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ENERGY DISTRIBUTION OF THE SOLUTIONS OF ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA $\boldsymbol{R}^3$

By

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Abstract. This paper deals with the asymptotic energy distributions for large times of the solutions of elastic wave propagation problems in stratified media $\boldsymbol{R}^3$. We construct asymptotic wave functions which approximate the solutions for large times and calculate the asymptotic energy of the solutions using these asymptotic wave functions. In particular, it is shown that the energy of Stoneley wave is asymptotically concentrated along the interface.

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§ 1. Introduction

Energy distribution of the solutions of various wave propagation problems has been studied by C. H. Wilcox ([10], [11], [12], [13]). He constructed asymptotic wave functions which approximate the solutions in the sense of L^2 for large times and calculated asymptotic energy distributions of the solutions in several domain by making use of these asymptotic wave functions.

The construction of asymptotic wave functions is based on an eigenfunction expansion theorem which is proved by the same author and on the method of stationary phase. J.C. Guillot [3] studied a Rayleich surface wave propagating along the free boundary of a transversely isotropic elastic half-space and showed that the energy of the Rayleich component of the solutions with finite energy is asymptotically concentrated along the boundary.

In this paper we shall derive energy distribution of the solutions of elastic wave propagation problems in plane-stratified media $\boldsymbol{R}^3$ using methods due to

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Wilcox. We construct asymptotic wave functions by using spectral integral representations of the solutions and the method of stationary phase. The integral representations are based on an eigenfunction expansion theory which was proved by the author [8] using methods due to S. Wakabayashi [9]. We calculate asymptotic energy of the solutions for large times of the interface problems for elastic waves and show that the energy of the Stoneley components of the solutions with finite energy is asymptotically concentrated along the interface.

We start with the mathematical formulation of the elastic wave propagation problem.

Consider the plane stratified medium $\boldsymbol{R}^3=\{x=(x_1, x_2, x_3); x_i\in\boldsymbol{R}\}$ with the planar interface $x_3=0$, which is defined by

$$(\lambda(x_3), \mu(x_3), \rho(x_3))=\begin{cases}(\lambda_1, \mu_1, \rho_1), & x_3<0,\\ (\lambda_2, \mu_2, \rho_2), & x_3>0.\end{cases}$$

Here $\lambda_1, \lambda_2, \mu_1, \mu_2$ are certain quantities called Lamé constants and $\rho_1, \rho_2>0$ are the densities.

We shall denote the lower half-space $\boldsymbol{R}^3_-=\{x\in\boldsymbol{R}^3; x_3<0\}$ by *medium* I and the upper half-space $\boldsymbol{R}^3_+=\{x\in\boldsymbol{R}^3; x_3>0\}$ by *medium* II, respectively, as in Figure 1.

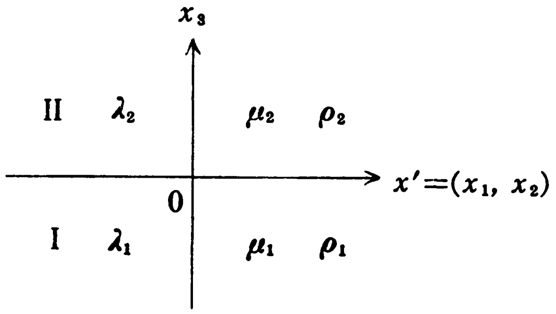

Figure 1. Stratified media I and II.

The propagation problem of elastic waves in the stratified medium is formulated as the following initial-interface value problem:

(1.1) $$\frac{\partial^2 u}{\partial t^2}(t, x)+Mu(t, x)=0,$$

(1.2) $$u(t, x)|_{x_3=-0}=u(t, x)|_{x_3=+0},$$

(1.3) $$\sigma_{k3}u(t, x)|_{x_3=-0}=\sigma_{k3}u(t, x)|_{x_3=+0},$$

(1.4) $$u(0, x)=f(x), \qquad \frac{\partial u}{\partial t}(0, x)=g(x),$$

where

$$(1.5)\qquad Mu=-\frac{\lambda(x_3)+\mu(x_3)}{\rho(x_3)}\nabla(\nabla\cdot u)-\frac{\mu(x_3)}{\rho(x_3)}\Delta u=\frac{1}{\rho(x_3)}\sum_{k,j=1}^{3}M_{kj}\frac{\partial^2 u}{\partial x_k\partial x_j},$$

$$(1.6)\qquad \sigma_{kj}u=\lambda(x_3)(\nabla\cdot u)\delta_{kj}+2\mu(x_3)\varepsilon_{kj}u,$$

$$(1.7)\qquad \varepsilon_{kj}u=\frac{1}{2}\left(\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\right).$$

(1.2) and (1.3) are called interface conditions, and (1.4) is called an initial condition.

The c^I_{kilj}, $c^{II}_{kilj}(i, j, k, l=1, 2, 3)$ are the stress-strain tensors given by

$$(1.8)\qquad \begin{aligned} c^I_{kilj}&=\lambda_1\delta_{ki}\delta_{lj}+\mu_1(\delta_{kl}\delta_{ij}+\delta_{kj}\delta_{il}),\\ c^{II}_{kilj}&=\lambda_2\delta_{ki}\delta_{lj}+\mu_2(\delta_{kl}\delta_{ij}+\delta_{kj}\delta_{il})\end{aligned}$$

with the properties

$$c^I_{kilj}=c^I_{iklj}=c^I_{kijl}=c^I_{ljki},$$

$$c^{II}_{kilj}=c^{II}_{iklj}=c^{II}_{kijl}=c^{II}_{ljki},$$

and δ_{ki} is the Kronecker delta. We assume that the constants c^I_{kilj}, c^{II}_{kilj} satisfy the following stability conditions

$$(1.9)\qquad \lambda_i+\mu_i>0,\qquad \mu_i>0,\qquad (i=1, 2),$$

which are equivalent to the conditions

$$(1.9)'\qquad \begin{aligned} \sum_{k,i,l,j=1}^{3}c^I_{kilj}s_{lj}\overline{s_{ki}}&\geqq\exists\delta_1\sum_{k,i=1}^{3}|s_{ki}|^2,\qquad \delta_1>0,\\ \sum_{k,i,l,j=1}^{3}c^{II}_{kilj}s_{lj}\overline{s_{ki}}&\geqq\exists\delta_2\sum_{k,i=1}^{3}|s_{ki}|^2,\qquad \delta_2>0,\end{aligned}$$

for all complex symmetric 3×3 matrices (s_{ki}), $s_{ki}=s_{ik}\in\boldsymbol{C}$ (cf. [4]).

We introduce the Hilbert space

$$(1.10)\qquad \mathcal{H}=L^2(\boldsymbol{R}^3, \boldsymbol{C}^3, \rho(x_3)dx)$$

with inner product

$$(u, v)=\int_{\boldsymbol{R}^3}u\cdot v\rho(x_3)dx,$$

where $u\cdot v$ denotes the usual scalar product in $\boldsymbol{C}^3: u\cdot v=\sum_{i=1}^{3}u_i\bar{v}_i$. It was shown in [8, Theorem 1.2] that the operator A on $\mathcal{H}$ with domain

$$\begin{aligned} D(A)=\{u\in H^2(\boldsymbol{R}^3_-, \boldsymbol{C}^3)\oplus H^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3);\ & u \text{ satisfies the interface conditions (1.2) and (1.3)}\\ & \text{in the sense of trace on } x_3=0\}\end{aligned}$$

and action defined by

$$(1.11)\qquad Au=Mu, \qquad u\in D(A)$$

is a self-adjoint operator on $\mathcal{H}$. Here

$$H^2(\boldsymbol{R}^3_\pm, \boldsymbol{C}^3)=\{u(x);\ D^\alpha_x u\in L^2(\boldsymbol{R}^3_\pm) \text{ for } 0\leqq\alpha\leqq 2\}$$

is a Hilbert space with innner product

$$(u, v)_2=\int_{\boldsymbol{R}^3_\pm}\sum_{|\alpha|\leq m} D^\alpha u(x)\cdot D^\alpha v(x)dx\ .$$

Every $u\in D(A)$ satisfies the interface conditions (1.2) and (1.3), so the mixed problem (1.1)-(1.4) may be reformulated as the problem of finding a function u: $\boldsymbol{R}\to\mathcal{H}$ such that

$$(1.12)\qquad \frac{\mathrm{d}^2u}{\mathrm{d}t^2}+Au=0 \qquad \text{for } \forall t\in\boldsymbol{R}\,,$$

$$(1.13)\qquad u(0)=f, \qquad \frac{\mathrm{d}u}{\mathrm{d}t}(0)=g\ .$$

The operator A is non-negative [8, Lemma 1.4] and the spectral theorem for self-adjoint operators (cf. [2]) implies that (1.12) and (1.13) has a (generalized) solution given by

$$(1.14)\qquad u(t)=(\cos tA^{1/2})f+(A^{-1/2}\sin tA^{1/2})g\,, \qquad t\in\boldsymbol{R}$$

for every pair $f, g\in\mathcal{H}$. u has derivatives $\mathrm{d}u/\mathrm{d}t$ and $\mathrm{d}^2u/\mathrm{d}t^2$ and is a strict solution of (1.12) if and only if $f\in D(A)$, $g\in D(A^{1/2})$.

Next we define the energy of solution u on a set $K\subset\boldsymbol{R}^3$ at time t for the elastic wave propagation problem by

$$(1.15)\qquad E(u, K, t)=\int_K\left(\left|\frac{\partial u}{\partial t}\right|^2\rho(x_3)-\sum_{k,j=1}^{3}M_{kj}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_k}\right)dx\ .$$

If u is a solution of (1.1)-(1.4), u satisfies the conservation laws of energy:

$$E(u, \boldsymbol{R}^3, t)=E(u, \boldsymbol{R}^3, 0)=\text{const.} \qquad \text{for } \forall t\in\boldsymbol{R},$$

where the constant may be finite or infinite. If one defines a sesquilinear form B in $\mathcal{H}$ by

$$D(B)=H^1(\boldsymbol{R}^3, \boldsymbol{C}^3)\subset\mathcal{H}$$

and

$$B(u, v)=-\sum_{k,j=1}^{3}\int_{\boldsymbol{R}^3}M_{kj}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_k}dx\,,$$

then it is easy to verify that B is closed and non-negative, and that A is the unique self-adjoint non-negative operator in $\mathcal{H}$ associated with B (cf. [5]). Then

$D(A^{1/2})=H^1(\boldsymbol{R}^3, \boldsymbol{C}^3)$ and for all $u\in D(A^{1/2})$ one has

$$\|A^{1/2}u\|^2=B(u, u)=-\sum_{k,j=1}^{3}\int_{\boldsymbol{R}^3}M_{kj}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_k}dx ,$$

where $\|\cdot\|$ is the norm in $\mathcal{H}$. It follows that

$$E(u, \boldsymbol{R}^3, t)=\left\|\frac{du}{dt}\right\|^2+\|A^{1/2}u\|^2=\|u\|_{\mathcal{E}}^2 . \tag{1.16}$$

Here the norm $\|u\|_{\mathcal{E}}$ is called the energy norm. If $f\in D(A^{2/1})$, $g\in\mathcal{H}$, then $u(t)\in D(A^{1/2})$, $du/dt\in\mathcal{H}$ for all $t\in\boldsymbol{R}$ and $u(t)$ satisfies

$$\|u(t)\|_{\mathcal{E}}^2=\|u(0)\|_{\mathcal{E}}^2<\infty \qquad \text{for } \forall t\in\boldsymbol{R} . \tag{1.17}$$

Therefore a necessary and sufficient condition for u to have this property is that the initial state f, g has finite energy:

$$f\in D(A^{1/2}), \qquad g\in\mathcal{H} . \tag{1.18}$$

Hereafter we consider only solutions with finite total energy.

When

$$f\in\mathcal{H} , \qquad g\in D(A^{-1/2}) ,$$

the solution u of the elastic wave propagation problem in $\mathcal{H}$, defined by (1.12) and (1.13), satisfies

$$u(t, x)=Re\{v(t, x)\} ,$$

where

$$v(t, \cdot)=e^{-itA^{1/2}}h , \qquad h=f+iA^{-1/2}g ,$$

then $v(t, x)$ has the following representation (see Section 2):

$$v(t, x)=\sum_{j\in M}v_{1j}^{\pm}(t, x)+\sum_{j\in M}v_{1j}^{St}(t, x)+\sum_{k\in N}v_{2k}^{\pm}(t, x)\in\mathcal{H} .$$

$v_{1j}^{\pm}(t, x)(j\in\{p_1, p_2\})$ are called Pressure (P) components, $v_{1j}^{\pm}(t, x)(j\in\{s_1, s_2\})$ are called Shear Vertical (SV) components, $v_{1j}^{St}(t, x)(j\in M=\{s_1, p_1, s_2, p_2\})$ are called Stoneley components and $v_{2k}^{\pm}(t, x)(k\in N=\{s_1, s_2\})$ are called Shear Horizontal (SH) components. We remark that if

$$\mathrm{Dis}(c_{s_\gamma}^2)>0 , \tag{1.19}$$

then the Stoneley components exist. Here $c_{s_\gamma}=\min\{c_{s_1}, c_{s_2}\}$ and $\mathrm{Dis}(z)$ is defined by (2.6) below (cf. Section 2, [8, Section 3]). This condition is determined by Lamé constants λ_i, μ_i and densities ρ_i $(i=1, 2)$.

Our main results are the following theorems. Theorem 1.1 shows that the energy of the Stoneley components $v_{1j}^{St}(t, x)(j\in M)$ of v is asymptotically concentrated along the interface $x_3=0$.

THEOREM 1.1. *We assume that*

$$f \in D(A^{1/2}) \cap \mathcal{H}, \qquad g \in \mathcal{H} \cap D(A^{-1/2}), \qquad \mathrm{Dis}(c_{s_T}^2) > 0,$$

then

$$\lim_{t\to\infty} E(v_{1j}^{St}, (C^-(\theta) \cup C^+(\theta)) \cap B(t, \vartheta(t)), t) = E(v_{1j}^{St}, \boldsymbol{R}^3, 0), \qquad j \in M,$$

where

$$C^-(\theta) = \{x \in \boldsymbol{R}_-^3 ; -\theta(|x'|) < x_3 < 0\},$$

$$C^+(\theta) = \{x \in \boldsymbol{R}_-^3 ; 0 < x_3 < \theta(|x'|)\},$$

$$B(t, \vartheta(t)) = \{x \in \boldsymbol{R}^3 ; c_{St}t - \vartheta(t) \leqq |x'| \leqq c_{St}t + \vartheta(t),\ x_3 \in \boldsymbol{R}\},$$

$$\vartheta(t):\ \lim_{t\to\infty} \vartheta(t) = \infty, \quad |\vartheta(t)| < 2c_{St}t,$$

$\theta(|x'|)$: $\lim_{|x'|\to\infty} \theta(|x'|) = \infty$, monotone increasing function,

c_{St}: propagation speed of Stoneley wave.

The next theorem shows that the P, SV, SH components $v_{1j}^{\pm}(t, x) (j \in M)$, $v_{2k}^{\pm}(t, x) (k \in N)$ behave like free waves.

THEOREM 12. *We assume that*

$$f \in D(A^{1/2}) \cap \mathcal{H}, \qquad g \in \mathcal{H} \cap D(A^{-1/2}),$$

then

$$\lim_{t\to\infty} E(v_{1j}^{\pm}, S_{s_1}(t, \vartheta) \cup S_{p_1}(t, \vartheta) \cup S_{s_2}(t, \vartheta) \cup S_{p_2}(t, \vartheta), t) = E(v_{1j}^{\pm}, \boldsymbol{R}^3, 0), \quad j \in M,$$

$$\lim_{t\to\infty} E(v_{2k}^{\pm}, S_{s_1}(t, \vartheta) \cup S_{s_2}(t, \vartheta), t) = E(v_{2k}^{\pm}, \boldsymbol{R}^3, 0), \qquad k \in N,$$

where

$$S_{s_1}(t, \vartheta(t)) = \{x \in \boldsymbol{R}_-^3 ; c_{s_1}t - \vartheta(t) \leqq |x| \leqq c_{s_1}t + \vartheta(t)\},$$

$$S_{p_1}(t, \vartheta(t)) = \{x \in \boldsymbol{R}_-^3 ; c_{p_1}t - \vartheta(t) \leqq |x| \leqq c_{p_1}t + \vartheta(t)\},$$

$$S_{s_2}(t, \vartheta(t)) = \{x \in \boldsymbol{R}_+^3 ; c_{s_2}t - \vartheta(t) \leqq |x| \leqq c_{s_2}t + \vartheta(t)\},$$

$$S_{p_2}(t, \vartheta(t)) = \{x \in \boldsymbol{R}_+^3 ; c_{p_2}t - \vartheta(t) \leqq |x| \leqq c_{p_2}t + \vartheta(t)\},$$

$$\vartheta(t):\ \lim_{t\to\infty} \vartheta(t) = \infty,$$

c_{p_1}, c_{p_2}: propagation speeds of P waves,

c_{s_1}, c_{s_2}: propagation speeds of SV and SH waves.

These theorems are obtained calculating the energy of the asymptotic wave functions $v_{1j}^{St\infty}(t, x)$, $v_{1j}^{\pm\infty}(t, x)$ $(j \in M)$, $v_{2k}^{\pm\infty}(t, x)$ $(k \in N)$ which defined by means of the stationary phase method.

The remainder of this paper is organized as follows. In Section 2, we give spectral integral representations of the solutions of the propagation problem by using the eigenfunction expansion theorem for A developed in [8]. In Section 3, we construct asymptotic wave functions of the Stoneley components by means of the method of stationary phase. We construct asymptotic wave functions of the P, SV, SH components in Section 4. In Section 5, we calculate the asymptotic energy distributions of the solutions for large times.

§ 2. Eigenfunction Expansions for A

The eigenfunction expansion theorem for A was developed in [8]. In this section it is applied to give spectral integral representations of the solutions of the elastic propagation problem. This section begins with a brief review of the structure and properties of the eigenfunctions and the expansion theorem.

Let $\eta'=(\eta_1, \eta_2)\in \boldsymbol{R}^2$ be the dual variables of $x'=(x_1, x_2)$ and let $F_{x'}$ denote the partial Fourier transformation with respect to x';

$$\hat{u}(\eta', x_3)=(F_{x'}u)(\eta', x_3)=\underset{R\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{|x'|\leq R} e^{-i(x_1\eta_1+x_2\eta_2)}u(x)dx'$$

for u in $\mathscr{H}$. Let

$$D(\hat{A})=F_{x'}D(A)=\{\hat{u}\ ;\ u\in D(A)\},$$

$$\hat{A}\hat{u}=F_{x'}AF_{\eta'}^{-1}\hat{u}\ ,\qquad \hat{u}\in D(\hat{A})\,.$$

For every $\eta'\neq 0$, let

$$(2.1)\qquad \mathrm{U}=\frac{1}{|\eta'|}\begin{pmatrix}\eta_1 & -\eta_2 & 0\\ \eta_2 & \eta_1 & 0\\ 0 & 0 & |\eta'|\end{pmatrix},\qquad \mathrm{C}=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

where U and C are unitary matrices and $|\eta'|=(\eta_1^2+\eta_2^2)^{1/2}$. Then we have

$$(2.2)\qquad Au=F_{\eta'}^{-1}\mathrm{UC}(A_1(\eta')\oplus A_2(\eta'))(\mathrm{UC})^{-1}F_{x'}u\qquad \text{for } u\in D(A),$$

where $A_1(\eta')$ and $A_2(\eta')$ are non-negative self-adjoint operators (see [8, Proposition 1.7], [1], [3]).

We can get an explicit representation of the Green function $G_1(x_3, y_3, \eta'\,;\zeta)$ for the operator $A_1(\eta')-\zeta I$ $(\zeta\notin \boldsymbol{R})$ from the expression of the solution for the following problem:

$$(2.3)\qquad (A_1(\eta', D)-\zeta)v(\eta', x_3)=f(\eta', x_3),$$

$$(2.4)\qquad v(\eta', x_3)|_{x_3=-0}=v(\eta', x_3)|_{x_3=+0}\,,$$

(2.5) $$B_1(\eta')v(\eta',\ x_3)|_{x_3=-0}=B_1(\eta')v(\eta',\ x_3)|_{x_3=+0}\ .$$

Here (2.4) and (2.5) are the interface conditions for $A_1(\eta',\ D)$ corresponding to (1.2) and (1.3). $A_1(\eta',\ D)$ $(D=(1/i)(\mathrm{d}/\mathrm{d}x_3))$ is the differential operators corresponding to the self-adjoint operator $A_1(\eta')$. Since the solution v of (2.3) should satisfy the interface conditions (2.4) and (2.5), the denominator of v has the Lopatinski determinant $\Delta(\eta',\ \zeta)$ as follows:

$$\Delta(\eta',\ \zeta)=|\eta'|^6\mathrm{Dis}(z),$$

(2.6) $$\begin{aligned}\mathrm{Dis}(z)=&\Big(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}+\frac{\mu_2 z}{c_{s_2}^2}\Big)^2+4(\mu_1-\mu_2)^2 a_1a_2b_1b_2\\&-a_1b_1\Big(2(\mu_1-\mu_2)+\frac{\mu_2 z}{c_{s_2}^2}\Big)^2-a_2b_2\Big(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}\Big)^2\\&-\frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^2}(a_1b_2+a_2b_1)z^2\ ,\end{aligned}$$

where

$$z=\frac{\zeta}{|\eta'|^2}\ ,$$

$$a_1=\sqrt{1-\frac{z}{c_{p_1}^2}},\qquad a_2=\sqrt{1-\frac{z}{c_{p_2}^2}},\qquad b_1=\sqrt{1-\frac{z}{c_{s_1}^2}},\qquad b_2=\sqrt{1-\frac{z}{c_{s_2}^2}}\ .$$

The squares of propagation speeds of shear(SV, SH) and pressure(P) waves are given by

(2.7) $$c_{s_i}^2=\frac{\mu_i}{\rho_i},\qquad c_{p_i}^2=\frac{\lambda_i+2\mu_i}{\rho_i},\qquad (i=1,\ 2),$$

respectively. From the conditions (1.9), the minimum speed of $\{c_{s_1},\ c_{p_1},\ c_{s_2},\ c_{p_2}\}$ is either c_{s_1} or c_{s_2}.

We can see that $\mathrm{Dis}(z)$ has the only one real zero when $\mathrm{Dis}(z)$ has zeros. Denote by c_{St}^2 its real zero. Then the zero of $\Delta(\eta',\ \zeta)$ is $c_{St}^2|\eta'|^2$ and is the origin of the Stoneley wave propagating along the interface $x_3=0$ in the elastic space R^3, and c_{St} is its speed.

By virtue of principle of the argument, the conditions for the existence of zeros of the Lopatinski determinant $\Delta(\eta',\ \zeta)=|\eta'|^6\mathrm{Dis}(z)$ (the existence of the Stoneley waves) are given as follows:

If $c_{s_1}<c_{s_2}$, then

(i) $\mathrm{Dis}(c_{s_1}^2)>0 \Longrightarrow$ The zero $\zeta=c_{St}^2|\eta'|^2$ of $\Delta(\eta',\ \zeta)$ in ζ exists in $[0,\ c_{s_1}^2|\eta'|^2)$ with order 1. More precisely, we shall prove in the proof of [8, Theorem 6.5] that $c_{St}\neq 0$.

(ii) $\mathrm{Dis}(c_{s_1}^2)=0 \Longrightarrow c_{St}=c_{s_1}$ and we shall consider this case under some some restricted conditions (cf. [8, Lemma 6.4]).

(iii) $\mathrm{Dis}(c_{s_1}^2)<0 \Longrightarrow \Delta(\eta', \zeta)$ has no zero.

If $c_{s_2}<c_{s_1}$, then we must replace $\mathrm{Dis}(c_{s_1}^2)$ by $\mathrm{Dis}(c_{s_2}^2)$.

We also obtain an explicit representation of the Green function $G_2(x_3, y_3, \eta' ; \zeta)$ for the operator $A_2(\eta')-\zeta I$ $(\zeta\notin \boldsymbol{R})$ by the same method as $G_1(x_3, y_3, \eta' ; \zeta)$. The Lopatinski determinant corresponding to the operator $A_2(\eta')-\zeta I$ $(\zeta\notin\boldsymbol{R})$ has no zero. By using the Green functions $G_1(x_3, y_3, \eta' ; \zeta)$ and $G_2(x_3, y_3, \eta' ; \zeta)$, we define

$$\psi_{1j}(x_3, \eta ; \zeta)=F_{y_3}^{-1}[G_1(x_3, y_3, \eta' ; \zeta)](\xi)(\lambda_j(\eta)-\zeta)P_j(\eta)\rho(x_3)^{-1}, \quad j\in M,$$

$$\psi_{1j}^{St}(x_3, \eta ; \zeta)=\frac{\zeta-c_{St}^2|\eta'|^2}{\zeta-\lambda_j(\eta)}\psi_{1j}(x_3, \eta ; \zeta), \quad j\in M,$$

$$\psi_{2k}(x_3, \eta ; \zeta)=F_{y_3}^{-1}[G_2(x_3, y_3, \eta' ; \zeta)](\xi)(\lambda_j(\eta)-\zeta)\rho(x_3)^{-1}, \quad k\in N.$$

Here $\eta=(\eta_1, \eta_2, \xi)=(\eta', \xi)$, $\lambda_j(\eta)=c_j^2|\eta|^2$ are the eigenvalues of $A_1(\eta')$, $P_j(\eta)$ are mutually orthogonal projections for $A_1(\eta')$, $\lambda_k(\eta)=c_k^2|\eta|^2$ are the eigenvalues of $A_2(\eta')$, $M=\{s_1, p_1, s_2, p_2\}$ and $N=\{s_1, s_2\}$. When $\zeta\to\lambda_j(\eta)\pm i0$, $\zeta\to c_{St}^2|\eta|^2$, and $\zeta\to\lambda_k(\eta)\pm i0$, the limits $\psi_{1j}^{\pm}(x_3, \eta)$, $\psi_{1j}^{St}(x_3, \eta)$, and $\psi_{2k}^{\pm}(x_3, \eta)$ exist and these limit functions are generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, respectively.

Using these generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, we define generalized eigenfunctions for A as follows:

$$\psi_{1j}^{\pm}(x, \eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{\pm}(x_3, \eta)\oplus O_{1\times 1}), \qquad j\in M, \tag{2.8}$$

$$\psi_{1j}^{St}(x, \eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{St}(x_3, \eta)\oplus O_{1\times 1}), \qquad j\in M, \tag{2.9}$$

$$\psi_{2k}^{\pm}(x, \eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2}\oplus\psi_{2k}^{\pm}(x_3, \eta)), \qquad k\in N, \tag{2.10}$$

where $O_{n\times n}$ denotes the $n\times n$ zero matrix.

Now we define the Fourier transform of $f\in\mathscr{H}$ with respect to these generalized eigenfunctions: $f\mapsto(\hat{f}_{1j}^{\pm}, \hat{f}_{1j}^{St}, \hat{f}_{2k}^{\pm})$,

$$\hat{f}_{1j}^{\pm}(\eta)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|x|\leq R}\psi_{1j}^{\pm}(x, \eta)^*f(x)\rho(x_3)dx, \qquad j\in M, \tag{2.11}$$

$$\hat{f}_{1j}^{St}(\eta)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|x|\leq R}\psi_{1j}^{St}(x, \eta)^*f(x)\rho(x_3)dx, \qquad j\in M, \tag{2.12}$$

$$\hat{f}_{2k}^{\pm}(\eta)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|x|\leq R}\psi_{2k}^{\pm}(x, \eta)^*f(x)\rho(x_3)dx, \qquad k\in N. \tag{2.13}$$

Theorem 2.1 corresponds to the Parseval and Plancherel formulas.

THEOREM 2.1. *We assume that* $\mathrm{Dis}(c_{s\tau}^2)>0$. *Let* $f, g\in\mathcal{H}$ *and* $0<a<b<\infty$. *Then we have*

$$(f, g)=\sum_{j\in M}\left(\int_{R^3}\hat{f}_{1j}^{\pm}(\eta)\cdot\hat{g}_{1j}^{\pm}(\eta)d\eta+\int_{R^3}\hat{f}_{1j}^{St}(\eta)\cdot\hat{g}_{1j}^{St}(\eta)d\eta\right)$$

$$+\sum_{k\in N}\int_{R^3}\hat{f}_{2k}^{\pm}(\eta)\cdot\hat{g}_{2k}^{\pm}(\eta)d\eta.$$

The first half of Theorem 2.2 expresses the Fourier inversion formula with respect to generalized eigenfunctions. The latter half gives the canonical form for A.

THEOREM 2.2. *We assume the same assumption as Theorem* 2.1.

(1) *For* $f\in\mathcal{H}$,

$$f(x)=\sum_{j\in M}\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|\eta|\le R}(\psi_{1j}^{\pm}(x, \eta)\hat{f}_{1j}^{\pm}(\eta)+\psi_{1j}^{St}(x, \eta)\hat{f}_{1j}^{St}(\eta))d\eta$$

$$+\sum_{k\in N}\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|\eta|\le R}\psi_{2k}^{\pm}(x, \eta)\hat{f}_{2k}^{\pm}(\eta)d\eta.$$

(2) *For* $f\in D(A)$,

$$Af(x)=\sum_{j\in M}\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|\eta|\le R}(\lambda_j(\eta)\psi_{1j}^{\pm}(x, \eta)\hat{f}_{1j}^{\pm}(\eta)+c_{St}^2|\eta'|^2\psi_{1j}^{St}(x, \eta)\hat{f}_{1j}^{St}(\eta))d\eta$$

$$+\sum_{k\in N}\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|\eta|\le R}\lambda_k(\eta)\psi_{2k}^{\pm}(x, \eta)\hat{f}_{2k}^{\pm}(\eta)d\eta,$$

and

$$(\widehat{Af})_{1j}^{\pm}(\eta)=\lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta), \qquad j\in M,$$

$$(\widehat{Af})_{1j}^{St}(\eta)=c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta), \qquad j\in M,$$

$$(\widehat{Af})_{2k}^{\pm}(\eta)=\lambda_k(\eta)\hat{f}_{2k}^{\pm}(\eta), \qquad k\in N.$$

Theorem 2.3 gives an explicit expression of the ranges $R(\Phi^{\pm})$.

THEOREM 2.3. *Assume the same assumption as Theorem* 2.1. *We define the mappings by*

$$\Phi_{1j}^{\pm}: \mathcal{H}\ni f \longrightarrow \hat{f}_{1j}^{\pm}(\eta)\in L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3), \qquad j\in M,$$

$$\Phi_{1j}^{St}: \mathcal{H}\ni f \longrightarrow \hat{f}_{1j}^{St}(\eta)\in L^2(\boldsymbol{R}^3, \boldsymbol{C}^3), \qquad j\in M,$$

$$\Phi_{2k}^{\pm}: \mathcal{H}\ni f \longrightarrow \hat{f}_{2k}^{\pm}(\eta)\in L^2(\boldsymbol{R}_{\pm}^3, \boldsymbol{C}^3), \qquad k\in N,$$

and put

$$\Phi^{\pm}=\sum_{j\in M}\Phi_{1j}^{\pm}\oplus\sum_{j\in M}\Phi_{1j}^{St}\oplus\sum_{k\in N}\Phi_{2k}^{\pm}.$$

Then we have

$$R(\Phi^{\pm})=\sum_{j\in M}\oplus(P_j(\eta)\oplus O_{1\times 1})L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3)\oplus\sum_{j\in M}\oplus(P_j(\eta)\oplus O_{1\times 1})L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3)$$
$$\oplus\sum_{k\in N}\oplus(O_{2\times 2}\oplus 1)L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3).$$

This implies that $\Phi^{\pm}$ are unitary operators in $\mathcal{H}$, and that the systems of generalized eigenfunctions $\{\psi_{1j}^{+},\ \psi_{1j}^{St},\ \psi_{2k}^{+}\}$ and $\{\psi_{1j}^{-},\ \psi_{1j}^{St},\ \psi_{2k}^{-}\}$ are complete, respectively.

The next theorem shows the utility of the eigenfunction expansion theorem for the operator A.

THEOREM 2.4. *Let $\Psi(\lambda)$ be a complex-valued bounded Lebesgue measurable function on $\sigma(A)=\{\lambda:\lambda\geqq 0\}$ and let $\Psi(A)$ be the corresponding operator defined by means of the spectral theorem.*

Then we have

$$(\widehat{\Psi(A)f})_{1j}^{\pm}(\eta)=\Psi(c_j^2|\eta|^2)\hat{f}_{1j}^{\pm}(\eta)\in(P_j(\eta)\oplus O_{1\times 1})L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3),\qquad j\in M,$$

$$(\widehat{\Psi(A)f})_{1j}^{St}(\eta)=\Psi(c_{St}^2|\eta'|^2)\hat{f}_{1j}^{St}(\eta)\in(P_j(\eta)\oplus O_{1\times 1})L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3),\qquad j\in M,$$

$$(\widehat{\Psi(A)f})_{2k}^{\pm}(\eta)=\Psi(c_k^2|\eta|^2)\hat{f}_{2k}^{\pm}(\eta)\in(O_{2\times 2}\otimes 1)L^2(\boldsymbol{R}_{\pm}^3,\ \boldsymbol{C}^3),\qquad k\in N.$$

It will be convenient to rewrite the solution (1.12)-(1.13) in the following form.

THEOREM 2.5. *Let f and g be real-valued functions such that*

$$f\in\mathcal{H},\qquad g\in D(A^{-1/2}), \tag{2.14}$$

and define

$$h=f+iA^{-1/2}g\in\mathcal{H}. \tag{2.15}$$

Then the solution in $\mathcal{H}$ defined by (1.14) *satisfies*

$$u(t,\ x)=Re\{v(t,\ x)\}, \tag{2.16}$$

where $v(t,\ x)$ is the complex-valued solution in $\mathcal{H}$ defined by

$$v(t,\ \cdot)=e^{-itA^{1/2}}h. \tag{2.17}$$

The proof of Theorem 2.5 is due to Wilcox [10, Theorem 2.3]. This theorem implies that the solution $u(t,\ x)$ of (1.12) and (1.13) is determined by $v(t,\ x)$.

Combining Theorem 2.4 and Theorem 2.5, we have the following:

THEOREM 2.6. *We assume that*

$$f\in\mathcal{H},\qquad g\in D(A^{-1/2}),\qquad \mathrm{Dis}(c_{s\gamma}^2)>0.$$

Then the solution of the elastic wave propagation problem, by (1.12) *and* (1.13) *has the representation*

$$v(t,x)=\sum_{j\in M}v_{1j}^{\pm}(t,x)+\sum_{j\in M}v_{1j}^{St}(t,x)+\sum_{j\in N}v_{2k}^{\pm}(t,x)\in\mathcal{H}, \tag{2.18}$$

where

$$v_{1j}^{\pm}(t,x)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|\eta|\le R}e^{-itc_j|\eta|}\psi_{1j}^{\pm}(x,\eta)\hat{h}_{1j}^{\pm}(\eta)d\eta,\qquad j\in M, \tag{2.19}$$

$$v_{1j}^{St}(t,x)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|\eta|\le R}e^{-itc_{St}|\eta'|}\psi_{1j}^{St}(x,\eta)\hat{h}_{1j}^{St}(\eta)d\eta,\qquad j\in M, \tag{2.20}$$

$$v_{2k}^{\pm}(t,x)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_{|\eta|\le R}e^{-itc_k|\eta|}\psi_{2k}^{\pm}(x,\eta)\hat{h}_{2k}^{\pm}(\eta)d\eta,\qquad k\in N, \tag{2.21}$$

and

$$\hat{h}_{1j}^{\pm}(\eta)=\hat{f}_{1j}^{\pm}(\eta)+i\frac{1}{c_j|\eta|}\hat{g}_{1j}^{\pm}(\eta)\in(P_j(\eta)\oplus O_{1\times1})L^2(\boldsymbol{R}_{\pm}^3,\boldsymbol{C}^3), \tag{2.22}$$

$$\hat{h}_{1j}^{St}(\eta)=\hat{f}_{1j}^{St}(\eta)+i\frac{1}{c_{St}|\eta'|}\hat{g}_{1j}^{St}(\eta)\in(P_j(\eta)\oplus O_{1\times1})L^2(\boldsymbol{R}^3,\boldsymbol{C}^3), \tag{2.23}$$

$$\hat{h}_{2k}^{\pm}(\eta)=\hat{f}_{2k}^{\pm}(\eta)+i\frac{1}{c_k|\eta|}\hat{g}_{2k}^{\pm}(\eta)\in(O_{2\times2}\oplus1)L^2(\boldsymbol{R}_{\pm}^3,\boldsymbol{C}^3). \tag{2.24}$$

§ 3. Transient Gulded (Stoneley) Waves

This section deals with the Stoneley components $v_{1j}^{St}(t,x)$ $(j\in M)$ defined by (2.20) and (2.23). It is shown in Section 5 below that these components are transient, in the sense that the energy in any bounded region tends to zero for large t, and are guided, in the sense that the energy concentrate near the boundary $x_3=0$. The proofs are based on asymptotic approximations for $v_{1j}^{St}(t,x)$ $(j\in M)$ for large t which are derived in this section.

In this section it is assumed that the initial data $f(x)$ and $g(x)$ are real-valued functions and $f\in\mathcal{H}$, $g\in D(A^{-1/2})$, and that the condition $\mathrm{Dis}(c_{s\gamma}^2)>0$ (i.e. existence of the Stoneley wave) is satisfied.

Substituting (2.9) in (2.20), we can represent the Stoneley components $v_{1j}^{St}(t,x)$ $(j\in M)$ in the form

$$v_{1j}^{St}(t,x)=\underset{R\to\infty}{\mathrm{l.i.m.}}\left(\frac{1}{2\pi}\right)\int_{|\eta|\le R}e^{i(x'\cdot\eta'-tc_{St}|\eta'|)}\mathrm{U}(\eta')\mathrm{C}(\phi_{1j}^{St}(x_3,\eta)\oplus0_{1\times1})\hat{h}_{1j}^{St}(\eta)d\eta, \tag{3.1}$$

where

$$U(\eta')=\frac{1}{|\eta'|}\begin{pmatrix}\eta_1 & -\eta_2 & 0\\ \eta_2 & \eta_1 & 0\\ 0 & 0 & |\eta'|\end{pmatrix},\quad C=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},\quad \eta=(\eta',\xi)=(\eta_1,\eta_2,\xi),$$

and $\psi_{1j}^{St}(x_3,\eta)$ is a generalized eigenfunction for the operator $A_1(\eta')$ (given by [8, (4.17) and (4.18)]).

The function $\psi_{1j}^{St}(x_3,\eta)$ and $\hat{h}_{1j}^{St}(\eta)$ $(j\in M)$ can be written in the form

$$\psi_{1j}^{St}(x_3,\eta)=\frac{|\eta'|}{\xi-ic_{0j}|\eta'|}e^{-c_{0j}|\eta||x_3|}\phi_{1j}^{St}(\eta')P_j(\eta),\tag{3.2}$$

$$\hat{h}_{1j}^{St}(\eta)=\frac{\sqrt{|\eta'|}}{\xi+ic_{0j}|\eta'|}k_{1j}^{St}(\eta'),\tag{3.3}$$

where

$$c_{0j}=\sqrt{1-\frac{c_{St}^2}{c_j^2}}\quad (0<c_{0j}<1).$$

Here $\phi_{1j}^{St}(\eta')$ is a bounded continuous function (see [8, (4.17) and (4.18)]) and $k_{1j}^{St}(\eta')\in L^2(\boldsymbol{R}^2, \boldsymbol{C}^3)$ because

$$\begin{aligned}\|\hat{h}_{1j}^{St}(\eta)\|_{L^2(R^3)}^2&=\int_{R^2}\int_R\left|\frac{\sqrt{|\eta'|}}{\xi+ic_{0j}|\eta'|}k_{1j}^{St}(\eta')\right|^2d\xi d\eta'\\&=\int_{R^2}|k_{1j}^{St}(\eta')|^2\left(\int_{-\infty}^{\infty}\frac{|\eta'|}{\xi^2+c_{0j}^2|\eta'|}d\xi\right)d\eta'\\&=\frac{\pi}{c_{0j}}\|k_{1j}^{St}(\eta')\|_{L^2(R^2)}^2.\end{aligned}\tag{3.4}$$

Then the integral in (3.1) is rewritten

$$\begin{aligned}v_{1j}^{St}(t,x)=\underset{R\to\infty}{\text{l.i.m.}}\left(\frac{1}{2\pi}\right)\int_{|\eta'|\le R}&e^{i(x'\cdot\eta'-tc_{St}|\eta'|)-c_{0j}|\eta'||x_3|}\\&\times U(\eta')C(\phi_{1j}^{St}(\eta')\oplus 0_{1\times1})Q(\eta')\sqrt{|\eta'|}k_{1j}^{St}(\eta')d\eta',\quad j\in M,\end{aligned}\tag{3.5}$$

where

$$Q(\eta')=\int_{\xi^2\le R^2-|\eta'|^2}\frac{|\eta'|}{\xi^2+c_{0j}^2|\eta'|^2}(P_j(\eta',\xi)\oplus 0_{1\times1})d\xi\tag{3.6}$$

$$\begin{aligned}P_{s_1}(\eta)=P_{s_2}(\eta)&=\frac{1}{|\eta|^2}\begin{pmatrix}\xi^2 & -|\eta'|\xi\\ -|\eta'|\xi & |\eta'|^2\end{pmatrix},\\ P_{p_1}(\eta)=P_{p_2}(\eta)&=\frac{1}{|\eta|^2}\begin{pmatrix}|\eta'|^2 & |\eta'|\xi\\ |\eta'|\xi & \xi^2\end{pmatrix}.\end{aligned}\tag{3.7}$$

We note that $U(\eta')C(\phi_{1j}^{St}(\eta')\oplus O_{1\times1})Q(\eta')$ is a bounded continuous function of η' in $\boldsymbol{R}^2$, because

$$|Q(\eta')| \leqq \int_{\xi^2 \leq R^2 - |\eta'|^2} \frac{|\eta'|}{\xi^2 + c_{0j}^2 |\eta'|^2} d\xi \leqq \frac{2}{c_{0j}} \int_0^\infty \frac{1}{\theta^2+1} d\theta = \frac{\pi}{c_{0j}} .$$

Now we consider the following integral

$$w(t, x) = \frac{1}{2\pi} \int_{R^2} e^{i(x'\cdot\eta' - tc_1|\eta'|) - c_2|\eta'||x_3|} \sqrt{|\eta'|}\, \phi(\eta') d\eta', \quad \phi \in \mathcal{D}(\boldsymbol{R}^2) = \mathcal{D}(\boldsymbol{R}^2, \boldsymbol{C}), \tag{3.8}$$

where c_1 and c_2 are positive constants and $\mathcal{D}(\boldsymbol{R}^2)$ denotes the usual Schwartz space.

Introducing plane polar coordinates (ν, ω) for η', we find

$$\begin{aligned} w(t, x) &= \frac{1}{2\pi} \int_0^\infty \int_{S^1} e^{i\nu(x'\cdot\omega - c_1 t) - \nu c_2 |x_3|} \nu^{3/2} \phi(\nu\omega) d\omega d\nu \\ &= \frac{1}{2\pi} \int_0^\infty e^{-i\nu c_1 t - \nu c_2 |x_3|} \nu^{3/2} J(x', \nu) d\nu , \end{aligned} \tag{3.9}$$

where

$$J(x', \nu) = \int_{S^1} e^{i\nu x'\cdot\omega} \phi(\nu\omega) d\omega . \tag{3.10}$$

In order to find the asymptoic behavior of $w(t, x)$ for $t \to \infty$, we calculated the asymptotic behavior of $J(x', \nu)$ for $|x'| \to \infty$ making use of the method of stationary phase.

The following theorem is a version of the method of stationary phase and give the asymptotic formula at infinity of the Fourier transform of a measure (with smooth density) concentrated on the hypersurface S^{n-1} (see [6, Section 5], [7, Section 4] for general C^∞ hypersurface S).

THEOREM 3.1. *Let S be the unit sphere of $\boldsymbol{R}^n$ $(n \geqq 2)$, μ be a C^∞ function defined on S. Then we have the following asymptotic formula:*

(3.11)

$$\begin{aligned} I(x) &= \int_S e^{i(x, s)} \mu(s) dS \\ &= \mu(\theta) \left(\frac{2\pi}{|x|} \right)^{(n-1)/2} e^{i(|x| - (\pi/4)(n-1))} + \mu(-\theta) \left(\frac{2\pi}{|x|} \right)^{(n-1)/2} e^{-i(|x| - (\pi/4)(n-1))} + q(x) \end{aligned}$$

as $|x| \to \infty$ along the ray $x = |x|\theta$, where $q(x)$ satisfies

$$\left(\frac{\partial}{\partial x} \right)^\nu q(x) = O(|x|^{-(n+1)/2}) \quad \text{as } |x| \to \infty \tag{3.12}$$

for each multi-index ν.

Applying this theorem to (3.10) we find

$$J(x', \nu)=\left(\frac{2\pi}{i\nu r}\right)^{1/2} e^{i\nu r}\phi(\nu\theta)+\left(\frac{2\pi}{-i\nu r}\right)^{1/2} e^{-i\nu r}\phi(-\nu\theta)+q_0(x', \nu), \tag{3.13}$$

where

$$x'=r\theta, \qquad r=|x'|\geqq 0, \qquad \theta\in S^1, \tag{3.14}$$

and we get

$$|q_0(x', \nu)|M_0|\nu x'|^{-3/2} \qquad \text{for } |x'|\geqq 1. \tag{3.15}$$

Here $M_0=M_0(\phi)$ is a positive constant which is independent of x', $\theta\in S^1$ and $\nu>0$. In (3.13) the square roots are defined by the convention that if $z=\pm i|z|$ then $z^{1/2}=e^{\pm i(\pi/4)}|z|^{1/2}$ with $|z|^{1/2}\geqq 0$.

Substituting (3.13) in (3.9), we obtain

$$\begin{aligned} w(t, x)=&(2\pi ir)^{-1/2}\int_0^\infty e^{i\nu(r-c_1t)-\nu c_2|x_3|}\nu\phi(\nu\theta)d\nu \\ &+(-2\pi ir)^{-1/2}\int_0^\infty e^{-i\nu(r-c_1t)-\nu c_2|x_3|}\nu\phi(-\nu\theta)d\nu \\ &+q_1(t, x) \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} q_1(t, x)=&(2\pi)^{-1}\int_0^\infty e^{-i\nu c_1 t-\nu c_2|x_3|}\nu^{3/2}q_0(x', \nu)d\nu, \\ |q_1(t, x)|\leqq&(2\pi)^{-1}M_0|x'|^{-3/2}\int_0^\infty e^{-\nu c_2|x_3|}d\nu \\ =&(2\pi)^{-1}M_0c_2^{-1}|x'|^{-3/2}|x_3|^{-1}. \end{aligned} \tag{3.17}$$

From (3.16), it follows that $q_1(t, x)$ is a continuous function of $x=(x', x_3)$. Therefore we have

$$|q_1(t, x)|\leqq M(1+|x'|^{3/2})^{-1}(1+|x_3|)^{-1} \qquad \text{for } x=(x', x_3)\in \boldsymbol{R}^3, \tag{3.18}$$

where $M=\max((2\pi)^{-1}M_0c_2^{-1}, \max_{t, |x|\leq 1}|q_1(t, x)|)$ is independent of t.

Let us define the functions $G_\phi^\pm(\tau, \theta, x_3)$ by

$$G_\phi^\pm(\tau, \theta, x_3)=(\pm 2\pi i)^{-1/2}\int_0^\infty e^{\pm i\nu\tau-\nu c_2|x_3|}\nu\phi(\pm\nu\theta)d\nu, \quad \tau, x_3\in\boldsymbol{R}, \ \theta\in S^1. \tag{3.19}$$

Then we have

$$\begin{aligned} w(t, x)=&\frac{G_\phi^+(r-c_1t, \theta, x_3)}{\sqrt{r}}+\frac{G_\phi^-(r+c_1t, \theta, x_3)}{\sqrt{r}}+q_1(t, x) \\ &x'=r\theta, \ r=|x'|\geqq 0, \ \theta\in S^1, \ x_3\in\boldsymbol{R}. \end{aligned} \tag{3.20}$$

We prepare the following four lemmas.

LEMMA 3.2. *For every $\phi(\eta')\in L^2(\boldsymbol{R}^2)$, $G_\phi^+(\tau, \theta, x_3)$ can be define and we have*

$$\|G_\phi^+(\tau, \theta, x_3)\|_{L^2(\boldsymbol{R}\times S\times \boldsymbol{R})}=\frac{1}{\sqrt{c_2}}\|\phi(\eta')\|_{L^2(\boldsymbol{R}^2)}\,. \tag{3.21}$$

PROOF. By Fubini's theorem, we have

$$\begin{aligned}
&\|G_\phi^+(\tau, \theta, x_3)\|^2_{L^2(\boldsymbol{R}\times S\times \boldsymbol{R})}\\
&=\int_{\boldsymbol{R}}\int_{S^1}\int_{\boldsymbol{R}}|G_\phi^+(\tau, \theta, x_3)|^2 d\tau d\theta dx_3=\int_{\boldsymbol{R}}\int_{S^1}\left(\int_{\boldsymbol{R}}|G_\phi^+(\tau, \theta, x_3)|^2 d\tau\right)d\theta dx_3\\
&=\int_{\boldsymbol{R}}\int_{S^1}\left(\int_{\boldsymbol{R}}|F_{\tau\to\nu}G_\phi^+(\tau, \theta, x_3)|^2 d\nu\right)d\theta dx_3 \qquad \text{(by Parseval's formula)}\\
&=\int_{\boldsymbol{R}}\int_{S^1}\left(\int_{-\infty}^{\infty}e^{-2\nu c_2|x_3|}|\nu\phi(\nu\theta)|^2 d\nu\right)d\theta dx_3\\
&=\int_{S^1}\int_{\boldsymbol{R}}\left(2\int_0^{\infty}e^{-2\nu c_2 x_3}\nu dx_3\right)|\phi(\nu\theta)|^2\nu d\nu d\theta\\
&=\frac{1}{c_2}\int_{S^1}\int_{\boldsymbol{R}}|\phi(\nu\theta)|^2\nu\, d\nu d\theta\\
&=\frac{1}{c_2}\|\phi(\eta')\|^2_{L^2(\boldsymbol{R}^2)}, \qquad \eta'=\nu\theta. \quad \square
\end{aligned}$$

LEMMA 3.3. *For every $\phi\in L^2(\boldsymbol{R}^2)$, we define $w_\phi^\infty(t, x)$ by*

$$w_\phi^\infty(t, x)=\frac{G_\phi^+\left(|x'|-c_1 t, \frac{x'}{|x'|}, x_3\right)}{\sqrt{|x'|}}\,. \tag{3.22}$$

Then we have

$$\begin{aligned}
\|w_\phi^\infty(t, \cdot)\|^2_{L^2(\boldsymbol{R}^3)}&\leqq\|G_\phi^+(\tau, \theta, x_3)\|^2_{L^2(\boldsymbol{R}\times S^1\times \boldsymbol{R})}\\
&=\frac{1}{c_2}\|\phi(\eta')\|^2_{L^2(\boldsymbol{R}^2)}\,.
\end{aligned} \tag{3.23}$$

PROOF.

$$\begin{aligned}
\|w_\phi^\infty(t, \cdot)\|^2_{L^2(\boldsymbol{R}^3)}&=\int_{\boldsymbol{R}}\int_{S^1}\int_0^{\infty}|G_\phi^+(r-c_1 t, \theta, x_3)|^2 dr d\theta dx_3\\
&=\int_{\boldsymbol{R}}\int_{S^1}\int_{-c_1 t}^{\infty}|G_\phi^+(\tau, \theta, x_3)|^2 d\tau d\theta dx_3\\
&\leqq\|G_\phi^+(\tau, \theta, x_3)\|^2_{L^2(\boldsymbol{R}\times S^1\times \boldsymbol{R})}\,. \quad \square
\end{aligned}$$

LEMMA 3.4. *The function $w(t, x)$ defined by* (3.8) *for $\phi\in\mathcal{D}(\boldsymbol{R}^2)$ can also be defined for any $\phi\in L^2(\boldsymbol{R}^2)$ and we have*

(3.24) $$\|w(t,\ \cdot)\|_{L^2(\boldsymbol{R}^3)}=\frac{1}{\sqrt{c_2}}\|\phi(\eta')\|_{L^2(\boldsymbol{R}^2)}\ .$$

PROOF. In fact,

$$\|w(t,\ \cdot)\|_{L^2(\boldsymbol{R}^3)}^2=\int_{\boldsymbol{R}}\Big(\int_{\boldsymbol{R}^2}|w(t,\ x)|^2dx'\Big)dx_3$$

$$=\int_{\boldsymbol{R}}\Big(\int_{\boldsymbol{R}^2}|e^{-itc_1|\eta'|-c_2|\eta'||x_3|}\sqrt{|\eta'|}\phi(\eta')|^2d\eta'\Big)dx_3$$

(Parseval's formula)

$$=\int_{\boldsymbol{R}^2}\Big(2\int_0^{\infty}|\eta'|e^{-2c_2|\eta'|x_3}dx_3\Big)|\phi(\eta')|^2d\eta'$$

$$=\frac{1}{c_2}\|\phi(\eta')\|_{L^2(\boldsymbol{R}^2)}^2\ .\quad\square$$

LEMMA 3.5. (See Wilcox [10, Lemma 2.7]). *Assume that $u(t, x)$ has the properties*

(3.25) $$u(t,\ \cdot)\in L^2(\boldsymbol{R}^n)\qquad \textit{for every } t>t_0,$$

(3.26) $$\lim_{t\to\infty}\|u(t,\ \cdot)\|_{L^2(K)}=0\qquad \textit{for every compact } K\subset\boldsymbol{R}^n,$$

(3.27) $$|u(t,\ x)|\leqq g(x)\in L^2(\boldsymbol{R}^n).$$

where t_0 is a constant. Then

(3.28) $$\lim_{t\to\infty}\|u(t,\ \cdot)\|_{L^2(\boldsymbol{R}^n)}=0\ .$$

THEOREM 3.6. *Let $w_\phi=w$ and w_ϕ^∞ be the functions defined by* (3.8) *and* (3.22) *for $\phi\in L^2(\boldsymbol{R}^2)$, respectively. Then*

(3.29) $$\lim_{t\to\infty}\|w_\phi(t,\ \cdot)-w_\phi^\infty(t,\ \cdot)\|_{L^2(\boldsymbol{R}^3)}=0\ .$$

PROOF. First we consider the case where $\phi\in\mathcal{D}(\boldsymbol{R}^2)$. Putting

(3.30) $$u(t,\ x)=w_\phi(t,\ x)-w_\phi^\infty(t,\ x),$$

we verify that (3.25)-(3.27) hold for $u(t, x)$. From (3.30), Lemma 3.3 and Lemma 3.4, $u(t, \cdot)\in L^2(\boldsymbol{R}^3)$ for every $t\in\boldsymbol{R}$. Next consider

$$w_\phi(t,\ x)=\frac{1}{2\pi}\int_{S^1}\int_0^{\infty}e^{-ic_1\nu t}\psi(x',\ \nu,\ \omega)d\nu d\omega\,,$$

where

$$\psi(x,\ \nu,\ \omega)=e^{i\nu x'\cdot\omega-\nu c_2|x_3|}\nu^{3/2}\phi(\nu\omega)\,.$$

Noting that ψ is a C^1 function of ν in $[0, \infty)$ for fixed (x', ω), we perform an integration by parts with respect to ν. Then we get the estimate

$$\sup_{x\in K}|w_{\phi}(t, x)|\leqq\frac{M_K}{|t|},$$

where M_K is a positive constant which depends on K and ϕ but does not depend on t. As for $w_{\phi}^{\infty}(t, x)$, we have for any $d>0$

$$\|w_{\phi}^{\infty}(t, \cdot)\|_{L^2(B_d)}^2\leqq\int_{-d}^{d}\int_{S^1}\int_0^d|G_{\phi}^{+}(r-c_1t, \theta, x_3)|^2 drd\theta dx_3$$

$$=\int_{-d}^{d}\int_{S^1}\int_{-\infty}^{\infty}\chi_{[-c_1t,\, d-c_1t]}(s)|G_{\phi}^{+}(s, \theta, x_3)|^2 dsd\theta dx_3,$$

where $B_d=\{x\,;\,|x|\leqq d\}$ and $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a, b]$. The last integral tends to zero when $t\to\infty$ by Lebesgue's dominated convergence theorem. Thus

$$\lim_{t\to\infty}\|u(t, \cdot)\|_{L^2(B_d)}=0 \qquad \text{as } t\to\infty .$$

From (3.20), (3.22) and (3.30), it follows that

$$u(t, x)=\frac{G_{\phi}^{-}\left(|x'|+c_1t, \frac{x'}{|x'|}, x_3\right)}{\sqrt{|x'|}}+q_1(t, x), \qquad x'=r\theta. \tag{3.31}$$

An integration by parts in (3.19) gives

$$G_{\phi}^{-}(\tau, \theta, x_3)=\frac{1}{\sqrt{-2\pi i}}\frac{1}{-(i\tau+c_2|x_3|)}\int_0^{\infty}e^{-i\nu\tau-\nu c_2|x_3|}\frac{\partial}{\partial\nu}(\nu\phi(-\nu\theta))d\nu .$$

From this we deduce the estimate

$$\left|\frac{G_{\phi}^{-}\left(|x'|+c_1t, \frac{x'}{|x'|}, x_3\right)}{\sqrt{|x'|}}\right|\leqq g(x)\in L^2(\boldsymbol{R}^3) \qquad \text{for } |t|\geqq 1, \tag{3.32}$$

where

$$g(x)=\begin{cases}\dfrac{M}{(c_1+|x'|+c_2|x_3|)\sqrt{|x'|}} & \text{for } |x'|\geqq 1,\\[2ex] \dfrac{M}{(c_1+c_2|x_3|)\sqrt{|x'|}} & \text{for } |x'|\leqq 1,\end{cases} \tag{3.33}$$

and M is a suitable constant. From (3.18), (3.31) and (3.32) with (3.33), we see that (3.27) holds for this $u(t, x)$.

Now we show (3.29) for general $\phi\in L^2(\boldsymbol{R}^2)$. For arbitrary $\varepsilon>0$, there exists $\phi_0\in\mathcal{D}(\boldsymbol{R}^2)$ such that $\|\phi-\phi_0\|_{L^2(\boldsymbol{R}^2)}<\varepsilon$, because $\mathcal{D}(\boldsymbol{R}^2)$ is dense in $L^2(\boldsymbol{R}^2)$. Then

$$\|w_\phi(t,\ \cdot)-w_\phi^\infty(t,\ \cdot)\|_{L^2(R^3)}$$
$$\leqq\|w_\phi(t,\ \cdot)-w_{\phi_0}(t,\ \cdot)\|_{L^2(R^3)}+\|w_\phi^\infty(t,\ \cdot)-w_{\phi_0}^\infty(t,\ \cdot)\|_{L^2(R^3)}$$
$$+\|w_{\phi_0}(t,\ \cdot)-w_{\phi_0}^\infty(t,\ \cdot)\|_{L^2(R^3)}$$
$$=\frac{2}{\sqrt{c_2}}\|\phi-\phi_0\|_{L^2(R^2)}+\|w_{\phi_0}(t,\ \cdot)-w_{\phi_0}^\infty(t,\ \cdot)\|_{L^2(R^3)}$$
$$\leqq\frac{2\varepsilon}{\sqrt{c_2}}+\|w_{\phi_0}(t,\ \cdot)-w_{\phi_0}^\infty(t,\ \cdot)\|_{L^2(R^3)}\ .$$

Since $\phi_0\in\mathcal{D}(\boldsymbol{R}^2)$, there exists $t_0>0$ such that for any $t\geqq t_0$

$$\|w_{\phi_0}(t,\ \cdot)-w_{\phi_0}^\infty(t,\ \cdot)\|_{L^2(R^3)}<\varepsilon\ .$$

Thus (3.29) holds for any $\phi\in L^2(\boldsymbol{R}^2)$. This completes the proof of Theorem 3.6. □

In order to state our main theorem in this section, we recall some relations. When $f\in\mathcal{H}$ and $g\in D(A^{-1/2})$, $h=f+iA^{-1/2}g\in\mathcal{H}$. Let $\hat{f}_{1j}^{St}$ and $\hat{g}_{1j}^{St}$ be the Fourier transforms of f and g with respect to the generalized eigenfunction ψ_{1j}^{St} of A, respectively. Then

$$\hat{h}_{1j}^{St}(\eta)=\hat{f}_{1j}^{St}(\eta)+i\frac{1}{c_{St}|\eta'|}\hat{g}_{1j}^{St}(\eta)\in L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3)$$

and the Stoneley components $v_{1j}^{St}(t,\ \cdot)\in L^2(\boldsymbol{R}^3,\ \boldsymbol{C}^3)$ $(j\in M)$ of the solution $v(t,\ x)$ of the elastic wave propagation problem defined by (1.12) and (1.13) can be represented in the form (3.5):

$$v_{1j}^{St}(t,\ x)=\underset{R\to\infty}{\text{l.i.m.}}\left(\frac{1}{2\pi}\right)\int_{|\eta'|\leq R}e^{i(x'\cdot\eta'-tc_{St}|\eta'|)-c_{0j}|\eta'||x_3|}$$
$$\times \mathrm{U}(\eta')\mathrm{C}(\phi_{1j}^{St}(\eta')\oplus 0_{1\times 1})Q(\eta')\sqrt{|\eta'|}\,k_{1j}^{St}(\eta')d\eta',\quad j\in M,$$

where $\phi_{1j}^{St}(\eta')$ and $k_{1j}^{St}(\eta')$ be the functions defined by (3.2) and (3.3), respectively, i. e.,

$$\psi_{1j}^{St}(x_3,\ \eta)=\frac{|\eta'|}{\xi-ic_{0j}|\eta'|}e^{-c_{0j}|\eta'||x_3|}\phi_{1j}^{St}(\eta')P_j(\eta),$$
$$\hat{h}_{1j}^{St}(x_3,\ \eta)=\frac{\sqrt{|\eta'|}}{\xi+ic_{0j}|\eta'|}k_{1j}^{St}(\eta').$$

By (3.4),

$$k_{1j}^{St}(\eta')\in L^2(\boldsymbol{R}^2,\ \boldsymbol{C}^3),$$

and

$$\mathrm{U}(\eta')\mathrm{C}(\phi_{1j}^{St}(\eta')\oplus 0_{1\times 1})Q(\eta')k_{1j}^{St}(\eta')\in L^2(\boldsymbol{R}^2,\ \boldsymbol{C}^3).$$

Taking as ϕ in Theorem 3.6 each component of the matrix function

$$U(\eta')C(\phi_{1j}^{St}(\eta')\oplus O_{1\times 1})Q(\eta')\hat{h}_{1j}^{St}(\eta'),$$

then we obtain the following main theorem in this section.

THEOREM 3.7. *We assume that*

$$f\in D(A^{1/2})\cap\mathcal{H},\quad g\in\mathcal{H}\cap D(A^{-1/2}),\quad \mathrm{Dis}(c_{s\gamma}^2)>0.$$

Let $v_{1j}^{St\infty}(t, x)$ $(j\in M)$ *be the functions defined by*

$$v_{1j}^{St\infty}(t, x)=\frac{G_{St}(r-c_{St}t,\ \theta,\ x_3)}{\sqrt{r}}\quad x'=r\theta,\ r=|x'|\geqq 0,\ \theta\in S^1, \tag{3.34}$$

where

$$G_{St}(\tau,\ \theta,\ x_3)=\underset{R\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi i}}\int_0^R e^{i\nu\tau-c_{0j}\nu|x_3|} \tag{3.35}$$

$$\times\ U(\nu\theta)C(\phi_{1j}^{St}(\nu\theta)\oplus O_{1\times 1})Q(\nu\theta)\sqrt{\nu}\,\hat{k}_{1j}^{St}(\nu\theta)\frac{1}{\rho(x_3)}d\nu.$$

Then we have

$$\lim_{t\to\infty}\|v_{1j}^{St}(t,\ \cdot)-v_{1j}^{St\infty}(t,\ \cdot)\|_{\mathcal{H}}=0. \tag{3.36}$$

$v_{1j}^{St\infty}(t, x)\in\mathcal{H}$ *is called asymptotic wave function for Stoneley component* $v_{1j}^{St}(t, x)$ *of the solution* $v(t, x)$.

§4. Transient Free (P, SV, SH) Waves

This section deals with the P, SV components $v_{1j}^{\pm}(t, x)$ $(j\in M)$ and the SH components $v_{2k}^{\pm}(t, x)$ $(k\in N)$ defined by (2.19) and (2.22), (2.21) and (2.24), respectively. It is shown in Section 5 below that each such components are transient and free in the sense that they behave like a diverging cylindrical wave when $t\to\infty$. The proof of these statements are based on asymptotic approximations of $v_{1j}^{\pm}(t, x)$ $(j\in M)$ and $v_{2k}^{\pm}(t, x)$ $(k\in N)$ for large t which are derived in this section.

In this section it is assumed as in Section 3 that the initial data $f(x)$ and $g(x)$ are real-valued functions. We study mainly the asymptotic behavior for large times of the component $v_{1p_1}^{+}(t, x)$, because the other components $v_{1p_1}^{-}(t, x)$, $v_{1j}^{\pm}(t, x)$ $(j\in\{s_1, p_2, s_2\})$ and $v_{2k}^{\pm}(t, x)$ $(k\in N=\{s_1, s_2\})$ can be handled in a quite similar way.

If $f\in\mathcal{H}$, $g\in D(A^{-1/2})$, the component $v_{1p_1}^{+}(t, x)$ has the following spectral integral representation

$$v_{1p_1}^+(t,\ x)=\underset{R\to\infty}{\mathrm{l.i.m.}}\Big(\frac{1}{2\pi}\Big)\int\!\!\int_{|\eta|\le R} e^{i(x'\cdot\eta'-tc_{p_1}|\eta|)}\mathrm{U}(\eta')\times\mathrm{C}(\psi_{1p_1}^+(x_3,\ \eta)\oplus 0_{1\times 1})\hat{h}_{1p_1}^+(\eta)d\eta\,. \tag{4.1}$$

Here $\mathrm{U}(\eta')$, C are the matrices defined by (2.1), $\hat{h}_{1p_1}^+(\eta)$ is defined by (2.22), and $\psi_{1p_1}^+(x_3,\ \eta)$ is a generalized eigenfunction of $A_1(\eta')$.

We now recall that $\psi_{1p_1}^+(x_3,\ \eta)$ has the following form:

$$\psi_{1p_1}^+(x_3,\ \eta)=\begin{cases}\psi_{1p_1}^{I+}(x_3,\ \eta), & x_3<0,\\ \psi_{1p_1}^{II+}(x_3,\ \eta), & x_3>0,\end{cases} \tag{4.2}$$

$$\psi_{1p_1}^{I+}(x_3,\ \eta)=\begin{cases}e^{i\xi x_3}\alpha_1(\eta)+e^{-i\xi x_3}\alpha_2(\eta)+e^{-i\xi_{s_1}(\eta,\lambda_{p_1})x_3}\alpha_3(\eta), & \xi>0,\\ 0, & \xi<0,\end{cases} \tag{4.3}$$

$$\psi_{1p_1}^{II+}(x_3,\ \eta)=\begin{cases}e^{i\xi_{p_2}(\eta,\lambda_{p_1})x_3}\alpha_4(\eta)+e^{i\xi_{s_2}(\eta,\lambda_{p_1})x_3}\alpha_5(\eta), & \xi>0,\\ 0, & \xi<0.\end{cases} \tag{4.4}$$

Here $\alpha_i(\eta)$ $(i=1,\ \cdots,\ 5)$ are bounded continuous 2×2 matrix functions of $\eta=(\eta',\ \xi)=(\eta_1,\ \eta_2,\ \xi)$ and

$$\xi_{s_1}(\eta,\ \lambda_{p_1})=\begin{cases}\pm\sqrt{\dfrac{c_{p_1}^2}{c_{s_1}^2}|\eta|^2-|\eta'|^2}, & c_{p_1}|\eta|>c_{s_1}|\eta'|,\\ i\sqrt{|\eta'|^2-\dfrac{c_{p_1}^2}{c_{s_1}^2}|\eta|^2}, & c_{p_1}|\eta|<c_{s_1}|\eta'|,\end{cases} \tag{4.5}$$

(cf. [8, (4.9), (4.10)]).

Then $v_{1p_1}^+(t,\ x)$ has for $x_3<0$ the form

$$v_{1p_1}^+(t,\ x)=\underset{R\to\infty}{\mathrm{l.i.m.}}\Big(\frac{1}{2\pi}\Big)\int\!\!\int_{|\eta|\le R,\xi>0} e^{i(x'\cdot\eta'-tc_{p_1}|\eta|)}\times\mathrm{U}(\eta')\mathrm{C}(\psi_{1p_1}^{I+}(x_3,\ \eta)\oplus 0_{1\times 1})\hat{h}_{1p_1}^+(\eta)d\eta, \tag{4.6}$$

and the decomposition

$$\begin{aligned}&=\underset{R\to\infty}{\mathrm{l.i.m.}}\Big(\frac{1}{2\pi}\Big)\int\!\!\int_{|\eta|\le R,\xi>0} e^{i(x'\cdot\eta'+\xi x_3-tc_{p_1}|\eta|)}\mathrm{U}(\eta')\mathrm{C}(\alpha_1(\eta)\oplus 0_{1\times 1})\hat{h}_{1p_1}^+(\eta)d\eta\\ &+\underset{R\to\infty}{\mathrm{l.i.m.}}\Big(\frac{1}{2\pi}\Big)\int\!\!\int_{|\eta|\le R,\xi>0} e^{i(x'\cdot\eta'-\xi x_3-tc_{p_1}|\eta|)}\mathrm{U}(\eta')\mathrm{C}(\alpha_2(\eta)\oplus 0_{1\times 1})\hat{h}_{1p_1}^+(\eta)d\eta\\ &+\underset{R\to\infty}{\mathrm{l.i.m.}}\Big(\frac{1}{2\pi}\Big)\int\!\!\int_{|\eta|\le R,\xi>0} e^{i(x'\cdot\eta'-\xi_{s_1}(\eta,\lambda_{p_1})x_3-tc_{p_1}|\eta|)}\mathrm{U}(\eta')\mathrm{C}(\alpha_3(\eta)\oplus 0_{1\times 1})\hat{h}_{1p_1}^+(\eta)d\eta\\ &=V_1(t,\ x)+V_2(t,\ x)+V_3(t,\ x)\qquad \text{if } x_3<0.\end{aligned} \tag{4.7}$$

Since we can decompose $v_{1p_1}^+(t,\ x)$ $(x_3>0)$ into a sum of integral expression of type $V_3(t,\ x)$ using (4.4), we consider $v_{1p_1}^+(t,\ x)$ only in $\boldsymbol{R}_-^3=\{x=(x',\ x_3),\ x'\in\boldsymbol{R}^2,\ x_3<0\}$.

First we consider $V_1(t, x)$. Let $Y(\xi)$ be the Heaviside function of ξ (i.e. $Y(\xi)=1$ for $\xi>0$ and $=0$ for $\xi<0$) and put

$$\Phi(\eta)=Y(\xi)\mathrm{U}(\eta')\mathrm{C}(\alpha_1(\eta)\oplus 0_{1\times 1})\hat{h}_{1p_1}^{+}(\eta). \tag{4.8}$$

Then $\Phi\in L^2(\boldsymbol{R}_+^3, \boldsymbol{C}^3)$. As in Section 3, we can extend the result obtained for $\Phi\in\mathcal{D}(\boldsymbol{R}_+^3, \boldsymbol{C}^3)$ to the result for $\Phi\in L^2(\boldsymbol{R}_+^3, \boldsymbol{C}^3)$ by using the fact that $\mathcal{D}(\boldsymbol{R}_+^3, \boldsymbol{C}^3)$ is dense in $L^2(\boldsymbol{R}_+^3, \boldsymbol{C}^3)$. Therefore it suffices to consider the integral

$$W_1(t, x)=\frac{1}{2\pi}\int_{\boldsymbol{R}^3} e^{i(x'\cdot\eta'+x_3\xi-tc_1|\eta|)}\phi(\eta)d\eta,\ \phi\in\mathcal{D}(\boldsymbol{R}_+^3)=\mathcal{D}(\boldsymbol{R}_+^3, \boldsymbol{C})\subset\mathcal{D}(\boldsymbol{R}^3, \boldsymbol{C}). \tag{4.9}$$

Then

$$\frac{\partial}{\partial\xi}(x'\cdot\eta'+x_3\xi-tc_1\sqrt{|\eta'|^2+\xi^2})=x_3-tc_1\frac{\xi}{\sqrt{|\eta'|^2+\xi^2}}<0$$

if $x_3<0$, $\xi>0$ and $t>0$.

This means that the phase function has no stationary point on $\operatorname{supp}\phi(\eta)$ and therefore we can see integrating by parts with respect to ξ that $W_1(t, x)$ tends to zero when $t\to\infty$ for fixed x and uniformly an each compact set $K\subset\boldsymbol{R}_-^3$. In order to find the asymptotic behavior of $W_1(t, x)$ as $|x|\to\infty$, we introduce spherical coordinates

$$\eta=\nu\omega,\quad \nu=|\eta|\geqq 0,\quad \omega\in S^2.$$

We find

$$W_1(t, x)=\frac{1}{2\pi}\int_0^\infty \nu^2 e^{-ic_1t\nu}J(x, \nu)d\nu, \tag{4.10}$$

where

$$J(x, \nu)=\int_{S^2} e^{i\nu x\cdot\omega}\phi(\nu\omega)d\omega. \tag{4.11}$$

By Theorem 3.1 we have

$$J(x, \nu)=\left(\frac{2\pi}{i\nu r}\right)e^{i\nu r}\phi(\nu\theta)+\left(\frac{2\pi}{-i\nu r}\right)e^{-i\nu r}\phi(-\nu\theta)+q(x, \nu), \tag{4.12}$$

where

$$x=r\theta,\quad r=|x|\geqq 0,\quad \theta\in S^2,$$

and

$$|q(x, \nu)|\leqq\frac{M_0}{|\nu r|^2}\quad \text{for } |\nu x|>0. \tag{4.13}$$

Note that if $x\in\boldsymbol{R}_-^3$, $\phi(\nu\theta)=0$ because $\theta_3<0$ and $\operatorname{supp}\phi\subset\boldsymbol{R}_+^3$. Now we define following Wilcox's procedure [10]

$$G_1^{\pm}(\tau, \theta)=\int_0^\infty e^{\pm i\nu\tau}(\pm i\nu)\phi(\pm\nu\theta)d\nu, \tag{4.14}$$

then we have

$$W_1(t, x)=\frac{G_1^-(r+c_1t, \theta)}{r}+q_1(t, x), \tag{4.15}$$

where

$$q_1(t, x)=\frac{1}{2\pi}\int_0^\infty \nu^2 e^{-ic_1t\nu}q(x, \nu)d\nu. \tag{4.16}$$

From (4.13), we get the estimate

$$|q_1(t, x)|\leqq\frac{M_1}{|x|^2}, \quad \text{for } |x|\geqq 1. \tag{4.17}$$

By the same argument as in Section 3 which is due to Wilcox, one shows that

$$\lim_{t\to\infty}\|W_1(t, \cdot)\|_{L^2(R_-^3)}=0. \tag{4.18}$$

As for $V_2(t, x)$, it suffices to consider $W_1(t, x', -x_3)$ for $x_3<0$. In fact,

$$\begin{aligned} W_2(t, x)&=\frac{1}{2\pi}\int_{R_+^3} e^{i(x'\cdot\eta'-x_3\xi-tc_1|\eta|)}\phi(\eta)d\eta, \quad \phi\in\mathcal{D}(R_+^3) \\ &=\frac{1}{2\pi}\int_{R^3} e^{i(x'\cdot\eta'+x_3\xi-tc_1|\eta|)}\phi(\eta', -\xi)d\eta'd\xi \\ &=W_1(t, x', -x_3). \end{aligned} \tag{4.19}$$

Note that if $x\in R_-^3$ i.e. $x=r\theta$, $\theta_3<0$, then $\phi(-\nu\theta', -(-\nu\theta_3))=0$. Hence we find

$$W_2(t, x)=\frac{G_2^+(r-c_1t, \theta)}{r}+q_1(t, x', -x_3), \tag{4.20}$$

where

$$G_2^+(\tau, \theta)=\int_0^\infty e^{i\nu\tau}(-i\nu)\phi(\nu\theta', -\nu\theta_3)d\nu. \tag{4.21}$$

In this case, we can also show that

$$\lim_{t\to\infty}\|W_2(t. \cdot)-W_2^\infty(t, \cdot)\|_{L^2(R_-^3)}=0, \tag{4.22}$$

where

$$W_2^\infty(t, x)=\frac{G_2^+(r-c_1t, \theta)}{r}, \quad x=r\theta. \tag{4.23}$$

Next we consider the following integral

$$W_3(t, x)=\frac{1}{2\pi}\int_{R_+^3} e^{i(x'\cdot\eta'-x_3\zeta(\eta)-tc_1|\eta|)}\phi(\eta)d\eta, \quad \phi\in\mathcal{D}(R_+^3), \tag{4.24}$$

where

$$(4.25)\qquad \zeta(\eta)=\begin{cases}\pm\sqrt{c_2^2|\eta|^2-|\eta'|^2}, & c_2|\eta|>|\eta'|,\\ i\sqrt{|\eta'|^2-c_2^2|\eta|^2}, & c_2|\eta|<|\eta'|.\end{cases}$$

We take a C^∞ partition of unity $\{\chi_1, \chi_2, \chi_3\}$ in a neighborhood of supp ϕ ($\bar{\Omega}\subset \boldsymbol{R}_+^3$) such that

$$(4.26)\qquad \begin{aligned}&0\leqq\chi_j(\eta)\leqq 1 \quad (j=1, 2, 3), \qquad \chi_j\in C_0^\infty(\Omega_j),\\ &\chi_1(\eta)+\chi_2(\eta)+\chi_3(\eta)=1 \quad \text{on } \Omega,\\ &\Omega_1=\left\{\eta\,;|\eta|^2>\frac{|\eta'|^2}{(c_2^2-\varepsilon)}\right\},\\ &\Omega_2=\left\{\eta\,;\frac{|\eta'|^2}{c_2^2+2\varepsilon}<|\eta|^2<\frac{|\eta'|^2}{c_2^2-2\varepsilon}\right\},\\ &\Omega_3=\left\{\eta\,;|\eta|^2<\frac{|\eta'|^2}{c_2^2+\varepsilon}\right\},\end{aligned}$$

where ε is a sufficiently small positive constant. Using this partition of unity, we decompose $W_3(t, x)$ as follows:

$$(4.27)\qquad \begin{aligned}W_3(t, x)&=\sum_{j=1}^3\frac{1}{2\pi}\int_{\Omega_j}e^{i(x'\cdot\eta'-x_3\zeta(\eta)-tc_1|\eta|)}\chi_j(\eta)\phi(\eta)d\eta\\ &=\sum_{j=1}^3 W_{3j}(t, x), \quad \text{respectively.}\end{aligned}$$

First consider

$$(4.28)\qquad W_{31}^{\pm}(t, x)=\frac{1}{2\pi}\int_{\Omega_1}e^{i(x'\cdot\eta'\pm x_3\sqrt{c_2^2|\eta|-|\eta'|}-tc_1|\eta|)}\chi_1(\eta)\phi(\eta)d\eta\,.$$

Making the change of variables $(\eta', \xi)\rightarrow(\eta', \lambda)$, $\lambda=\sqrt{c_2^2|\eta|^2-|\eta|^2}$, we get

$$(4.29)\qquad \begin{aligned}W_{31}^{\pm}(t, x)=\frac{1}{2\pi}&\int_{\{(\eta',\lambda);\,\eta'\in R^2,\,\lambda>0\}}e^{i(x'\cdot\eta'\pm x_3\lambda-t\frac{c_1}{c_2}\sqrt{|\eta'|^2+\lambda^2})}\\ &\times J(\eta', \lambda)\chi_1(\eta', \xi(\eta', \lambda))\phi(\eta', \xi(\eta', \lambda))d\eta'd\lambda,\end{aligned}$$

where

$$(4.30)\qquad J(\eta', \lambda)=\frac{\partial(\eta_1, \eta_2, \xi)}{\partial(\eta_1, \eta_2, \lambda)}=\frac{\lambda}{c_2^2\xi}, \quad \xi=\xi(\eta', \lambda)=\frac{1}{c_2}\sqrt{\lambda^2-(c_2^2-1)|\eta'|^2}.$$

This transformation is non-singular on a suitable neighborhood of supp $\chi_1\phi$. Noting that

$$J(\eta', \lambda)\chi_1(\eta', \xi(\eta', \lambda))\phi(\eta', \xi(\eta', \lambda))\in\mathcal{D}(\{(\eta', \lambda)\,;\,\eta'\in\boldsymbol{R}^2, \lambda>0\}),$$

we see that (4.29) is an integral of the same type of (4.9) and (4.19). Therefore we can show that

(4.31) $$\lim_{t\to\infty}\|W_{31}^{+}(t,\cdot)\|_{L^2(R_-^3)}=0,$$

and

(4.32) $$\lim_{t\to\infty}\|W_{31}^{-}(t,\cdot)-W_{31}^{-\infty}(t,\cdot)\|_{L^2(R_-^3)}=0.$$

Here

(4.33) $$W_{31}^{-\infty}(t,x)=\frac{G_{31}^{+}\left(r-\frac{c_1}{c_2}t,\theta\right)}{r},\quad x=r\theta,$$

and

(4.34) $$G_{31}^{+}(\tau,\theta)=\int_0^\infty e^{i\nu\tau}(-i\nu)J(\nu\theta',-\nu\theta_3)$$
$$\times\chi_1(\nu\theta',\xi(\nu\theta',-\nu\theta_3))\phi(\nu\theta',\xi(\nu\theta',-\nu\theta_3))d\nu.$$

Next consider

(4.35) $$W_{33}(t,x)=\frac{1}{2\pi}\int_{R_+^3}e^{i(x'\cdot\eta'-tc_1|\eta|)+x_3\sqrt{|\eta'|^2-c_2^2|\eta|^2}}\chi_3(\eta)\phi(\eta)d\eta.$$

Making use of spherical coordinates in (t,x)-space:

(4.36) $$t=r\theta_0,\quad x_j=r\theta_j\ (j=1,2,3),\quad r=\sqrt{t^2+|x|^2}\geqq 0,\quad \theta\in S^3.$$

We write the integral in (4.35) as follows:

(4.37) $$W_{33}(t,x)=\frac{1}{2\pi}\int_{R_+^3}e^{ir(\theta'\cdot\eta'-c_1\theta_0|\eta|)+r\theta_3\sqrt{|\eta'|^2-c_2^2|\eta|^2}}\chi_3(\eta)\phi(\eta)d\eta.$$

Then

(4.38) $$p(\theta,\eta)=\theta'\cdot\eta'-c_1\theta_0|\eta|-i\theta_3\sqrt{|\eta'|^2-c_2^2|\eta|^2},\quad \eta=(\eta',\xi),$$

is a complex phase function such that $\operatorname{Im}p(\theta,\eta)>0$ on $\operatorname{supp}\chi_3(\eta)\phi(\eta)$ when $\theta_3<0$. Since

$$\frac{\partial p}{\partial\eta_k}=\theta_k-c_1\theta_0\frac{\eta_k}{|\eta|}-i\theta_3\frac{(1-c_2^2)\eta_k}{\sqrt{|\eta'|^2-c_2^2|\eta|^2}},\quad k=1,2,$$
$$\frac{\partial p}{\partial\xi}=-c_1\theta_0\frac{\xi}{|\eta|}+i\theta_3\frac{c_2^2\xi}{\sqrt{|\eta'|^2-c_2^2|\eta|^2}}.$$

We find that

(4.39) $$\sum_{k=1}^{2}\left|\frac{\partial p}{\partial\eta_k}\right|^2+\left|\frac{\partial p}{\partial\xi}\right|^2=\sum_{k=1}^{2}\left(\theta_k-c_1\theta_0\frac{\eta_k}{|\eta|}\right)^2+c_1^2\theta_0^2\frac{\xi^2}{|\eta|^2}$$
$$+\theta_3^2\frac{(1-c_2^2)^2|\eta'|^2+c_2^4\xi^2}{(1-c_2^2)|\eta'|^2-c_2^2\xi^2}\geqq\exists\delta>0$$

on a suitable neighborhood of $\operatorname{supp}\chi_3\phi$. Then for the operator L

(4.40) $$L=\left(i\sum_{k=1}^{2}\left|\frac{\partial p}{\partial\eta_k}\right|^2+\left|\frac{\partial p}{\partial\xi}\right|^2\right)^{-1}\left(\sum_{k=1}^{m}\frac{\partial\bar p}{\partial\eta_k}\frac{\partial}{\partial\eta_k}+\frac{\partial\bar p}{\partial\xi}\frac{\partial}{\partial\xi}\right),$$

we have

$$e^{irp(\theta,\eta)}=\frac{1}{r}L[e^{irp(\theta,\eta)}], \tag{4.41}$$

where $\bar{p}$ denotes the complex conjugate of p. Using repeatedly this relation, we find

$$\begin{aligned} W_{33}(t, x) &= \frac{1}{2\pi r}\int_{R^3_+} L[e^{irp(\theta,\eta)}]\chi_3(\eta)\phi(\eta)d\eta \\ &= \frac{1}{2\pi r}\int_{R^3_+} e^{irp(\theta,\eta)}{}^tL(\chi_3(\eta)\phi(\eta))d\eta \\ &= \cdots \\ &= \frac{1}{2\pi r^l}\int_{R^3_+} e^{irp(\theta,\eta)}({}^tL)^l(\chi_3(\eta)\phi(\eta))d\eta . \end{aligned}$$

Here ${}^tL$ denotes the transpose operator of L. From this expression, we get the estimate

$$|W_{33}(t, x)|\leqq\frac{M_{p,\phi,l}}{(t^2+|x|^2)^{l/2}}, \tag{4.42}$$

where $M_{p,\phi,l}$ is a positive constant. Taking $l>[n/2]+1$, we deduce from the estimate (4.42) that

$$\lim_{t\to\infty}\|W_{33}(t,\cdot)\|_{L^2(R^3_-)}=0. \tag{4.43}$$

Now consider $W_{32}(t, x)$. From (2.19), we see that the linear operator

$$L^2(R^3_+, C^3)\ni Y(\xi)\hat{h}^+_{1p_1}(\eta)\mapsto v^+_{1p_1}(t,\cdot)\in L^2(R^3_-, C^3)$$

is continuous uniformly in $t\in R$. From (4.9) and (4.19), we have

$$\|W_1(t,\cdot)\|_{L^2(R^3_-)}=\sqrt{2\pi}\|e^{-itc_1|\eta|}\phi(\eta)\|_{L^2(R^3_+)},$$

$$\|W_2(t,\cdot)\|_{L^2(R^3_-)}=\sqrt{2\pi}\|e^{-itc_1|\eta|}\phi(\eta)\|_{L^2(R^3_+)}.$$

From these relations, it follows that the linear operators

$$L^2(R^3_+, C^3)\ni\hat{h}^+_{1p_1}(\eta)\mapsto V_1(t,\cdot)\in L^2(R^3_-, C^3),$$

$$L^2(R^3_+, C^3)\in\hat{h}^+_{1p_1}(\eta)\mapsto V_2(t,\cdot)\in L^2(R^3_-, C^3),$$

are continuous uniformly in $t\in R$. Hence the linear operator

$$L^2(R^3_+, C^3)\ni\hat{h}^+_{1p_1}(\eta)\mapsto V_3(t,\cdot)\in L^2(R^3_-, C^3)$$

is also continuous uniformly in $t\in R$ and we have

$$V_3(t, x)=W_{32}(t, x) \quad \text{for } \phi(\eta)=\chi_3(\eta)\mathrm{U}(\eta')\mathrm{C}(\alpha_3(\eta)\oplus O_{1\times1})\hat{h}^+_{1p_1}(\eta).$$

Thus for arbitrary $\delta>0$, there exists a $R>0$ for which

$$\|W_{32}(t, x)\|_{L^2(R^3_-\cap\{x;|x|\geq R\}, C^3)}<\delta \quad \text{for } \forall t\in \boldsymbol{R}.$$

Taking ε small enough, we have from (4.26) and (4.27)

$$\|W_{32}(t, \cdot)\|_{L^2(R^3_-\cap\{x;|x|\leq R\}, C^3)}<\delta.$$

Note that

$$\Phi(\eta)=Y(\xi)\mathrm{U}(\eta')\mathrm{C}(\alpha_i(\eta)\oplus 0_{1\times 1})\hat{h}^+_{1p_1}(\eta)\in\mathcal{D}(\{\eta\in \boldsymbol{R}^3_+;\ \eta'\neq 0\}, \boldsymbol{C}^3)$$

when

$$\hat{h}^+_{1p_1}(\eta)\in\mathcal{D}(\{\eta\in \boldsymbol{R}^3_+;\ \eta'\neq 0\}, \boldsymbol{C}^3),$$

and define $v^{+\infty}_{1p_1}(t, x)$ by

$$v^{+\infty}_{1p_1}(t, x)=\frac{G^+_2(r-c_{p_1}t, \theta)}{r}+\frac{G^+_{31}(r-c_{s_1}t, \theta)}{r}, \quad x\in \boldsymbol{R}^3_-,$$

where G^+_2 and G^+_{31} are the functions defined by (4.21) and (4.34) for $\phi(\eta)=\Phi(\eta)$ $\in\mathcal{D}(\{\eta\in \boldsymbol{R}^3_+;\ \eta'\neq 0\}, \boldsymbol{C}^3)$. Then we conclude that

$$\lim_{t\to\infty}\|v^+_{1p_1}(t, \cdot)-v^{+\infty}_{1p_1}(t, \cdot)\|_{L^2(R^3_-, C^3)}=0 \tag{4.44}$$

when $\hat{h}^+_{1p_1}(\eta)\in\mathcal{D}(\{\eta\in \boldsymbol{R}^3_+;\ \eta'\neq 0\}, \boldsymbol{C}^3)$.

For general $\hat{h}^+_{1p_1}(\eta)\in L^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3)$, we can show that (4.44) also holds. In fact, from the continuity of linear operators

$$L^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3)\ni\hat{h}^+_{1p_1}(\eta)\to v^+_{1p_1}(t, \cdot)\in L^2(\boldsymbol{R}^3_-, \boldsymbol{C}^3),$$

$$L^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3)\ni\Phi(\eta)\to v^{+\infty}_{1p_1}(t, \cdot)\in L^2(\boldsymbol{R}^3_-, \boldsymbol{C}^3),$$

$$L^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3)\ni\hat{h}^+_{1p_1}(\eta)\to\Phi(\eta)$$
$$=Y(\xi)\mathrm{U}(\eta')\mathrm{C}(\alpha_i(\eta)\oplus 0_{1\times 1})\hat{h}^+_{1p_1}(\eta)\in L^2(\boldsymbol{R}^3_-, \boldsymbol{C}^3),$$

and from the fact that $\mathcal{D}(\{\eta\in \boldsymbol{R}^3_+;\ \eta'\neq 0\}, \boldsymbol{C}^3)$ is dense in $L^2(\boldsymbol{R}^3_+, \boldsymbol{C}^3)$ by the same argument in Section 3.

Therefore, the principal result of this section states as follows:

THEOREM 4.1. *We assume that*

$$f\in D(A^{1/2})\cap\mathcal{H}, \quad g\in\mathcal{H}\cap D(A^{-1/2}).$$

Let $v^{+\infty}_{1j}(t, x)$ $(j\in M)$ *be the functions defined by*

$$v^{+\infty}_{1j}(t, x)=\begin{cases}\dfrac{G^+_{s_1}(r-c_{s_1}t, \theta)}{r}+\dfrac{G^+_{p_1}(r-c_{p_1}t, \theta)}{r}, & x_3<0,\\[2ex] \dfrac{G^+_{s_2}(r-c_{s_2}t, \theta)}{r}+\dfrac{G^+_{p_2}(r-c_{p_2}t, \theta)}{r}, & x_3>0,\end{cases} \tag{4.45}$$

for $t\in \boldsymbol{R}$, $x=r\theta$, $r=|x|\geq 0$, $\theta\in S^2$, *where if* $l=j$, *then*

$$G_l^+(\tau,\theta)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_0^R e^{i\nu\tau}(-i\nu)Y(-\nu\theta_3)\mathrm{U}(\nu\theta')\mathrm{C}\times(\alpha(\nu\theta',-\nu\theta_3)\oplus 0_{1\times1})\hat{h}_{1j}^+(\nu\theta',-\nu\theta_3)\frac{1}{\rho(x_3)}d\nu, \tag{4.46}$$

and if $l\neq j$, *then*

$$G_l^+(\tau,\theta)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_0^R e^{i\nu\tau}(-i\nu)J(\nu\theta',-\nu\theta_3)\chi_1(\nu\theta',\xi(\nu\theta',-\nu\theta_3))\times Y(\xi(\nu\theta'-\nu\theta_3))\mathrm{U}(\nu\theta')\mathrm{C}(\alpha(\nu\theta',\xi(\nu\theta',-\nu\theta_3))\oplus 0_{1\times1})\times\hat{h}_{1j}^+(\nu\theta',\xi(\nu\theta',-\nu\theta_3))\frac{1}{\rho(x_3)}d\nu \tag{4.47}$$

for $\eta=\nu\omega$, $\nu\geqq0$, $\omega\in S^2$. *Here* α*'s are bounded continuous* 2×2 *matrix functions,*

$$J(\eta',\lambda)=\frac{c_l\lambda}{c_j\sqrt{\lambda^2-\left(\frac{c_j^2}{c_l^2}-1\right)|\eta'|^2}}, \tag{4.48}$$

and χ_1 *satisfies*

$$0\leqq\chi_1(\eta)\leqq1,\quad \chi_1\in C_0^\infty(\Omega_1),\quad \Omega_1=\left\{\eta\,;\,|\eta|^2>\frac{|\eta'|^2}{\frac{c_j^2}{c_l^2}-\varepsilon}\right\}. \tag{4.49}$$

Then we have

$$\lim_{t\to\infty}\|v_{1j}^+(t,\cdot)-v_{1j}^{+\infty}(t,\cdot)\|_{\mathcal{H}}=0. \tag{4.50}$$

$v_{1j}^{+\infty}(t,x)\in\mathcal{H}$ *are called asymptotic wave functions P, SV components* $v_{1j}^+(t,x)$ *of the solution* $v(t,x)$.

Moreover let $v_{2k}^{+\infty}(t,x)$ $(k\in N)$ *be the functions defined by*

$$v_{2k}^{+\infty}(t,x)=\begin{cases}\dfrac{G_{s_1}^+(r-c_{s_1}t,\theta)}{r}, & x_3<0,\\[2mm] \dfrac{G_{s_2}^+(r-c_{s_2}t,\theta)}{r}, & x_3>0,\end{cases} \tag{4.51}$$

for $t\in\boldsymbol{R}$, $x=r\theta$, $r=|x|\geqq0$, $\theta\in S^2$, *where if* $l=k$, *then*

$$G_l^+(\tau,\theta)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_0^R e^{i\nu\tau}(-i\nu)Y(-\nu\theta_3)\mathrm{U}(\nu\theta')\mathrm{C}\times(0_{2\times2}\oplus\beta(\nu\theta',-\nu\theta_3))\hat{h}_{2k}^+(\nu\theta',-\nu\theta_3)\frac{1}{\rho(x_3)}d\nu, \tag{4.52}$$

and if $l\neq k$, *then*

$$G_l^+(\tau,\theta)=\underset{R\to\infty}{\mathrm{l.i.m.}}\int_0^R e^{i\nu\tau}(-i\nu)J(\nu\theta',-\nu\theta_3)\chi_1(\nu\theta',\xi(\nu\theta',-\nu\theta_3))\times Y(\xi(\nu\theta',-\nu\theta_3))\mathrm{U}(\nu\theta')\mathrm{C}(0_{2\times2}\oplus\beta(\nu\theta',\xi(\nu\theta',-\nu\theta_3)))\times\hat{h}_{2k}^+(\nu\theta',\xi(\nu\theta',-\nu\theta_3))\frac{1}{\rho(x_3)}d\nu \tag{4.53}$$

for $\eta=\nu\omega$, $\nu\geqq 0$, $\omega\in S^2$. *Here* β*'s are bounded continuous functions, and* J *and* χ_1 *are defined by* (4.48) *and* (4.49), *respectively. Then we have*

$$\lim_{t\to\infty}\|v_{2k}^{+}(t,\cdot)-v_{2k}^{+\infty}(t,\cdot)\|_{\mathscr{H}}=0. \tag{4.54}$$

$v_{2k}^{+\infty}(t, x)\in\mathscr{K}$ *are called asymptotic wave functions for SH component* $v_{2k}^{+}(t, x)$ *of the solution* $v(t, x)$.

REMARK. As to $v_{1j}^{-\infty}(t, x)$ $(j\in M)$ and $v_{2k}^{-\infty}(t, x)$ $(k\in N)$, we obtain similar asymptotic wave functions by obvious modification.

Proof of Theorem 4.1 is the same as the proof of Theorem 3.7.

§5. The Asymptotic Energy Distributions for Large Times

In this section we calculate asymptotic energy distributions of the solutions of the elastic propagation problem when $t\to\infty$, by using the asymptotic wave functions $v_{1j}^{St\infty}(t, x)$, $v_{1j}^{\pm\infty}(t, x)$ $(j\in M)$, $v_{2k}^{\pm\infty}(t, x)$ $(k\in N)$ which constructed in Section 3 and 4.

In this section, as in Section 3 and 4, it is assumed that $f(x)$ and $g(x)$ are real-valued functions.

THEOREM 5.1. *Suppose that the solution* $u(t)$ *of* (1.12) *and* (1.13) *defined by* (1.14) *has the property*

$$\lim_{t\to\infty}\|u(t)\|_{\mathscr{H}}=0,$$

for any initial data $f\in D(A^{1/2})\cap\mathscr{H}$, $g\in\mathscr{H}\cap D(A^{-1/2})$. *Then, for the solution* $u(t)$ *of* (1.12) *and* (1.13) *with initial data*

$$f\in D(A^{1/2})\cap\mathscr{H},\quad g\in\mathscr{H}\cap D(A^{-1/2}), \tag{5.1}$$

we have

$$\lim_{t\to\infty}E(u, \boldsymbol{R}^3, t)=\lim_{t\to\infty}\|u(t)\|_E=0. \tag{5.2}$$

PROOF. From the condition (5.1) and (2.8),

$$A^{1/2}u(t)=A^{1/2}e^{-itA^{1/2}}(f+iA^{-1/2}g)=e^{-itA^{1/2}}(A^{1/2}f+ig)\in\mathscr{H},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t)=-iA^{1/2}e^{-itA^{1/2}}(f+iA^{-1/2}g)=-ie^{-itA^{1/2}}(A^{1/2}f+ig)\in\mathscr{H}.$$

Thus $(\mathrm{d}/\mathrm{d}t)u(t)$ is the solution of (1.12) for $f'=A^{1/2}f\in\mathscr{H}$ and $g'=A^{1/2}g\in D(A^{-1/2})$. Then by assumption

$$\lim_{t\to\infty}\|A^{1/2}u(t)\|_{\mathcal{H}}=0 \quad \text{and} \quad \lim_{t\to\infty}\left\|\frac{d}{dt}u(t)\right\|_{\mathcal{H}}=0.$$

Hence

$$\lim_{t\to\infty}E(u, \boldsymbol{R}^3, t)=\lim_{t\to\infty}\left(\left\|\frac{d}{dt}u(t)\right\|_{\mathcal{H}}^2+\|A^{1/2}u(t)\|_{\mathcal{H}}^2\right)=0. \quad \square$$

Let $f\in D(A^{1/2})\cap\mathcal{H}$, $g\in\mathcal{H}\cap D(A^{-1/2})$, $\mathrm{Dis}(c_{S\gamma}^2)>0$, and $v(t, x)$ be the corresponding solution of (1.12) and (1.13). We define the asymptotic wave functions $v_{1jl}^{St\infty}(t, x)$ $(l=0, 1, 2, 3)$ by

$$v_{1jl}^{St\infty}(t, x)=\frac{G_{St}^l\left(x'-c_{St}t, \frac{x'}{|x'|}, x_3\right)}{|x'|^{1/2}}, \quad x'\neq 0, \tag{5.3}$$

$$G_{St}^0(\tau, \theta, x_3)=\underset{R\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi i}}\int_0^R e^{i\nu\tau-c_{0j}\nu|x_3|}(-ic_{St}\nu) \times \mathrm{U}(\nu\theta)\mathrm{C}(\phi_{1j}^{St}(\nu\theta)\oplus O_{1\times 1})Q(\nu\theta)\sqrt{\nu}\,\hat{h}_{1j}^{St}(\nu\theta)\frac{1}{\rho(x_3)}d\nu, \tag{5.4}$$

$$G_{St}^l(\tau, \theta, x_3)=\underset{R\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi i}}\int_0^R e^{i\nu\tau-c_{0j}\nu|x_3|}(i\nu\theta_l) \times \mathrm{U}(\nu\theta)\mathrm{C}(\phi_{1j}^{St}(\nu\theta)\oplus O_{1\times 1})Q(\nu\theta)\sqrt{\nu}\,\hat{h}_{1j}^{St}(\nu\theta)\frac{1}{\rho(x_3)}d\nu, \tag{5.5}$$

$(l=1, 2)$

$$G_{St}^3(\tau, \theta, x_3)=\underset{R\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi i}}\int_0^R e^{i\nu\tau-c_{0j}\nu|x_3|}(\sigma c_{0j}\nu) \times \mathrm{U}(\nu\theta)\mathrm{C}(\phi_{1j}^{St}(\nu\theta)\oplus O_{1\times 1})Q(\nu\theta)\sqrt{\nu}\,\hat{h}_{1j}^{St}(\nu\theta)\frac{1}{\rho(x_3)}d\nu, \tag{5.6}$$

where σ is 1 or -1 according as $x_3<0$ or $x_3>0$. Then we have

THEOREM 5.2. *Assume that*

$$f\in D(A^{1/2})\cap\mathcal{H}, \quad g\in\mathcal{H}\cap D(A^{-1/2}), \quad \mathrm{Dis}(c_{S\gamma}^2)>0.$$

Then

$$\lim_{t\to\infty}\left\|\frac{\partial}{\partial t}v_{1j}^{St}(t, \cdot)-v_{1j0}^{St\infty}(t, \cdot)\right\|_{\mathcal{H}}=0, \quad j\in M, \tag{5.7}$$

$$\lim_{t\to\infty}\left\|\frac{\partial}{\partial x_k}v_{1j}^{St}(t, \cdot)-v_{1jl}^{St\infty}(t, \cdot)\right\|_{\mathcal{H}}=0 \quad l=1, 2, 3, \quad j\in M. \tag{5.8}$$

The proof of Theorem 5.2 is the same as that of Theorem 3.2 except for obvious modifications.

The calculation of asymptotic energy distributions are based on the next lemma.

LEMMA 5.3. *Assume that*

$$f\in D(A^{1/2})\cap\mathcal{H},\quad g\in\mathcal{H}\cap D(A^{-1/2}),\quad \mathrm{Dis}(c_{sr}^2)>0.$$

Let

$$B(t,\vartheta(t))=\{x\in\boldsymbol{R}^3\,;\,c_{St}t-\vartheta(t)\leqq|x'|c_{St}t+\vartheta(t),\ x_3\in\boldsymbol{R}\},\tag{5.9}$$

where $\vartheta(t)$ *is any functions of* $t\in\boldsymbol{R}$ *which satisfy*

$$0\leqq\vartheta(t)\leqq\infty\quad \textit{for } \forall t\in\boldsymbol{R}.\tag{5.10}$$

Then we have

$$\begin{aligned}(5.11)\qquad & E(v_{1j}^{St\infty},B(t,\vartheta(t)),t)\\ &=\int_{\boldsymbol{R}}\int_{-\vartheta(t)}^{\vartheta(t)}\|G_{St}^0(r,\cdot,x_3)\|_{L^2(S^1)}^2dr\rho(x_3)dx_3\\ &\quad-\int_{\boldsymbol{R}}\int_{-\vartheta(t)}^{\vartheta(t)}\int_{S^1}\sum_{k,l=1}^3 M_{kl}G_{St}^k(r,\theta,x_3)\cdot G_{St}^l(r,\theta,x_3)d\theta drdx_3.\end{aligned}$$

PROOF. From the definition of the energy (1.15)

$$E(v_{1j}^{St\infty},B,t)=\int_B\Big(|v_{1j0}^{St\infty}|\rho(x_3)-\sum_{k,l=1}^3 M_{kl}v_{1jk}^{St\infty}\cdot v_{1jl}^{St\infty}\Big)dx.\tag{5.12}$$

By the change of variable $r-c_{St}=r'$, the first term of the right-hand side of (5.12) is

$$\begin{aligned}&\|v_{1j0}^{St\infty}(t,\cdot)\|_{\mathcal{H}(B(t,\vartheta(t)))}^2\\ &=\int_{-\infty}^{\infty}\int_{c_{St}t-\vartheta(t)}^{c_{St}t+\vartheta(t)}\int_{S^1}|G_{St}^0(r-c_{St}t,\theta,x^3)|^2d\theta dr\rho(x_3)dx_3\\ &=\int_{-\infty}^{\infty}\int_{-\vartheta(t)}^{\vartheta(t)}\int_{S^1}|G_{St}^0(r,\theta,x_3)|^2d\theta dr\rho(x_3)dx_3.\end{aligned}$$

By introducing the spherical coordinates $x'=r\theta$, $r=|x'|\geqq0$, $\theta\in S^1$, and by the change of variable $r-c_{St}t=r'$, the second term of the right-hand side of (5.12) is

$$\begin{aligned}&-\sum_{k,l=1}^3\int_{B(t,\vartheta(t))}M_{kl}v_{1jk}^{St\infty}(t,x)\cdot v_{1jl}^{St\infty}(t,x)dx\\ &=-\sum_{k,l=1}^3\int_{c_{St}t-\infty}^{\infty}\int_{c_{St}t-\vartheta(t)}^{\vartheta(t)}\int_{S^1}M_{kl}G_{St}^k(r-c_{St}t,\theta,x_3)\cdot G_{St}^l(r-c_{St}t,\theta,x_3)d\theta drdx_3\\ &=-\sum_{k,l=1}^3\int_{-\infty}^{\infty}\int_{-\vartheta(t)}^{\vartheta(t)}\int_{S^1}M_{kl}G_{St}^k(r,\theta,x_3)\cdot G_{St}^l(r,\theta,x_3)d\theta drdx_3.\end{aligned}$$

Thus we have (5.11). □

The following theorem shows that asymptotic energy distributions concern the asymptotic concentration of energy in expanding spherical region $B(t,\vartheta(t))$.

THEOREM 5.4. *Assume that*

$$f\in D(A^{1/2})\cap\mathscr{H},\quad g\in\mathscr{H}\cap D(A^{-1/2}),\quad \mathrm{Dis}(c_{s\gamma}^{2})>0.$$

Let $\vartheta(t)$ *satisfy* (5.10) *and also*

$$\lim_{t\to\infty}\vartheta(t)=\infty. \tag{5.13}$$

Then

$$\lim_{t\to\infty}E(v_{1j}^{St},\ B(t,\ \vartheta(t)),\ t)=E(v_{1j}^{St},\ \boldsymbol{R}^{3},\ 0), \tag{5.14}$$

and hence

$$\lim_{t\to\infty}E(v_{1j}^{St},\ \boldsymbol{R}^{3}\backslash B(t,\ \vartheta(t)),\ t)=0. \tag{5.15}$$

Proof. From the triangle inequality

$$|E(v_{1j}^{St},\ B,\ t)^{1/2}-E(v_{1j}^{St\infty},\ B,\ t)^{1/2}|\leqq E(v_{1j}^{St}-v_{1j}^{St\infty},\ B,\ t)^{1/2}.$$

Theorem 5.1 implies

$$\lim_{t\to\infty}E(v_{1j}^{St}-v_{1j}^{St\infty},\ B,\ t)^{1/2}\leqq\lim_{t\to\infty}\|v_{1j}^{St}-v_{1j}^{St\infty}\|_{E}=0.$$

Lemma 5.3 implies

$$\begin{aligned}\lim_{t\to\infty}E(v_{1j}^{St\infty},\ B,\ t)^{1/2}&=\int_{\boldsymbol{R}}\int_{-\infty}^{\infty}\|G_{St}^{0}(r,\ \cdot,\ x_3)\|_{L^2(S^1)}^{2}dr\rho(x_3)dx_3\\&\quad-\int_{\boldsymbol{R}}\int_{-\infty}^{\infty}\int_{S^1}\sum_{k,l=1}^{3}M_{kl}G_{St}^{k}(r,\ \theta,\ x_3)\cdot G_{St}^{l}(r,\ \theta,\ x_3)d\theta drdx_3\\&=E(v_{1j}^{St\infty},\ \boldsymbol{R}^{3},\ 0).\end{aligned}$$

This gives (5.14) and (5.15). □

The next corollary shows the transiency of the Stoneley components $v_{1j}^{St}(t,\ x)$ $(j\in M)$ in the sense that the energy in any bounded region tends to 0 for $t\to\infty$.

COROLLARY 5.5. *Assume that*

$$f\in D(A^{1/2})\cap\mathscr{H},\quad g\in\mathscr{H}\cap D(A^{-1/2}),\quad \mathrm{Dis}(c_{s\gamma}^{2})>0.$$

Let $K\subset\boldsymbol{R}^{3}$ *be any bounded set. Then we have*

$$\lim_{t\to\infty}E(v_{1j}^{St},\ K,\ t)=0. \tag{5.16}$$

Proof. By the boundedness of $K\subset\boldsymbol{R}^{3}$, there exists $r>0$ such that

$$K\subset\Omega_{r}=\{x\in\boldsymbol{R}^{3},\ |x|\leqq r\}.$$

In theorem 5.4, if we take

$$-\vartheta(t)=r-c_{St}t\geqq -c_{St}t,$$

then

$$K\subset\Omega_r\subset \mathbf{R}^3\backslash B(t,\vartheta(t)) \quad \text{for } \forall t\in\mathbf{R}.$$

Hence

$$0\leqq E(v_{1j}^{St},K,t)\leqq E(v_{1j}^{St},\mathbf{R}^3\backslash B(t,\vartheta(t)),t),$$

so (5.16) follows from Theorem 5.4. □

The main result of this paper is the following theorem. This theorem shows that the energy of the Stoneley components $v_{1j}^{St}(t,x)$ $(j\in M)$ of $v(t,x)$ is asymptotically concentrated along the interface $x_3=0$.

THEOREM 5.6. *Assume that*

$$f\in D(A^{1/2})\cap\mathcal{H},\quad g\in\mathcal{H}\cap D(A^{-1/2}),\quad \mathrm{Dis}(c_{s\gamma}^2)>0.$$

Then we have

$$\lim_{t\to\infty}E(v_{1j}^{St},(C^-(\theta)\cup C^+(\theta))\cap B(t,\vartheta(t)),t)=E(v_{1j}^{St},\mathbf{R}^3,0),\quad j\in M, \tag{5.17}$$

where

$$C^-(\theta)=\{x\in\mathbf{R}_-^3;\ -\theta(|x'|)<x_3<0\}, \tag{5.18}$$

$$C^+(\theta)=\{x\in\mathbf{R}_-^3;\ 0<x_3<\theta(|x'|)\}, \tag{5.19}$$

$$B(t,\vartheta(t))=\{x\in\mathbf{R}^3;\ c_{St}t-\vartheta(t)\leqq|x'|\leqq c_{St}t+\vartheta(t),\ x_3\in\mathbf{R}\}, \tag{5.20}$$

$$\vartheta(t):\ \lim_{t\to\infty}\vartheta(t)=\infty,\ |\vartheta(t)|<2c_{St}t, \tag{5.21}$$

$$\theta(|x'|):\ \lim_{|x'|\to\infty}\theta(|x'|)=\infty,\ \text{monotone increasing function.} \tag{5.22}$$

Proof. It suffices to show that

$$\lim_{t\to\infty}E(v_{1j}^{St\infty},\mathbf{R}^3\backslash(((C^-(\theta)\cup C^+(\theta))\cap B(t,\vartheta(t)),t)=0. \tag{5.23}$$

Because the triangle inequality and Theorem 5.1 imply

$$\lim_{t\to\infty}|E(v_{1j}^{St},K,t)^{1/2}-E(v_{1j}^{St\infty},K,t)^{1/2}|=0$$

for any $K\subset\mathbf{R}^3$. Note that

$$\begin{aligned}(5.24)\qquad &\mathbf{R}^3\backslash(((C^-(\theta)\cup C^+(\theta))\cap B(t,\vartheta(t)),t)\\ &=\{(\{x\in\mathbf{R}_-^3,\ x_3\leqq-\theta(|x'|)\}\cup\{x\in\mathbf{R}_+^3,\ \theta(|x'|)\leqq x_3\})\cap B(t,\vartheta(t))\}\\ &\quad\cup\{|x'|\leqq c_{St}t-\vartheta(t),\ c_{St}t+\vartheta(t)\leqq|x'|,\ x_3\in\mathbf{R}\}\\ &=G_1\cup G_2,\end{aligned}$$

and

$$E(v_{1j}^{St\infty}, G_i, t)=\int_{G_i}\Big(|v_{1j0}^{St\infty}|\rho(x_3)-\sum_{k,l=1}^{3}M_{kl}v_{1jk}^{St\infty}\cdot v_{1jl}^{St\infty}\Big)dx, \quad i=1, 2. \tag{5.25}$$

We consider the first term of the right-hand side of (5.25). By the change of variable $r'=r-c_{St}t$,

$$\|v_{1j0}^{St\infty}(t, \cdot)\|_{\mathcal{H}(G_1)}^2=\int_{c_{St}t-\vartheta(t)}^{c_{St}t+\vartheta(t)}\int_{S^1}\Big(\int_{\theta(r)}^{\infty}+\int_{-\infty}^{-\theta(r)}\Big)|G_{St}^0(r-c_{St}t, \theta, x_3)|^2\rho(x_3)dx_3d\theta dr$$

$$=\int_{-\vartheta(t)}^{+\vartheta(t)}\int_{S^1}\Big(\int_{\theta(r+c_{St}t)}^{\infty}+\int_{-\infty}^{-\theta(r+c_{St}t)}\Big)|G_{St}^0(r, \theta, x_3)|^2\rho(x_3)dx_3d\theta dr.$$

The conditions (5.21) and (5.22) implies

$$\lim_{t\to\infty}\theta(r+c_{St}t)\geqq\lim_{t\to\infty}\theta(-\vartheta(t)+2_{St}t)=\infty.$$

Hence

$$\lim_{t\to\infty}\|v_{1j0}^{St\infty}(t, \cdot)\|_{\mathcal{H}(G_1)}^2=0.$$

By the change of variable $r'=r-c_{St}t$,

$$\|v_{1j0}^{St\infty}(t, \cdot)\|_{\mathcal{H}(G_2)}^2=\int_{R}\Big(\int_{c_{St}t+\theta(r)}^{\infty}+\int_{-\infty}^{c_{St}t-\theta(r)}\Big)\int_{S^1}|G_{St}^0(r-c_{St}t, \theta, x_3)|^2d\theta dr\rho(x_3)dx_3$$

$$=\int_{R}\Big(\int_{-\theta(r)}^{\infty}+\int_{-\infty}^{-\theta(r)}\Big)\int_{S^1}|G_{St}^0(r, \theta, x_3)|^2d\theta dr\rho(x_3)dx_3.$$

From the condition (5.21), we have

$$\lim_{t\to\infty}\|v_{1j0}^{St\infty}(t, \cdot)\|_{\mathcal{H}(G_2)}^2=0.$$

The second term of the right-hand side of (5.25) can be treated similarly. This completes the proof of Theorem 5.6. □

Finally, we consider the P, SV, SH components $v_{1j}^{\pm}(t, x)$ $(j\in M)$ and $v_{2k}^{\pm}(t, x)$ $(k\in N)$. The next theorem shows that the P, SV, SH components behave like free waves.

THEOREM 5.7. *Assume that*

$$f\in D(A^{1/2})\cap\mathcal{H}, \quad g\in\mathcal{H}\cap D(A^{-1/2}).$$

Then we have

$$\lim_{t\to\infty}E(v_{1j}^{\pm}, S_{s_1}(t, \vartheta)\cup S_{p_1}(t, \vartheta)\cup S_{s_2}(t, \vartheta)\cup S_{p_2}(t, \vartheta), t) \tag{5.26}$$
$$=E(v_{1j}^{\pm}, \boldsymbol{R}^3, 0), \quad j\in M,$$

$$\lim_{t\to\infty}E(v_{2k}^{\pm}, S_{s_1}(t, \vartheta)\cup S_{s_2}(t, \vartheta), t)=E(v_{2k}^{\pm}, \boldsymbol{R}^3, 0), \quad k\in N, \tag{5.27}$$

where

(5.28) $$S_{s_1}(t, \vartheta(t))=\{x\in \boldsymbol{R}^3_-\ ;\ c_{s_1}t-\vartheta(t)\leqq|x|\leqq c_{s_1}t+\vartheta(t)\},$$

(5.29) $$S_{p_1}(t, \vartheta(t))=\{x\in \boldsymbol{R}^3_-\ ;\ c_{p_1}t-\vartheta(t)\leqq|x|\leqq c_{p_1}t+\vartheta(t)\},$$

(5.30) $$S_{s_2}(t, \vartheta(t))=\{x\in \boldsymbol{R}^3_+\ ;\ c_{s_2}t-\vartheta(t)\leqq|x|\leqq c_{s_2}t+\vartheta(t)\},$$

(5.31) $$S_{p_2}(t, \vartheta(t)=\{x\in \boldsymbol{R}^3_+\ ;\ c_{p_2}t-\vartheta(t)\leqq|x|\leqq c_{p_2}t+\vartheta(t)\},$$

(5.32) $$\vartheta(t) : \lim_{t\to\infty}\vartheta(t)=\infty .$$

The proof of this theorem is obtained by using Theorem 4.2 and modified Theorem 5.4.

The next corollary shows the transiency of the P, SV components $v^{\pm}_{1j}(t, x)$ $(j\in M)$ and the SH components $v^{\pm}_{2k}(t, x)$ $(k\in N)$ in the sense that the energy in any bounded region tends to 0 for $t\to\infty$.

COROLLARY 5.8. *Assume that*

$$f\in D(A^{1/2})\cap\mathcal{H},\quad g\in\mathcal{H}\cap D(A^{-1/2}).$$

Let $K\subset \boldsymbol{R}^3$ *be any bounded set. Then we have*

(5.33) $$\lim_{t\to\infty}E(v^{\pm}_{1j}, K, t)=0,$$

(5.34) $$\lim_{t\to\infty}E(v^{\pm}_{2k}, K, t)=0.$$

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