^2 - (trH_{n+p}^2)^2 \tag{3.3}$$

Applying Lemma 1 to (3.3) we have

$$\sum_{\alpha\beta} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta}(trH_\alpha H_\beta)^2 \geq -\frac{3}{2}(\sum_{\alpha\neq n+p} trH_\alpha^2)^2 - (trH_{n+p}^2)^2$$
$$= -\frac{3}{2}(S - trH_{n+p}^2)^2 - (trH_{n+p}^2)^2$$
$$= -\frac{3}{2}(S - nH^2)^2 - n^2H^4$$
$$= -\frac{3}{2}S^2 + 3nH^2S - \frac{5}{2}n^2H^4.$$

This completes the proof of Lemma 2.

Using (3.1) we can get

$$\sum_{ijk\alpha}(h_{ijk}^\alpha)^2 \geq \sum_{ik}(h_{iik}^{n+p})^2 = n\sum_i(\nabla_i H)^2. \tag{3.4}$$

It is obvious that

$$\frac{1}{2}\Delta H^2 = H\Delta H + \sum_i(\nabla_i H)^2. \tag{3.5}$$

Therefore, using Lemma 1, (3.4) and (3.5) when $p > 1$ by (2.15) we have

$$\frac{1}{2}\Delta S \geq \sum_{ijk\alpha}(h_{ijk}^\alpha)^2 + nH\Delta H + n(C+H^2)S - n^2H^2C - \frac{3}{2}S^2$$
$$\geq n\sum_i(\nabla_{iH})^2 + nH\Delta H + n(C+H^2)S - \frac{3}{2}S^2 - n^2H^2C$$
$$= \frac{1}{2}n\Delta H^2 + n(C+H^2)S - \frac{3}{2}S^2 - n^2H^2C \tag{3.6}$$

Since M is compact, form (3.6) we have

$$\int_M [n(C+H^2)S - \frac{3}{2}S^2 - n^2H^2C]dV \leq 0.$$

On the other hand, from the first inequality of (3.6), we know that if

$$nH\Delta H + n(C+H^2)S - \frac{3}{2}S^2n^2H^2C \geq 0 \tag{3.7}$$

and M is compact, then the second fundamental form h_{ij}^α is parallel and S is constant. In particular, if the equality of (3.7) holds and $C = 1$, then we see that the equality of (3.2) holds. So by Lemma 1 (3.7) implies that all $H_\alpha = 0$ (i.e. M is totally geodesic) or there exist two of H_α different from zero. In this case, by Lemma 1 we may therefore assume that

$$H_{n+1} = f\begin{pmatrix} 1 & 0 & \vdots & \\ 0 & -1 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & & \vdots & 0 \end{pmatrix}, \quad H_{n+2} = g\begin{pmatrix} 0 & 1 & \vdots & \\ 1 & 0 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & & \vdots & 0 \end{pmatrix}, f, g \neq 0.$$

Hence we have

$$trH_{n+1} = trH_{n+2} = 0. \tag{3.8}$$

Using (3.8) we find that $\sum_{\alpha} trH_{\alpha}h_{ij}^{\alpha} = 0$ and $H = 0$ identically. So by Lemma 1 the equality of (3.7) implies that M is totally geodesic or $n = 2$ and M is a veronese surface of $S^4(1)$.

On the other hand, when $p > 2$ using Lemma 2, (3.4) and (3.5) from (2.15) we get

$$\frac{1}{2}\Delta S \geq \sum_{ijk\alpha}(h_{ijk}^{\alpha})^2 + nH\Delta H + n(C+H^2)S - n^2H^2C - \frac{3}{2}S^2 + 3nH^2S - \frac{5}{2}n^2H^4$$

$$\geq \frac{1}{2}n\Delta H^2 + n(C+4H^2)S - \frac{3}{2}S^2 - n^2H^2C - \frac{5}{2}n^2H^4 \tag{3.9}$$

Thus, when M is compact by (3.9) we obtain

$$\int_M \left[n(C+4H^2)S - \frac{3}{2}S^2 - n^2H^2C - \frac{5}{2}n^2H^4 \right] dV \leq 0.$$

From the first inequality of (3.9), we see that if

$$nH\Delta H + n(C+4H^2)S - \frac{3}{2}S^2 - n^2H^2C - \frac{5}{2}n^2H^4 \geq 0. \tag{3.10}$$

then the second fundamental form h_{ij}^{α} is parallel and S is constant. In particular, for $p > 2$ when the equality of (3.10) holds and $C = 1$, by Lemma 1 we find that the equality of (3.2) holds and M is totally geodesic. This completes the proofs of Theorems.

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