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COIL ENLARGEMENTS OF ALGEBRAS

By

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Abstract. Let A be a finite dimensional, basic and connected algebra over an algebraically closed field k. We define a notion of weakly separating family in the Auslander-Reiten quiver of A which generalises the notion of a separating tubular family introduced by C. M. Ringel. Given an algebra A having a weakly separating family $\mathcal{T}$ of stable tubes, we say that an algebra B is a coil enlargement of A using modules from $\mathcal{T}$ if B is obtained from A by an iteration of admissible operations performed either on a stable tube of $\mathcal{T}$, or on a coil obtained from a stable tube of $\mathcal{T}$ by means of the operations done so far. The purpose of this paper is to describe the module category of B. We also give a criterion for the tameness of B if A is a tame concealed algebra.

Introduction.

Let k be an algebraically closed field, and A be a basic and connected finite dimensional k-algebra (associative, with an identity). We are interested in the study of the category mod A of finitely generated right A-modules. Among the nice features this category may possess is the existence of separating tubular families, introduced by Ringel in [12]. A well-known example of a class of algebras having a separating tubular family is the class of tame concealed algebras: in this case, the family consists of stable tubes. Further, Ringel introduced a notion of extension or coextension by branches using modules from a separating tubular family, then he showed that this process does not affect the existence of separating tubular families, so that the tilted algebras of euclidean type and the tubular algebras also possess such families [12]. Separating tubular families also occur in the module categories of wild algebras: this is the case, for instance, for all wild canonical algebras.

In [2, 3], the first two authors introduced the notion of admissible opera-

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tions which generalised that of branch extensions or coextensions. These allowed to define and describe components of the Auslander-Reiten quiver, called coils and multicoils, then a class of algebras, called multicoil algebras. Multicoil algebras are tame and actually of polynomial growth [2] (4.6), and this class of algebras seems to be of fundamental interest in the study of simply connected algebras of polynomial growth (see, for instance, [14, 16]). In particular, it follows from [14], [2] and [12] that if A is a strongly simply connected algebra of polynomial growth, then the support algebra of any indecomposable A-module is either a tilted algebra or a coil enlargement of a tame concealed algebra.

Our approach in this paper is different. We generalise the notion of separating tubular family as follows: a family of standard, pairwise orthogonal components $\mathcal{T}=(\mathcal{T}_i)_{i\in I}$ of the Auslander-Reiten quiver of A will be called a weakly separating family if the indecomposable modules not in $\mathcal{T}$ split into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that there is no non-zero morphism from $\mathcal{Q}$ to $\mathcal{P}$, from $\mathcal{Q}$ to $\mathcal{T}$, or from $\mathcal{T}$ to $\mathcal{P}$, while any non-zero morphism from $\mathcal{P}$ to $\mathcal{Q}$ factors through the additive subcategory generated by $\mathcal{T}$. A similar notion of weakly separating subcategory has been introduced in [8]. Denoting by ind A a full subcategory of mod A consisting of a complete set of non-isomorphic indecomposable A-modules, we express the foregoing properties by writing ind $A=\mathcal{P}\vee\mathcal{T}\vee\mathcal{Q}$. Given an algebra A having a weakly separating family $\mathcal{T}$ of stable tubes, we say that an algebra B is a coil enlargement of A using modules from $\mathcal{T}$ if B is obtained from A by an iteration of admissible operations performed either on a stable tube of $\mathcal{T}$, or on a coil obtained from a stable tube of $\mathcal{T}$ by means of the operations done so far. We also introduce numerical invariants c_B^- and c_B^+ which measure respectively the number of corays and rays inserted in the tubes of $\mathcal{T}$ by this sequence of admissible operations, and generalise respectively the notions of coextension and extension types. The aim of the present paper is to give a precise description of the module category of a coil enlargement algebra. Our results are summarised in the theorem.

THEOREM. *Let A be an algebra with a weakly separating family $\mathcal{T}$ of stable tubes and B be a coil enlargement of A using modules from $\mathcal{T}$. Then:*

(a) *B has a weakly separating family $\mathcal{T}'$ of coils obtained from the stable tubes of $\mathcal{T}$ by the corresponding sequence of admissible operations;*

(b) *there is a unique maximal branch coextension B^- of A which is a full subcategory of B, and c_B^- is the coextension type of B^-;*

(c) *there is a unique maximal branch extension B^+ of A which is a full sub-*

category of B, and c_B^+ is the extension type of B^+;

(d) ind $B=\mathcal{P}'\vee\mathcal{T}'\vee Q'$, *where $\mathcal{P}'$ consists of indecomposoble B^--modules, and Q' consists of indecomposable B^+-modules.*

If, in particular, A is a tame concealed algebra and $\mathcal{T}$ is its separating tubular family, we obtain handy criteria allowing to verify whether or not B is tame. Namely, we show that B is tame if and only if B^- and B^+ are tame, if and only if B is a multicoil alegebra, or if and only if the Tits form of B is weakly non-negative. This yields a class of tame algebras of finite global dimension for which all indecomposable modules are known and which satisfies the Tits form criterion (see [10]).

In [17], the third author shows how to iterate this process to obtain a larger class of tame algebras of finite global dimension satisfying the Tits form criterion.

Our paper is organised as follows. After a brief introductory section (1), in which we fix the notation and recall the relevant definitions, section (2) is devoted to the study of weakly separating families. We show in (2.7) part (a) of the above theorem, that is, the existence of weakly separating families is preserved by admissible operations. In section (3), we study the maximal branch enlargements which are full convex subcategories of a coil enlargment, proving in (3.5) parts (b) and (c) of the theorem. In section (4) we complete the description of the module category of a coil enlargement and prove the criteria for tameness of a coil enlargement of a tame concealed algebra.

1. Notation and preliminary definitions.

1.1. Throughout this paper, k will denote a fixed algebraically closed field. An algebra A will always mean a basic, connected, associative finite dimensionnal k-algebra with an identity. Thus there exists a connected bound quiver (Q_A, I) and an isomorphism $A\cong kQ_A/I$. Equivalently, $A=kQ_A/I$ may be considered as a k-linear category, of which the object class A_0 is the set of points of Q_A, and the morphism set $A(x, y)$ from x to y is the quotient of the k-vector space $kQ_A(x, y)$ of all formal linear combinations of paths in Q_A from x to y by the subspace $I(x, y)=I\cap kQ_A(x, y)$, see [6]. A full subcategory C of A is called convex if any path in A with source and target in C lies entirely in C.

By an A-module is always meant a finitely generated right A-module. We shall denote by mod A the category of A-modules and by ind A a full subcategory consisting of a complete set of non-isomorphic indecomposable A-modules. For a full subcategory $\mathcal{C}$ of mod A, we denote by add $\mathcal{C}$ the additive full subcategory of mod A consisting of the direct sums of indecomposable direct summands of the objects in $\mathcal{C}$. For two full subcategories $\mathcal{C}$ and $\mathcal{C}'$ of mod A, the notation $\mathrm{Hom}_A(\mathcal{C}, \mathcal{C}')=0$ will mean that $\mathrm{Hom}_A(M, M')=0$ for all M in $\mathcal{C}$, and M' in $\mathcal{C}'$. For a point i in Q_A, we denote by $S(i)$ the corresponding simple A-module, and by $P(i)$ (or $I(i)$) the projective cover (or injective envelope, respectively) of $S(i)$. The support of an A-module M is the full subcategory Supp M of A with object class $\{i \in A_0 \mid \mathrm{Hom}_A(P(i), M) \neq 0\}$. If C is a full convex subcategory of A such that A is obtained from C by a sequence of one-point extensions (or coextensions), we denote by $M|_C$ the restriction of an A-module M to C that is, the largest submodule (or quotient module, respectively) of M that is a C-module.

1.2. We shall use freely properties of the Auslander-Reiten translations $\tau = DTr$ and $\tau^{-1} = TrD$ and the Auslander-Reiten quiver Γ (mod A) of A, for whidh we refer to [5, 12]. We shall agree to identify points in Γ (mod A) with the corresponding indecomposable A-modules, and components with the corresponding full subcategories of ind A. A component Γ of Γ (mod A) is called standard if Γ is equivalent to its mesh category $k(\Gamma)$, see [6].

A translation quiver Γ is called a tube [7, 12] if it contains a cyclical path and its underlying topological space is homeomorphic to $S^1 \times \boldsymbol{R}^+$ (where S^1 is the unit circle, and $\boldsymbol{R}^+$ is the non-negative real line). A tube has only two types of arrows: pointing to infinity or pointing to the mouth. This also applies to sectional paths, that is, paths $x_0 \to x_1 \to \cdots \to x_m$ in Γ such that $x_{i-1} \neq \tau x_{i+1}$ for all $0<i<m$. A maximal sectional path consisting of arrows pointing to infinity (or to the mouth) is called a ray (or a coray, respectively). Tubes containing neither projectives nor injectives are called stable. It was shown in [15] that every standard component of Γ (mod A) with infinitely many τ-orbits is in fact a stable tube.

1.3. The one-point extension of the algebra A by the module M_A is the algebra $A[M] = \begin{bmatrix} A & 0 \\ M & k \end{bmatrix}$ with the usual addition and multiplication of matrices. The quiver of $A[M]$ contains Q_A as a full subquiver and an additional (extension) point that is a source. The $A[M]$-modules are usually identified with triples (V, X, φ), where V is a k-vector space, X an A-module and $\varphi: V \to \mathrm{Hom}_A(M, X)$ a k-linear map. An $A[M]$-linear map $(V, X, \varphi) \to (V', X', \varphi')$ is then identified with a pair (f, g), where $f: V \to V'$ is k-linear, $g: X \to X'$ is

A-linear and such that $\varphi' f = \mathrm{Hom}_A(M, g)\varphi$. One defines dually the one-point coextension $[M]A$ of A by M.

2. Weakly separating families.

2.1. DEFINITION. Let A be an algebra. A family $\mathcal{T} = (\mathcal{T}_i)_{i \in I}$ of components of $\Gamma(\mathrm{mod}\, A)$ is called a **weakly separating family** in $\mathrm{mod}\, A$ if the indecomposable A-modules not in $\mathcal{T}$ split into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that:

(WS1) the components $(\mathcal{T}_i)_{i \in I}$ are standard and pairwise orthogonal;

(WS2) $\mathrm{Hom}_A(\mathcal{Q}, \mathcal{P}) = \mathrm{Hom}_A(\mathcal{Q}, \mathcal{T}) = \mathrm{Hom}_A(\mathcal{T}, \mathcal{P}) = 0$; and

(WS3) any morphism from $\mathcal{P}$ to $\mathcal{Q}$ factors through $\mathrm{add}\, \mathcal{T}$.

Clearly, this definition is a straightforward generalisation of the definition of separating tubular families in [12] (3.1). Thus, every separating tubular family is a weakly separating family, but the converse is not true as we shall see in (2.8) below. We also note that, if $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{Q}$ are as in the definition, then $\mathcal{P}$ is closed under predecessors and $\mathcal{Q}$ is closed under successors. If $\mathcal{T}$ is a weakly separating family in $\mathrm{mod}\, A$, and $\mathcal{P}$, $\mathcal{Q}$ are as in the definition, we shall say that $\mathcal{T}$ separates (weakly) $\mathcal{P}$ from $\mathcal{Q}$ and write $\mathrm{ind}\, A = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$: this terminology is justified by the following lemma.

LEMMA. *Let A be an algebra, and $\mathcal{T}$ be a weakly separating family in* $\mathrm{mod}\, A$, *separating $\mathcal{P}$ from $\mathcal{Q}$. Then $\mathcal{P}$ and $\mathcal{Q}$ are uniquely determined by $\mathcal{T}$.*

PROOF. The proof of [12] (3.1) (4) p. 120 applies mutatis mutandis. We shall however repeat it here for the convenience of the reader. We start by defining a sequence of full subcategories of $\mathrm{ind}\, A$ as follows:

$$\mathcal{P}_0 = \{M \in \mathrm{ind}\, A \mid \mathrm{Hom}_A(M, \mathcal{T}) \neq 0,\ M \notin \mathcal{T}\}$$

and, for $i \geqq 1$,

$$\mathcal{P}_{2i-1} = \{M \in \mathrm{ind}\, A \mid \mathrm{Hom}_A(\mathcal{P}_{2i-2}, M) \neq 0,\ \mathrm{Hom}_A(\mathcal{T}, M) = 0\}$$

$$\mathcal{P}_{2i} = \{M \in \mathrm{ind}\, A \mid \mathrm{Hom}_A(M, \mathcal{P}_{2i-1}) \neq 0\}.$$

We shall prove by induction on i that $\mathcal{P}_0 \subseteqq \mathcal{P}_1 \subseteqq \cdots \subseteqq \mathcal{P}_i \subseteqq \mathcal{P}_{i+1} \subseteqq \cdots \subseteqq \mathcal{P}$. Clearly, $\mathrm{Hom}_A(\mathcal{Q}, \mathcal{T}) = 0$ implies $\mathcal{P}_0 \subseteqq \mathcal{P}$. Assume inductively that $\mathcal{P}_{2i-2} \subseteqq \mathcal{P}$, we shall show that $\mathcal{P}_{2i-2} \subseteqq \mathcal{P}_{2i-1}$. Since $\mathrm{Hom}_A(\mathcal{T}, \mathcal{P}) = 0$, we have $\mathrm{Hom}_A(\mathcal{T}, \mathcal{P}_{2i-2}) = 0$. On the other hand, $M \in \mathcal{P}_{2i-2}$ implies $\mathrm{Hom}_A(\mathcal{P}_{2i-2}, M) \neq 0$. Consequently, $\mathcal{P}_{2i-2} \subseteqq \mathcal{P}_{2i-1}$. We claim that $\mathcal{P}_{2i-1} \subseteqq \mathcal{P}$. Indeed, if this is not the case, there exists a module $M \in \mathcal{P}_{2i-1}$ which belongs to $\mathcal{T} \vee \mathcal{Q}$. Hence there exists $L \in \mathcal{P}_{2i-2} \subseteqq \mathcal{P}$ and a non-zero morphism $L \to M$ which can be factored through $\mathrm{add}\, \mathcal{T}$, then

$\mathrm{Hom}_A(\mathcal{T}, M)\neq 0$, a contradiction since $M\in\mathcal{P}_{2i-1}$. This shows our claim. Similarly, one proves that $\mathcal{P}_{2i-1}\subseteq\mathcal{P}_{2i}\subseteq\mathcal{P}$ for every i.

We shall now prove that $\mathcal{P}$ coincides with some $\mathcal{P}_i$. Assume first that P_A is an indecomposable projective module lying in $\mathcal{P}$. Since A is connected, there exists a sequence of indecomposable projectives $P_0, P_1, \cdots, P_{2i-1}, P_{2i}=P$ with $\mathrm{Hom}_A(P_0, \mathcal{T})\neq 0$, $\mathrm{Hom}_A(P_{2l-2}, P_{2l-1})\neq 0$ and $\mathrm{Hom}_A(P_{2l}, P_{2l-1})\neq 0$ for all $1\leqq l\leqq i$. We may clearly choose such a sequence with i minimal. We claim that all P_l in this sequence belong to $\mathcal{P}$. Indeed, if this is not the case, let t be maximal with $P_t\notin\mathcal{P}$. Then t is odd (for, otherwise, $\mathrm{Hom}_A(P_t, P_{t+1})\neq 0$ gives $P_{t+1}\notin\mathcal{P}$, a contradiction to the choice of t). Now $P_{t+1}\in\mathcal{P}$ and $\mathrm{Hom}_A(P_{t+1}, P_t)\neq 0$ imply that any non-zero morphism from P_{t+1} to P_t factors through $\mathrm{add}\,\mathcal{T}$. Hence $\mathrm{Hom}_A(P_{t+1}, \mathcal{T})\neq 0$ and we obtain a (strictly) shorter sequence by deleting $P_0, \cdots, P_t$: a contradiction to the minimality of i. This shows our claim that $P_l\in\mathcal{P}$ for all l. Now, let $M\in\mathcal{P}$. There exists an indecomposable projective module P_A with $\mathrm{Hom}_A(P, M)\neq 0$. Then $P\in\mathcal{P}$, and the previous argument implies that there exists l such that $P\in\mathcal{P}_l$. Consequently, $M\in\mathcal{P}_{l+1}$. This shows that $\mathcal{P}$ coincides with some $\mathcal{P}_i$ and hence is uniquely determined by $\mathcal{T}$. Consequently, $\mathcal{T}$ also determines uniquely Q. □

2.2. We recall the notion of admissible operations [2, 3]. Let A be an algebra and Γ be a standard component of $\Gamma\,(\mathrm{mod}\,A)$. For an indecomposable module X in Γ, called the pivot, three admissible operations are defined, depending on the shape of the support of $\mathrm{Hom}_A(X, -)|_\Gamma$ (this is by definition the subcategory of Γ consisting of the indecomposable modules M such that $\mathrm{Hom}_A(X, M)\neq 0$ and the morphisms $f: M\to N$ such that $\mathrm{Hom}_A(X, f)\neq 0$). These admissible operations yield in each case a modified algebra A' of A, and a modified component Γ' of Γ:

ad1) If the support of $\mathrm{Hom}_A(X, -)|_\Gamma$ is of the form:

$$X=X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

X is called an ad1)—pivot, we set $A'=(A\times D)[X\oplus Y_1]$, where D is the full $t\times t$ lower triangular matrix algebra, and Y_1 is the unique indecomposable projective-injective D-module. In this case, Γ' is obtained from Γ and $\Gamma\,(\mathrm{mod}\,D)$ by inserting a rectangle consisting of the modules $Z_{ij}=\left(k, X_i\oplus Y_j, \begin{pmatrix}1\\1\end{pmatrix}\right)$ for $i\geqq 0$, $1\leqq j\leqq t$, and $X'_i=(k, X_i, 1)$ for $i\geqq 0$, where Y_j, $1\leqq j\leqq t$, denote the indecomposable injective D-modules. If $t=0$, we set $A'=A[X]$, and the rectangle reduces to the ray formed by the modules of the form X'_i.

ad2) If the support of $\mathrm{Hom}_A(X, -)|_\Gamma$ is of the form:

$$Y_t \longrightarrow \cdots \longleftarrow Y_1 \longleftarrow X=X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

with $t \geqq 1$ (so that X is injective), X is called an ad2)-pivot, we set $A'=A[X]$. In this case, Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij}=\left(k,\ X_i \oplus Y_j,\ \binom{1}{1}\right)$ for $i \geqq 1$, $1 \leqq j \leqq t$ and $X_i'=(k, X_i, 1)$ for $i \geqq 0$.

ad3) If the support of $\mathrm{Hom}_A(X, -)|_\Gamma$ is of the form:

$$\begin{array}{ccccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow \cdots \longrightarrow & Y_t & & & & \\ \uparrow & & \uparrow & & \uparrow & & & & \\ X=X_0 & \longrightarrow & X_1 & \longrightarrow \cdots \longrightarrow & X_{t-1} & \longrightarrow & X_t & \longrightarrow & \cdots \end{array}$$

with $t \geqq 2$ (so that X_{t-1} is injective), X is called an ad3)-pivot, we set $A'=A[X]$. In this case, Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij}=\left(k,\ X_i \oplus Y_j,\ \binom{1}{1}\right)$ for $i \geqq 1$, $1 \leqq j \leqq i$ and $i>t$, $1 \leqq j \leqq t$, and $X_i'=(k, X_i, 1)$ for $i \geqq 0$.

It is shown in [3] that the component of Γ (mod A') containing X is Γ' and that, under suitable assumptions (which will always be satisfied in the present paper), Γ' is standard. The parameter t (which, in the notation above, is the number of modules of the form Y_j) is called the parameter of the operation: it is such that the number of rays in the rectangle of Γ' inserted by the admissible operation equals $t+1$. The dual operations ad1*) ad2*) ad3*) are also called admissible, and the parameter t then measures the number of corays inserted. We recall the following definition from [2, 3].

DEFINITION. A translation quiver Γ is called a **coil** if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \cdots, \Gamma_m=\Gamma$ such that Γ_0 is a stable tube and, for each $0 \leqq i < m$, Γ_{i+1} is obtained from Γ_i by an admissible operation.

We are now able to define the class of algebras we shall study in this work.

Definition. Let A be an algebra, and $\mathcal{T}$ be a weakly separating family of stable tubes of Γ (mod A). An algebra B is called a **coil enlargement** of A using modules from $\mathcal{T}$ if there is a finite sequence of algebras $A=A_0, A_1, \cdots, A_m=B$ such that, for each $0 \leqq j < m$, A_{j+1} is obtained from A_j by an admissible operation with pivot either on a stable tube of $\mathcal{T}$ or on a coil of Γ (mod A_j), obtained from a stable tube of $\mathcal{T}$ by means of the sequence of admissible operations done so far. The sequence $A=A_0, A_1, \cdots, A_m=B$ is then called an **admissible sequence.**

For instance, the representation-infinite tilted algebras of euclidean type, and the tubular algebras are, by [12] (4.9) (5.2), coil enlargements of a tame concealed algebra using only operations ad1) and ad1*). In this example, the size of the coils is measured by a numerical invariant, called the extension or coextension type (see [12] (4.7)), whose definition can be generalised as follows.

2.3 DEFINITION. Let B be a coil enlargement of A using modules from the weakly separating family $\mathcal{T}=(\mathcal{T}_i)_{i\in I}$ of stable tubes. The **coil type** $c_B=(c_B^-, c_B^+)$ of B is a pair of functions $c_B^-, c_B^+: I\to \boldsymbol{N}$ defined by induction on $0\leqq j<m$, where $A=A_0, A_1, \cdots, A_m=B$ is an admissible sequence.

(i) $c_A=c_0=(c_0^-, c_0^+)$ is the pair of functions $c_0^-=c_0^+$ such that, for each $i\in I$, the common value of $c_0^-(i)$ and $c_0^+(i)$ is the rank of the stable tube $\mathcal{T}_i$.

(ii) Assume $c_{A_{j-1}}=c_{j-1}=(c_{j-1}^-, c_{j-1}^+)$ is known, and let t_j be the parameter of the admissible operation from A_{j-1} to A_j, then $c_{A_j}=c_j=(c_j^-, c_j^+)$ is the pair of functions defined by:

$$c_j^-(i)=\begin{cases} c_{j-1}^-(i)+t_j+1 & \text{if the operation is ad1*) ad2*) or ad3*) with} \\ & \text{pivot in the coil of } \Gamma(\operatorname{mod} A_{j-1}) \text{ arising from } \mathcal{T}_i, \\ c_{j-1}^-(i) & \text{otherwise} \end{cases}$$

and

$$c_{j-1}^+(i)=\begin{cases} c_{j-1}^+(i)+t_j+1 & \text{if the operation is ad1) ad2) or ad3) with} \\ & \text{pivot in the coil of } \Gamma(\operatorname{mod} A_{j-1}) \text{ arising from } \mathcal{T}_i, \\ c_{j-1}^+(i) & \text{otherwise} \end{cases}$$

It follows from the definition that the coil type does not depend on the admissible sequence leading from A to B since, for each $i\in I$, $c_B^+(i)$ and $c_B^-(i)$ measure respectively the total number or rays and corays inserted in the tube $\mathcal{T}_i$ by the sequence of admissible operations.

If all but at most finitely many values of each of the functions c_B^- and c_B^+ equal 1, we shall replace each by a (finite) sequence, containing at least two terms and including all those which are larger than 1. To enable us to compare the number of rays and corays inserted in any individual tube, we shall use the following additional conventions:

(1) The finite sequences for c_B^- and c_B^+ contain exactly the same number of terms, where we agree to add to either sequence as many 1's as necessary.

(2) c_B^- is a non-decreasing sequence, that is, if $c_B^-=(c_B^-(i_1), \cdots, c_B^-(i_s))$ then $c_B^-(i_1)\leqq c_B^-(i_2)\leqq \cdots \leqq c_B^-(i_s)$.

(3) c_B^+ is the sequence consisting of the values of c_B^+ corresponding to the

values of c_B^-, that is, if c_B^- is as in (2), then $c_B^+=(c_B^+(i_1), \cdots, c_B^+(i_s))$.

2.4. The main theorem of this section asserts that, if A is an algebra with a weakly separating family $\mathcal{T}$ of stable tubes, and B is a coil enlargement of A using modules from $\mathcal{T}$, then the family $\mathcal{T}'$ of coils of $\Gamma(\mathrm{mod}\, B)$ obtained from the stable tubes of $\mathcal{T}$ is weakly separating in $\mathrm{mod}\, B$. In order to show this result, we shall need three lemmata. We shall always use the notation of (2.2).

LEMMA. *Let A be an algebra, Γ be a standard component of $\Gamma(\mathrm{mod}\, A)$ and $X\in\Gamma$ be the pivot of an admissible operation. Let A' be the modified algebra and Γ' be the modified component. Any indecomposable A'-module whose restriction to A has an indecomposable direct summand of the form X_i, for some $i\geqq 0$, belongs to Γ'.*

PROOF. We may assume, by duality, that the admissible operation is one of ad1), ad2), ad3). For an A'-module M, we let M_0 denote its restriction to $A\times D$, if the operation is ad1), or to A if it is ad2) or ad3). Denoting by e the extension point of A', we represent A'-modules by triples (M_e, M_0, φ_M), where M_e is a finite dimensional k-vector space and φ_M is a k-linear map from M_e to $\mathrm{Hom}_{A\times D}(X\oplus Y_1, M_0)$ or to $\mathrm{Hom}_A(X, M_0)$, respectively.

Let $M=(M_e, M_0, \varphi_M)$ be an indecomposable A'-module such that M_0 has an indecomposable direct summand of the form X_i for some $i\geqq 0$. We can assume $M_e\neq 0$ (otherwise the indecomposability of M implies that $M\cong X_i$, and there is nothing to show). Let $p: M_0\to X_i$ be a projection morphism with section $q: X_i\to M_0$. There exists a morphism $f=(f_e, f_0): M\to X_i'$ with $f_e\neq 0$ and $f_0=p$ (for, if this is not the case, $f=(0, p): M\to X_i$ is a retraction with section $(0, q)$). We may choose i to be minimal with this property. If f is an isomorphism there is nothing to show. Assume thus that this is not the case.

In the case ad1), f factors through the right minimal almost split morphism ending in X_i'. Using the minimality of $i\geqq 1$, we obtain a morphism $g=(g_e, g_0)$: $M\to Z_{it}$ with $g_e\neq 0$. In the cases ad2), ad3), if $i=0$, then $\mathrm{Im}\, f\nsubseteq X_0=\mathrm{rad}\, P(e)$. Hence f is a retraction and $M\cong X_0'=P(e)$, a contradiction to the assumption that f is not an isomorphism. If $i>0$, f factors through the right minimal almost split morphism ending in X_i'. In ad2), using the minimality of $i\geqq 1$, we obtain a morphism $g=(g_e, g_0): M\to Z_{it}$ with $g_e\neq 0$. In ad3), we obtain similarly, for $1\leqq i\leqq t$, a morphism $g=(g_e, g_0): M\to Z_{ii}$ with $g_e\neq 0$ and, for $i>t$, a morphism $g=(g_e, g_0): M\to Z_{it}$ with $g_e\neq 0$. Using an obvious descending induction on $i+j$, we thus show that there exists an isomorphism $M\to Z_{ij}$ for some j. □

2.5. Before the next lemma, we observe that, if Γ is a coil in $\Gamma(\text{mod } A)$, and X is an ad2)-pivot in Γ such that the support of $\text{Hom}_A(X, -)|_\Gamma$ is of the form

$$Y_t \longleftarrow \cdots \longleftarrow Y_1 \longleftarrow X=X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

then it follows from [3], Theorem (A), that each of the modules Y_i is injective (thus, in the notation of [3] (2.3), $\Gamma=\Gamma^*$).

LEMMA. *Let A be an algebra with a family $\mathcal{T}$ of coils weakly separating $\mathcal{P}$ from Q, Γ be a coil in $\mathcal{T}$ and X be an ad2)-pivot in Γ. Let $A'=A[X]$, where e denotes the extension point. Let $\mathcal{P}'$, $\mathcal{T}'$, Q' be the classes in* ind A' *defined as follows:*

(i) $\mathcal{P}'=\mathcal{P}$;

(ii) *$\mathcal{T}'$ consists of all indecomposables $M_{A'}$ such that $M_e=0$ and $M=M|_A$ is in $\mathcal{T}$, or $M_e\neq 0$ and $M|_A$ has an indecomposable direct summand of the form X_i, for some $i\geqq 0$; and*

(iii) *Q' consists of all indecomposables $M_{A'}$ such that $M_e=0$ and $M=M|_A$ is in Q, or $M=(k, 0, 0)$, or $M_e\neq 0$ and the indecomposable direct summands of $M|_A$ belong either to the set $\{Y_1, \cdots, Y_t\}$ or to the support of* $\text{Hom}_A(X, -)|_Q$.

Then ind $A'=\mathcal{P}'\vee\mathcal{T}'\vee Q'$, *and $\mathcal{T}'$ separates weakly $\mathcal{P}'$ from Q'.*

PROOF. Let M be an indecomposable A'-module. If $M_e=0$, then $M=M|_A$ and $M\in\mathcal{P}\vee\mathcal{T}\vee Q$. Hence $M\in\mathcal{P}'\vee\mathcal{T}'\vee Q'$. If $M_e\neq 0$, and $M|_A=0$ then $M=(k, 0, 0)$ and $M\in Q'$. If $M_e\neq 0$ and $M|_A\neq 0$, the indecomposable direct summands of $M|_A$ belong to $\{X_i | i\geqq 0\}\cup\{Y_1, \cdots, Y_t\}\cup Q$, since each of these summands receives a non-zero morphism from X. Hence $M\in\mathcal{T}'\vee Q'$.

By (2.4), $\mathcal{T}'=\Gamma'\vee\mathcal{T}_0$, where $\mathcal{T}_0$ consists of all the components of $\mathcal{T}$ distinct from Γ. Hence, by [3] (2.3), all the components of $\mathcal{T}'$ are standard and pairwise orthogonal. Clearly, $\text{Hom}_{A'}(\mathcal{T}'\vee Q', \mathcal{P}')=0$. Also, $\text{Hom}_{A'}(Q', \mathcal{T}_0)=0$. Let now $M\in Q'$ and $N\in\Gamma'$. It is easily seen that $\text{Hom}_{A'}(M, N)=0$ in each of the following four cases:

1) $M_e=0$;

2) $M=(k, 0, 0)$;

3) $M_e\neq 0$, $N_e=0$ and $N\notin\{Y_1, \cdots, Y_t\}$; and

4) $M_e\neq 0$, $N_e\neq 0$ and $N=X_i'=(k, X_i, 1)$ for some $i\geqq 0$.

Let $M_e\neq 0$ and $N=Y_j$ for some $1\leqq j\leqq t$, or $N=Z_{ij}$ for some $i\geqq 0$, $1\leqq j\leqq t$. Then any non-zero morphism $f\in\text{Hom}_{A'}(M, N)$ factors through the right minimal almost split morphism ending in N, and an obvious induction on j, or $i+j$, respectively, yields $f=0$ because $\text{Hom}_A(M_0, X_0)=0$. This completes the proof that $\mathrm{Hom}_{A'}(Q', \Gamma'')=0$.

Consider now a morphism $f: M \to N$ with $M \in \mathcal{P}'$, $N \in Q'$. Since $\mathcal{P}'=\mathcal{P} \subseteq \mathrm{ind}\, A$, we have $\mathrm{Im}\, f \subseteq N|_A$. If $N|_A$ is indecomposable, it lies in Q. If not, its indecomposable direct summands belong to $Q \cup \{Y_1, \cdots, Y_t\}$ by (2.4). Since $\{Y_1, \cdots, Y_t\} \subseteq \mathcal{T} \subseteq \mathcal{T}'$, and $\mathcal{T}$ is weakly separating, it follows that f factors through a module in add $\mathcal{T}'$. □

2.6. Let Γ be a coil in $\Gamma(\mathrm{mod}\, A)$ and X be an ad3)-pivot in Γ such that the support of $\mathrm{Hom}_A(X, -)|_\Gamma$ is of the form

$$\begin{array}{ccccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow \cdots\cdots \longrightarrow & Y_t & & & & \\ \uparrow & & \uparrow & & \uparrow & & & & \\ X=X_0 & \longrightarrow & X_1 & \longrightarrow \cdots\cdots \longrightarrow & X_{t-1} & \longrightarrow & X_t & \longrightarrow & \cdots\cdots \end{array}$$

with $t \geqq 2$ and X_{t-1} injective. Consider the subquiver Γ' of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1} (1 < i \leqq t)$ if they exist, and denote by Γ^* the connected component of Γ' containing X (see [3] (2.4)). By [3], Theorem (A), if an indecomposable module M belongs to Γ but not to Γ^*, then $M \cong \tau_A^{-s} Y_j$ for some $1 \leqq s \leqq t-j$, so that it belongs to the coray of Γ passing through Y_{j+s}.

LEMMA. *Let A be an algebra with a family $\mathcal{T}$ of coils weakly separating $\mathcal{P}$ from Q, Γ be a coil in $\mathcal{T}$ and X be an ad3)-pivot in Γ. Let $A'=A[X]$, where e denotes the extension point. Let $\mathcal{P}'$, $\mathcal{T}'$, Q' be the classes in* ind A' *defined as follows:*

(i) $\mathcal{P}'=\mathcal{P}$;

(ii) *$\mathcal{T}'$ consists of all indecomposables $M_{A'}$ such that $M_e=0$ and $M=M|_A$ is in $(\mathcal{T} \setminus \Gamma) \cup \Gamma^*$, or $M_e \neq 0$ and $M|_A$ has an indecomposable direct summand of the form X_i, for some $i \geqq 0$; and*

(iii) *Q' consists of all indecomposables $M_{A'}$ such that $M_e=0$ and $M=M|_A$ is in $Q \cup (\Gamma \setminus \Gamma^*)$, or $M=(k, 0, 0)$, or $M_e \neq 0$ and the indecomposable direct summands of $M|_A$ belong either to the set $\{Y_1, \cdots, Y_t\}$ or to the support of* $\mathrm{Hom}_A(X, -)|_Q$.

Then ind $A'=\mathcal{P}' \vee \mathcal{T}' \vee Q'$, *and $\mathcal{T}'$ separates weakly $\mathcal{P}'$ from Q'.*

PROOF. Similar to the proof of (2.5), except for the modules in $\Gamma \setminus \Gamma^*$ which in ind A lie in $\mathcal{T}$ and in ind A' lie in Q'. For these modules, we observe that:

1) $\mathrm{Hom}_A(M, N)=0$ whenever M belongs to $\Gamma \setminus \Gamma^*$ and N belongs to Γ^*; and

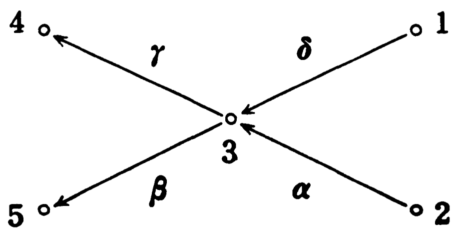
2) any morphism from M to N, where M belongs to $\mathcal{P}'=\mathcal{P}$ and N belongs

to $\Gamma\backslash\Gamma^*$, factors through a module in Γ' (namely, one of the modules Y_j, with $1\leqq j\leqq t$). □

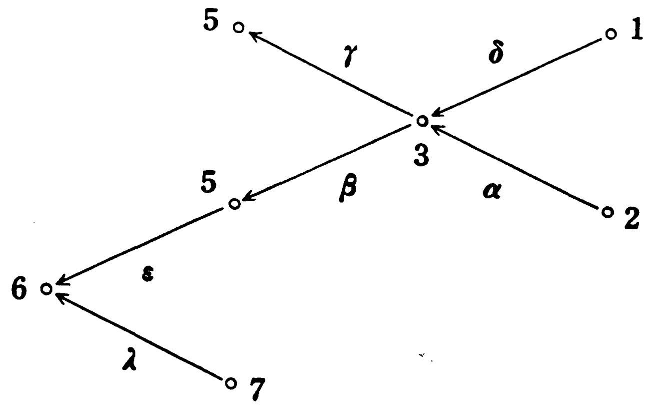
2.7. THEOREM. *Let A be an algebra with a family $\mathcal{T}$ of stable tubes weakly separating $\mathcal{P}$ from $\mathcal{Q}$, and let B be a coil enlargement of A using modules from $\mathcal{T}$. Then* $\operatorname{mod} B$ *has a family $\mathcal{T}'$ of coils, weakly separating $\mathcal{P}'$ from $\mathcal{Q}'$.*

PROOF. Let $A=A_0, A_1, \cdots, A_{m-1}, A_m=B$ be an admissible sequence. We prove the statement by induction on $0\leqq i\leqq m$. It holds for $i=0$ by the hypothesis on A. Assume that it holds for some $0\leqq i<m$. That it also holds for $i+1$ follows from [12] (4.7) (1) p. 230, it the admissible operation used in passing from A_i to A_{i+1} is ad1) or ad1*), and from (2.5) (2.6) and their duals in the remaining cases. □

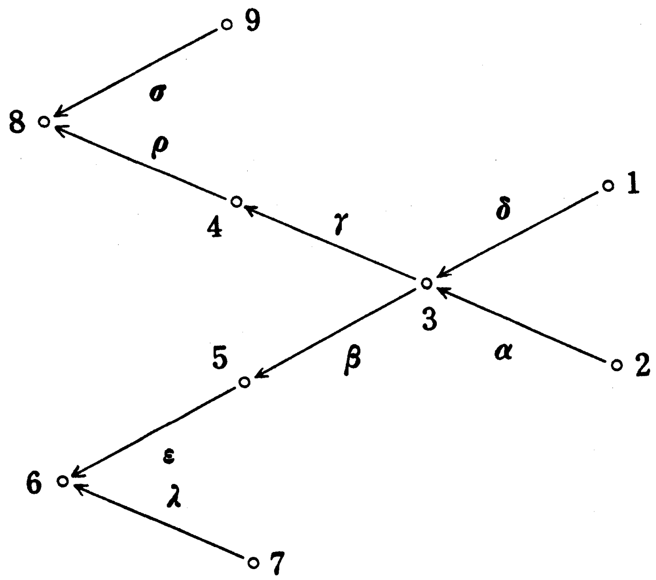
2.8. EXAMPLE. Let $A=A_0$ be the tame hereditary algebra given by the quiver



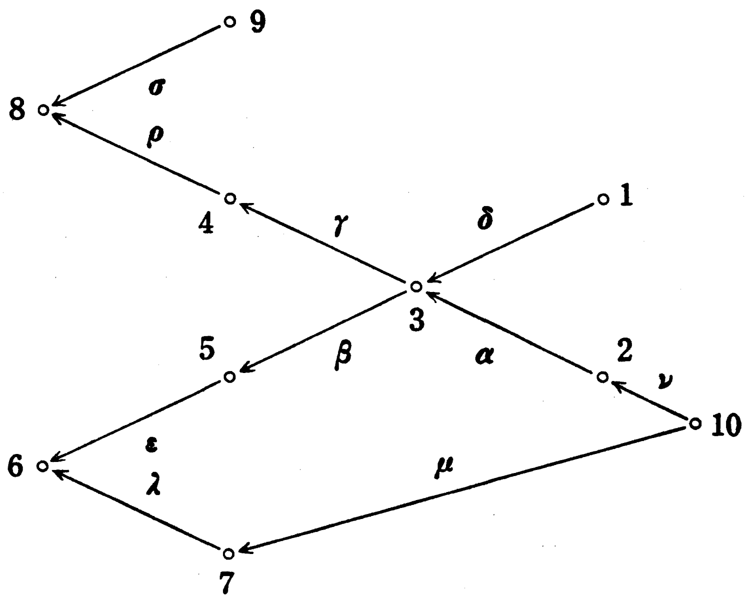
Its type is $c_A=((2, 2, 2), (2, 2, 2))$. The algebra A_1 given by the quiver



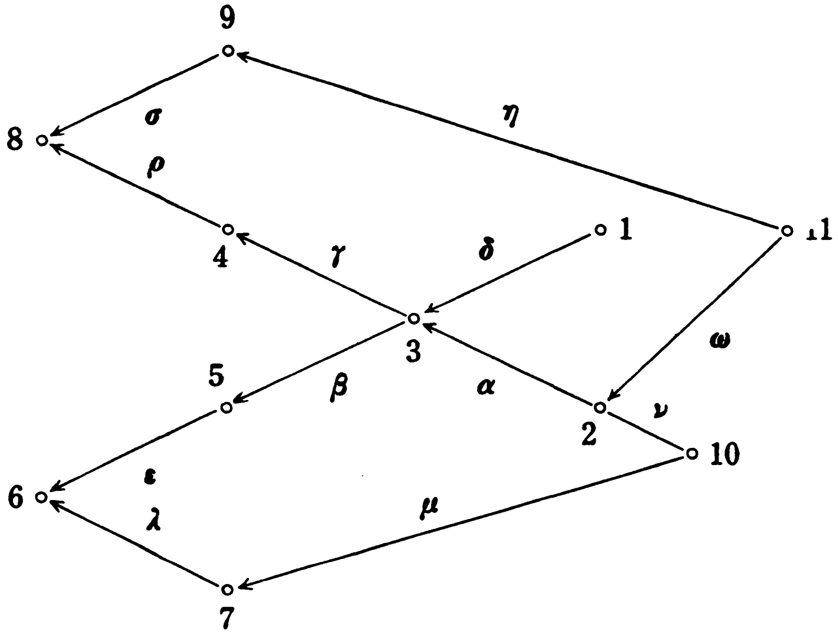
bound by $\delta\beta\varepsilon=0$, is obtained from A by an admissible operation of type ad1*) with pivot the simple regular A-module of dimension-vector $\begin{smallmatrix}0&&0\\&1&\\1&&1\end{smallmatrix}$. Its type is $c_{A_1}=((2, 2, 4), (2, 2, 2))$. Then A_1 is a tilted algebra of euclidean type $\tilde{D}_6$ having a complete slice in its postprojective component and a (unique) coinserted tube. The algebra A_2 given by the quiver



bound by $\delta\beta\varepsilon=0$, $\delta\gamma\rho=0$, is obtained from A_1 by an admissible operation of type ad1*) with pivot in a stable tube of $\Gamma(\mathrm{mod}\, A_1)$, having dimension-vector $\begin{smallmatrix} & 1 & & 0 \\ & & 1 & \\ & 0 & & 1 \\ 0 & & & \\ & 0 & & \end{smallmatrix}$. Its type is $c_{A_2}=((2, 4, 4), (2, 2, 2))$, and it is a tubular algebra. By [12], both $\mathrm{mod}\, A_1$ and $\mathrm{mod}\, A_2$ have separating tubular families. The algebra A_3 given by the quiver



bound by $\delta\beta\varepsilon=0$, $\nu\alpha\gamma=0$, $\mu\lambda=\nu\alpha\beta\varepsilon$, $\delta\gamma\rho=0$, is obtained from A_2 by an admissible operation of type ad2), with pivot the indecomposable A_2-module having dimension-vector $\begin{smallmatrix} & 0 & & \\ 0 & & & \\ & 0 & & 0 \\ & & 1 & \\ & 1 & & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{smallmatrix}$. Its type is $c_{A_3}=((2, 4, 4), (2, 4, 2))$. The separating tubular family of mod A_2 arising from the family of stable tubes of mod A becomes, by (2.7), a weakly separating family in mod A_3, containing a non-trivial coil (actually, a quasi-tube, in the sense of [13]). Finally, the algebra A_4 given by the quiver



bound by $\delta\beta\varepsilon=0$, $\nu\alpha\gamma=0$, $\mu\lambda=\nu\alpha\beta\varepsilon$, $\delta\gamma\rho=0$, $\omega\alpha\beta=0$, $\eta\sigma=\omega\alpha\gamma\rho$, is obtained from A_3 by an admissible operation of type ad2). Its type is $c_{A_4}=((2, 4, 4), (2, 4, 4))$. The weakly separating family of coils in mod A_3 becomes in mod A_4 a weakly separating family $\mathcal{T}_4$ of coils. However, $\mathcal{T}_4$ is not a separating family in the sense of [12] (3.1). Indeed, let M and N be the indecomposable A_4-modules given by

$$M(a)=\begin{cases} k & \text{if } a=3, 4, 5, 6, 7, 8, 9\,; \\ 0 & \text{if } a=1, 2, 10, 11\,; \end{cases}$$

and

$$N(a)=\begin{cases} k & \text{if } a=2, 3, 7, 9, 10, 11\,; \\ 0 & \text{if } a=1, 4, 5, 6, 8\,; \end{cases}$$

with the obvious morphisms. By (2.5), $M\in\mathcal{P}_4$ and $N\in\mathcal{Q}_4$, where $\mathcal{T}_4$ weakly separates $\mathcal{P}_4$ from $\mathcal{Q}_4$. On the other hand, the morphism $f: M\to N$ defined by $f_a=1_k$ if $M(a)=N(a)=k$, and $f_a=0$ otherwise has for image the semisimple module $S=S(3)\oplus S(7)\oplus S(9)$. Each of the simple summands of S lies in a different coil in $\mathcal{T}_4$, so that, while f factors through add $\mathcal{T}_4$, it does not factor through each coil in $\mathcal{T}_4$.

3. Maximal branch enlargements inside a coil enlargement.

3.1. Let A be an algebra with a weakly separating family $\mathcal{T}$ of stable tubes, and B be a coil enlargement of A using modules from $\mathcal{T}$. By (2.7), ind B $=\mathcal{P}'\vee\mathcal{T}'\vee\mathcal{Q}'$, where $\mathcal{T}'$ is a family of coils weakly separating $\mathcal{P}'$ from $\mathcal{Q}'$. We want to give a finer description of the full subcategories $\mathcal{P}'$ and $\mathcal{Q}'$ of ind B. For this purpose, we shall show that the admissible sequence leading from A to B can be replaced by another, which consists of a block of operations of type ad1*), followed by a block of operations of types ad1), ad2), ad3), and, dually, that it can be replaced by another admissible sequence, which consists of a block of operations of type ad1), followed by a block of operations of types ad1*), ad2*), ad3*). This is the aim of the following technical lemmata, the first of which gives a sufficient condition for two admissible operations to commute.

LEMMA. *Let A be an algebra with a weakly separating family $\mathcal{T}$ of coils, and A' be obtained from A by applying two admissible operations using modules from $\mathcal{T}$. If:*

(i) *the pivot of the second operation belongs to no ray, or coray, inserted by the first; and*

(ii) *in case the second operation is of type ad3) or ad3*) and is applied first to A, the pivot of the first still belongs to the family of coils obtained from $\mathcal{T}$;*

then, denoting by A'' the algebra obtained from A by applying the two operations in the reverse order, $A'\cong A''$.

PROOF. Since the admissible operations consist of one-point extensions or coextensions, it is easily seen that both algebras have the same bound quiver. □

In particular, this lemma covers the case of two consecutive operations ad2) and ad1*) (or ad3) and ad1*)), since the pivot of ad1*) must be a coray module, and therefore it cannot belong to the rectangle inserted by ad2) (or ad3), respectively).

3.2. Lemma (3.1) also covers the case of two consecutive operations of types ad1) with pivot $X=X_0$ and ad1*) with pivot $Y\not\cong X_0'=(k, X, 1)$ (in the notation of (2.2)). Indeed, assume that Supp $\mathrm{Hom}_A(X, -)|_{\Gamma}$ consists of an infinite sectional path starting with X (so that X can be chosen as ad1)-pivot) and $\text{Supp Hom}_A(-, X)|_\Gamma$ consists of an infinite sectional path ending with X (so that X can also be chosen as ad1*)-pivot). Apply the operation ad1) with pivot $X=X_0$, then X_0' is the only module in the rectangle inserted by ad1) that can be an ad1*)-pivot. If however $Y \cong X_0'$, we can apply the following lemma to replace the given sequence by a sequence consisting of ad1*) with pivot X followed by ad1) with pivot $X_0''=(X, k, 1)$.

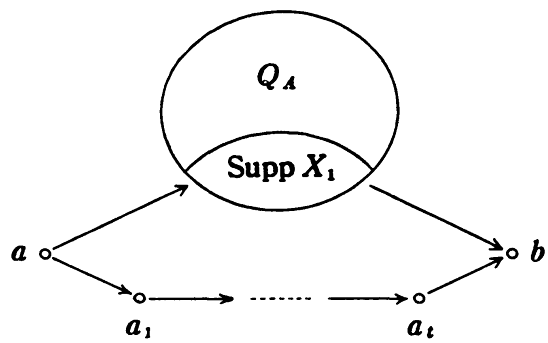
LEMMA. *Let A be an algebra with a weakly separating family $\mathcal{T}$ of coils, and X be an indecomposable in a coil of $\mathcal{T}$ which is an* ad1) *and* ad1*) *pivot. Let A' be the algebra obtained by first applying* ad1) *with pivot X, then* ad1*) *with pivot $X'=(k, X, 1)$, and A'' be the algebra obtained by first applying* ad1*) *with pivot X, followed by* ad1) *with pivot $X''=(X, k, 1)$. Then $A' \cong A''$.*

PROOF. Clearly, both algebras have the same bound quiver. □

3.3. Let A be an algebra with a weakly separating family $\mathcal{T}$ of coils, and Y be an indecomposable in a coil of $\mathcal{T}$ which is an ad1) and ad1*) pivot. Let A_1 be obtained from A by applying ad1) with pivot Y, and A_2 be obtained from A_1 by applying ad2*) with pivot $X=P(a)$, where a is the extension point of A_1. Let Γ be the standard coil of $\Gamma(\text{mod } A_1)$ containing X. Then the support of $\text{Hom}_A(-, X)|_\Gamma$ is of the form

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0=X \longleftarrow Y_1 \longleftarrow \cdots \longleftarrow Y_t$$

with $t \geq 1$ and $X_1=Y$. Then $Y_1, \cdots, Y_t$ are indecomposable projective A_1-modules corresponding respectively to points $a_1, \cdots, a_t$ in the quiver Q_{A_1} of A_1. Let b be the coextension point of $A_2=[X]A_1$. The bound quiver of A_2 is of the form



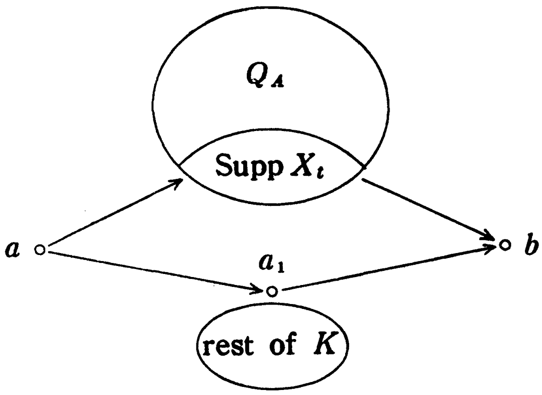
with $A_2(a, b)$ one-dimensional. Let A' be the full convex subcategory of A_2 consisting of all points except a. Then A' is the coextension of A at X_1 by the coextension branch K consisting of the points $b, a_t, \cdots, a_1$, that is, in the notation of [12] (4.7), we have $A' \cong [K, X_1]A$ and $A_2 \cong A'[I(b)]$, where $I(b)$ denotes the indecomposable injective A'-module corresponding to b. That ad1) followed by ad2*) can be replaced by ad1*) followed by ad2) is the content of the next lemma whose proof follows from the discussion above. For the notion (and notation) of branch extension, we again refer the reader to [12] (4.7).

LEMMA. *Let A be an algebra with a weakly separating family $\mathcal{T}$ of coils, and Y be an indecomposable in a coil of $\mathcal{T}$ which is an ad1) and ad1*)-pivot. Let a be the extension point of $A[Y]$ and K be the branch $a \to a_1 \to \cdots \to a_t$. Let b be the coextension point of $[Y]A$ and K' be the branch $a_1 \to \cdots \to a_t \to b$. Then $[P(a)](A[Y, K]) \cong ([K', Y]A)[I(b)]$.* □

3.4. Let A_1 be an algebra with a weakly separating family $\mathcal{T}$ of coils, and X be an indecomposable in a coil Γ of $\mathcal{T}$ which is an ad3*)-pivot. The support of $\operatorname{Hom}_{A_1}(-, X)|_\Gamma$ is of the form

$$\begin{array}{ccccccccc} & & Y_t & \longrightarrow \cdots \longrightarrow & Y_2 & \longrightarrow & Y_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow X_t & \longrightarrow & X_{t-1} & \longrightarrow \cdots \longrightarrow & X_1 & \longrightarrow & X_0 = X \end{array}$$

with $t \geq 2$. We shall assume for the time being that A_1 was obtained from an algebra A by applying r consecutive operations of type ad1), the first of which had $Y = X_t$ as a pivot, and these operations built up a branch K in A_1 with points $a, a_1, \cdots, a_t$, so that X_{t-1} and Y_t are the indecomposable projective A_1-modules corresponding respectively to a and a_1, and both Y_1 and $\tau_{A_1}^{-1} Y_1$ are ray modules in Γ. Let $A_2 = [X]A_1$ and let b denote the coextension point of A_2. The bound quiver of A_2 is of the form



with $A_2(a, b)$ one-dimensional. If follows from our assumptions that $X|_A = X_t$ and $X|_K$ is the indecomposable injective K-module in a_1. Let A' be the full convex subcategory of A_2 consisting of all points except a. Then $A' \cong [K', X_t]A$, where K' is the branch with points $b, a_1, \cdots, a_t$ and $A_2 = A'[Z]$, where Z is the indecomposable A'-module such that $Z|_A = X_t$ and $Z|_{K'}$ is the indecomposable projective K'-module in a_1. Let Γ' be the standard coil of Γ (mod A') containing Z. It follows from the shape of the bound quiver of A' and the description of the indecomposable module Z that the support of $\mathrm{Hom}_{A'}(Z, -)|_{\Gamma'}$ is of the form

$$\begin{array}{ccccccccccc} U_1 & \longrightarrow & U_2 & \longrightarrow \cdots\cdots \longrightarrow & U_t & & & & & & \\ \uparrow & & \uparrow & & \uparrow & & & & & & \\ Z=Z_0 & \longrightarrow & Z_1 & \longrightarrow \cdots\cdots \longrightarrow & Z_{t-1} & \longrightarrow & Z_t = X_t & \longrightarrow & Z_{t+1} & \longrightarrow & \cdots\cdots \end{array}$$

with $t \geqq 2$. Since $A_2 = [X]A_1$ and X, Z belong to the standard coil containing X_t in Γ(mod A_2), we get that $U_1 = Y_t, U_2 = Y_{t-1}, \cdots, U_t = Y_1$. That the sequence of operations of type ad1) that build up K followed by ad3*) (with pivot X) can be replaced by the sequence of operations of type ad1*) that build up K' followed by ad3) (with pivot Z) is the content of the next lemma, whose proof follows from the discussion above.

LEMMA. *Let A be an algebra with a weakly separating family $\mathcal{T}$ of coils and Y be an indecomposable in a coil of $\mathcal{T}$ which is an ad1) and ad1*)-pivot. Let c be the root of a branch of length t, and let K and K' be the branches constructed as follows: K consists of a root a, the branch in c and an arrow $a \to c$, while K' consists of a root b, the branch in c and an arrow $c \to b$. Let X be the indecomposable $A[Y, K]$-module such that $X|_A = Y$ and $X|_K$ is the indecomposable injective K-module in c, and let Z be the indecomposable $[K', Y]A$-module such that $Z|_A = Y$ and $Z|_{K'}$ is the indecomposable projective K'-module in c. Then $[X](A[Y, K]) \cong ([K', Y]A)[Z]$.* □

3.5. We are now able to prove the main result of this section.

THEOREM. *Let A be an algebra with a weakly separating family $\mathcal{T}$ of stable tubes, and B be a coil enlargement of A using modules from $\mathcal{T}$. Then:*

a) *There is a unique maximal branch coextension B^- of A which is a full convex subcategory of B, and c_B^- is the coextension type of B^-.*

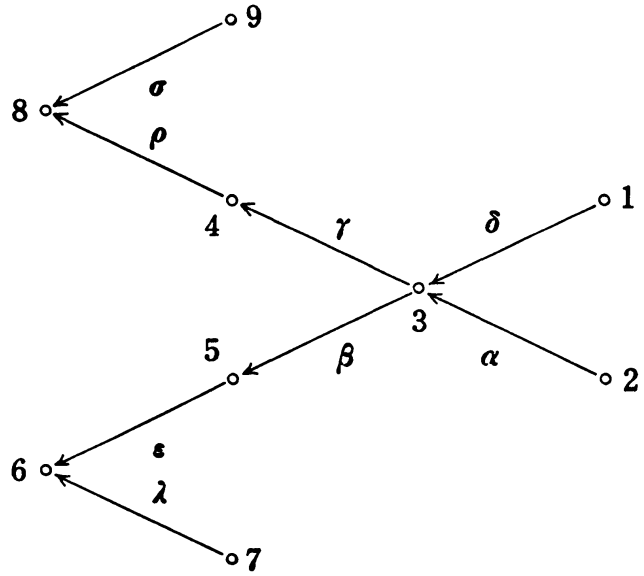
b) *There is a unique maximal branch extension B^+ of A which is a full convex subcategory of B, and c_B^+ is the extension type of B^+.*

PROOF. We shall only prove a), since the proof of b) is dual. We shall first prove that the admissible sequence leading from A to B can be replaced by another consisting of a block of operations of type ad1*) followed by a block of operations of types ad1), ad2), ad3). This is done by induction on the number n of operations in this admissible sequence. If $n=0$, there is nothing to prove. Assume $n>0$, and let $A=A_0, A_1, \cdots, A_{n-1}, A_n=B$ be the corresponding sequence of algebras, where we assume the statement holds for A_{n-1}. If the n^{th} operation is of type ad1), ad2) or ad3), there is nothing to show. If it is of type ad1*), we are able, by (3.1) and (3.2), to replace the given sequence by one of the required form. If it is of type ad2*), there must be in the sequence an operation of type ad1) that gives rise to the pivot X of ad2*) and the operation done between these two must not affect the support of Hom$(-, X)$ restricted to the coil containing X. By (3.1), all these operations commute with ad2*), so we can apply ad2*) after ad1) and then, using (3.3), replace these two operations by one of type ad1*) followed by one of type ad2). Using again (3.1) and (3.2), we are able to replace the given sequence by one of the required form. There remains to consider the case where the n^{th} operation is of type ad3*). There must be in the sequence at least one operation of type ad1) that gives rise to the pivot X of ad3*) and to the modules $Y_1, \cdots, Y_t$ in the support of Hom$(-, X)$ restricted to the coil containing X (in the notation of (2.2)). The operations done after must not affect this support. By (3.1), these operations commute with ad3*), and the operations of type ad1) that give rise to $X, Y_1, \cdots, Y_t$ can be done consecutively so that, by (3.4), we are able to replace these operations of type ad1) followed by ad*) by some operations of type ad1*) followed by an operation of type ad3). Another application of (3.1), (3.2) yields a sequence of the required form. This completes the proof of our claim.

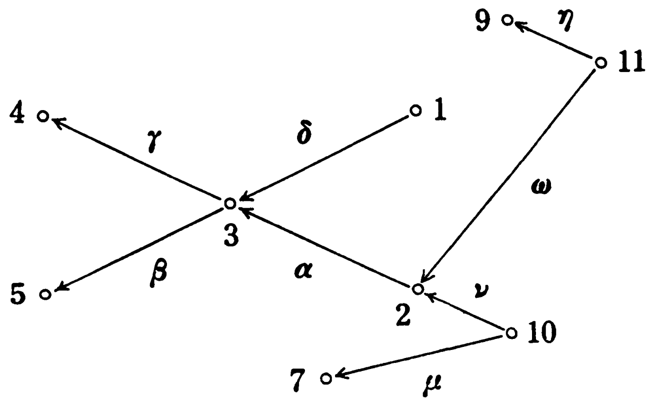
Let now B^- be the branch coextension of A determined by the block of operations of type ad1*) in the new admissible sequence. Since the remaining block in the sequence consists of operations of types ad1) ad2) ad3), that is, one-point extensions or, in the case ad1), branch extensions, it is clear that B^- is a branch coextension of A maximal with respect to the property of being a full convex subcategory of B. Furthermore, c_B^- is the coextension type of B^- because, if $\mathcal{T}=(\mathcal{T}_i)_{i\in I}$, then, for each $i\in I$, $c_B^-(i)$ equals the rank of $\mathcal{T}_i$ plus the number of corays inserted in $\mathcal{T}_i$ by the sequence of admissible operations of type ad1*) (see (2.3)).

There remains to show the uniqueness of B^-. Let B^* be a branch coextension of A inside B. We first note that, by construction of B^-, all the coextension points of A inside B must belong to B^-. Now, if b is a point in B^*, it must belong to a coextension branch of A inside B, hence, since the root of this branch belongs to B^-, the point b itself must belong to B^- (by construction of the latter). This shows that B^* is contained in B^- and completes our proof. □

3.6. EXAMPLE. In (2.8), for $B=A_4$, the algebra B^- is given by the quiver



bound by $\delta\beta\varepsilon=0$ and $\delta\gamma\rho=0$. Its type is indeed $c_B^-=(2, 4, 4)$. The algebra B^+ is given by the quiver



bound by $\nu\alpha\gamma=0$ and $\omega\alpha\beta=0$. Its type is indeed $c_B^+=(2, 4, 4)$.

4. The module category of a coil enlargement.

4.1. We shall now complete the description of the module category of a coil enlargement of an algebra having a weakly separating family of stable tubes. We shall use the following notation. Let K be a branch whose root is denoted by b. Then K is a tilted algebra of type A_n and there exist a complete slice Σ of $\Gamma(\bmod K)$ consisting of the indecomposable K-modules M such that there exists a sectional path $P(b)\to\cdots\to M$, and a complete slice Σ' of $\Gamma(\bmod K)$ consisting of the indecomposable K-modules M' such that there exists a sectional path $M'\to\cdots\to I(b)$. We shall denote by $\mathcal{L}(K)$ the set of all objects in $\operatorname{ind} K$ which are (not necessarily proper) predecessors of Σ', and by $\mathcal{R}(K)$ the set of all objects in $\operatorname{ind} K$ which are (not necessarily proper) successors of Σ. Thus, in the notation of [12] (4.7) (1),

$$\mathcal{R}(K)=\{M\in\operatorname{ind} K \mid \langle l_K, \underline{\dim}\, M\rangle>0\}$$

$$\mathcal{L}(K)=\{M\in\operatorname{ind} K \mid \langle\underline{\dim}\, M, l_K\rangle>0\}.$$

The main result of this section generalises [12] (4.7) (1) p. 230.

THEOREM. *Let A be an algebra with a family $\mathcal{T}=(\mathcal{T}_i)_{i\in I}$ of stable tubes, weakly separating $\mathcal{P}$ from $\mathcal{Q}$. Let B be a coil enlargement of A using modules from $\mathcal{T}$, and $B^-={}_{j=1}^{s}[K_j^*, E_j^*]A$, $B^+=A[E_i, K_i]_{i=1}^{r}$. Let $\mathcal{P}'$ be the class of all indecomposables M_B such that either $M|_A$ is non-zero and in $\mathcal{P}$, or else* Supp M *is contained in some K_j^* and $M\in\mathcal{L}(K_j^*)$. Let $\mathcal{Q}'$ be the class of all indecomposables N_B such that either $N|_A$ is non-zero and in $\mathcal{Q}$, or else* Supp N *is contained in some K_i and $N\in\mathcal{R}(K_i)$. Then there exists a family $\mathcal{T}'=(\mathcal{T}_i')_{i\in I}$ of coils in $\Gamma(\bmod B)$ such that* $\operatorname{ind} B=\mathcal{P}'\vee\mathcal{T}'\vee\mathcal{Q}'$, *$\mathcal{P}'$ consists of B^--modules, and $\mathcal{Q}'$ consists of B^+-modules.*

PROOF. We have seen, in the proof of (3.5), that the sequence of admissible operations leading from A to B can be replaced by a sequence consisting of a block of operations of type, ad1*), that determines B^-, followed by a block of operations of types ad1), ad2) or ad3). Dually, it can be replaced by a sequence consisting of a block of operations of type ad1), that determines B^+, followed by a block of operations of types ad1*), ad2*) or ad3*).

Using the first admissible sequence and (2.7) together with [12] (4.7) (1) p. 230, we obtain that $\operatorname{ind} B=\mathcal{P}'\vee\mathcal{T}'\vee\mathcal{Q}_1$, where $\mathcal{P}'$ is the class of all indecomposable B^--modules M such that either $M|_A$ is non-zero and in $\mathcal{P}$, or else Supp M is contained in some branch K_j^* and $M\in\mathcal{L}(K_j^*)$, and $\mathcal{T}'$ is the weakly separating family of coils obtained from $\mathcal{T}$ by applying the admissible operations in the sequence mentioned above. Using the second admissible sequence and the obvious fact that both sequences give rise to the same weakly separating family of coils, we obtain that $\operatorname{ind} B=\mathcal{P}_1\vee\mathcal{T}'\vee\mathcal{Q}'$, where $\mathcal{Q}'$ is the class of all indecomposable B^+-modules N such that either $N|_A$ is non-zero and in $\mathcal{Q}$, or else Supp N is contained in some branch K_i and $N\in\mathcal{R}(K_i)$. By (2.1), $\mathcal{P}'=\mathcal{P}_1$, $\mathcal{Q}'=\mathcal{Q}_1$ and the proof is complete. □

REMARKS. Since $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a (finite) sequence of admissible operations, only finitely many stable tubes of $\mathcal{T}$ are affected by these operations, and the remaining, when considered as stable tubes in $\mathcal{T}'$, consist of B^--modules (or of B^+-modules). Moreover, the non-stable coils in $\mathcal{T}'$ may contain infinitely many non-isomorphic indecomposable modules which are neither B^--modules nor B^+-modules. Indeed, these correspond to the points of intersection of the inserted rays with the inserted corays. In particular, for each $d \in \boldsymbol{N}$, all but at most finitely many non-isomorphic indecomposable modules in $\mathcal{T}'$ of dimension d are B^--modules or B^+-modules.

4.2. We now consider the case where A is a tame concealed algebra. We shall obtain a criterion for the tameness of a coil enlargement B of A using modules from its (separating) family of stable tubes. We shall need the following definitions. An algebra B is called cycle-finite if, for any cycle in mod B, no morphism on the cycle lies in the infinite power of the radical of mod B (see [1]). Multicoil algebras are defined and studied in [2, 3]. For the notions of tame, domestic, linear growth, polynomial growth and the Tits form of an algebra, we refer the reader to [13]. Let B be a coil enlargement of an algebra A having a weakly separating family of stable tubes. Its type $c_B=(c_B^-, c_B^+)$ is called tame if each of the sequences c_B^- and c_B^+ equals one of the following: (p, q), $1 \leq p \leq q$, $(2, 2, r)$, $2 \leq r$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ or $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$, $(2, 2, 2, 2)$.

COROLLARY. *Let A be a tame concealed algebra and $\mathcal{T}$ be its separating tubular family. Let B be a coil enlargement of A using modules from $\mathcal{T}$. The following conditions are equivalent:*

(a) *B is tame;*

(b) *B^- and B^+ are tame;*

(c) *B is a multicoil algebra;*

(d) *B is of polynomial growth;*

(e) *B is (domestic or) of linear growth;*

(f) *B is cycle-finite;*

(g) *c_B is tame;*

(h) *the Tits form q_B of B is weakly non-negative.*

Moreover, B is domestic if and only if both B^- and B^+ are tilted algebras of euclidean type.

PROOF. (a)⇒(b) Clear, since B^- and B^+ are full convex subcategories of B.

(b)⇒(c) follows from (4.1), since if B^- and B^+ are tame, they are multicoil algebras, and those B-modules which are neither B^--modules nor B^+-modules must belong to a weakly separating family of coils.

(c)⇒(f) follows from the definition of multicoil algebras.

(f)⇒(a) [1] (1.4).

(b)⇒(g) [11] (3.3), [9] (2.1).

(g)⇒(b) [12] (4.9) (2) p. 246 and (5.2) (4) p. 276.

(a)⇒(h) [10] (1.3).

(h)⇒(g) since B^- and B^+ are full convex subcategories of B, each of the Tits forms q_{B^-} and q_{B^+} is weakly non-negative; by [11] (3.3), c_B is tame.

(c)⇒(d) [2] (4.6).

(d)⇒(a) trivial.

(b)⇒(e) By [4] (2.3) and [9] (2.1), B^- and B^+ are both of linear growth. Applying (4.1), B itself is of linear growth.

(e)⇒(a) trivial.

The last assertion follows from [4] (2.3) and [12] (4.9) (1) p. 241. □

4.3 EXAMPLE. In (2.8) (3.6), the algebra B is tame and non-domestic (but of linear growth). In fact, it follows from (4.1) that if we denote

$$\operatorname{ind} B^- = \mathcal{P}_0^- \vee \mathcal{T}_0^- \vee \Big(\bigvee_{\gamma\in\mathbf{Q}^+} \mathcal{T}_\gamma^-\Big) \vee \mathcal{T}_\infty^- \vee \mathcal{Q}_\infty^-,$$

and

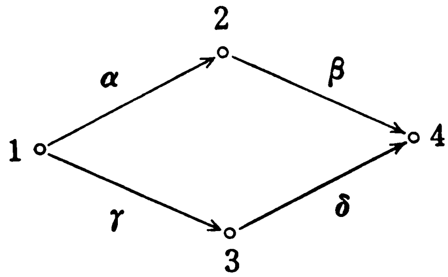
$$\operatorname{ind} B^+ = \mathcal{P}_0^+ \vee \mathcal{T}_0^+ \vee \Big(\bigvee_{\gamma\in\mathbf{Q}^+} \mathcal{T}_\gamma^+\Big) \vee \mathcal{T}_\infty^+ \vee \mathcal{Q}_\infty^+,$$

then

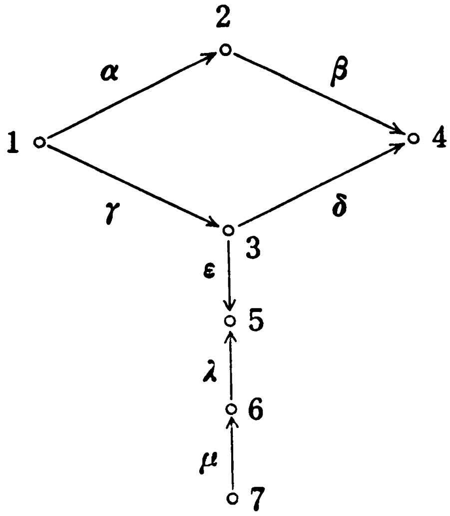
$$\operatorname{ind} B = \mathcal{P}' \vee \mathcal{T}' \vee \mathcal{Q}'.$$

where $\mathcal{P}' = \mathcal{P}_0^- \vee \mathcal{T}_0^- \vee \Big(\bigvee_{\gamma\in\mathbf{Q}^+} \mathcal{T}_\gamma^-\Big)$ and $\mathcal{Q}' = \Big(\bigvee_{\delta\in\mathbf{Q}^+} \mathcal{T}_\delta^+\Big) \vee \mathcal{T}_\infty^+ \vee \mathcal{Q}_\infty^+$. The family $\mathcal{T}'$ is obtained from $\mathcal{T}_\infty^+$ (or else from $\mathcal{T}_0^+$) by applying two operations of type ad2) (or ad2*), respectively). In fact, $\mathcal{T}'$ consists of all but two of the stable tubes of $\Gamma(\operatorname{mod} A)$ and two non-trivial coils (actually, quasi-tubes).

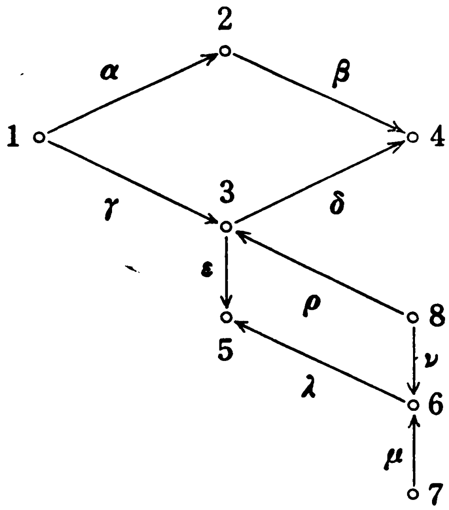
4.4 EXAMPLE. Let $A = A_0$ be the tame hereditary algebra given by the quiver



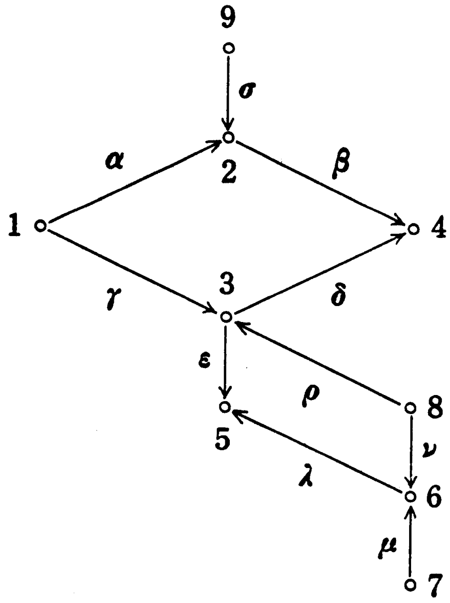
Its type is $c_A=((2, 2), (2, 2))$. The algebra A_1 given by the quiver



bound by $\gamma\varepsilon=0$ is obtained from A by an admissible operation of type ad1*) with pivot the simple regular A-module of dimension-vector $0\begin{smallmatrix}0\\1\end{smallmatrix}0$. Its coil type is $c_{A_1}=((2, 5), (2, 2))$. Then A_1 is a tilted algebra of type $\tilde{A}_6$. The algebra A_2 given by the quiver

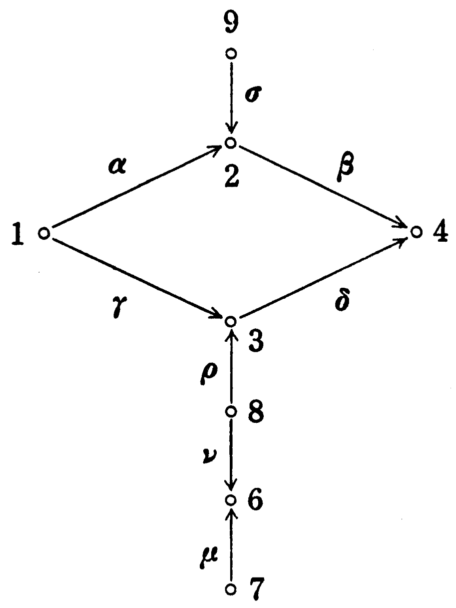


bound by $\gamma\varepsilon=0$, $\rho\delta=0$ and $\rho\varepsilon=\nu\lambda$ is obtained from A_1 by an admissible operation of type ad3) with pivot the indecomposable A_1-module of dimension-vector $\begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ & 1 & \\ & & 1 \\ & & 0 \end{smallmatrix}$. Its type is $c_{A_2}=((2, 5), (2, 5))$. The algebra $B=A_3$ given by the quiver



bound by $\gamma\varepsilon=0$, $\rho\delta=0$, $\sigma\beta=0$, $\rho\varepsilon=\nu\lambda$ is obtained from A_2 by an operation of type ad1) with pivot the indecomposable A_2-module of dimension vector $\begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \\ & 0 & \\ 0 & 0 & \\ & 0 & \\ & 0 & \end{smallmatrix}$. Its type is $c_B=((2, 5), (3, 5))$.

The algebra B^+ is given by the quiver



bound by $\sigma\beta=0$, $\rho\delta=0$. Its extension type is $c_B^+=(3, 5)$. Clearly, B^+ is a tilted algebra of type $\tilde{A}_7$. The algebra B^- coincides with the tilted algebra A_1, and its coextension type is $c_B^-=(2, 5)$. Since both B^- and B^+ are domestic, it follows from (4.2) that so is B.

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