TSUKUBA J. MATH.
Vol. 19 No. 2 (1995), 387—395

REFLEXIVE MODULES OVER QF-3′ RINGS*

By

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Abstract. We characterize reflexive modules over QF-3′ rings using a linear compactness condition relative to the Lambek torsion theory, and we also give a necessary and sufficient condition for a left QF-3′ maximal quotient ring to be right QF-3′.

1. Introduction.

The problem of finding the reflexive modules over generalizations of QF rings (and, in particular, over QF-3 rings) has a long tradition. One of the first contributions is due to Morita [10], who determined the finitely generated reflexive modules over a right artinian QF-3 ring and, some years later, Masaike [8] extended this result by giving a characterization of reflexive modules over QF-3 rings with ACC (or DCC) on left annihilators. On the other hand, Müller [11] proved that if ${}_RU_S$ is a bimodule that induces a Morita duality, then the U-reflexive modules are precisely the linearly compact modules and this applies, in particular, to the case in which $R=U$ is a PF ring. Recently, Masaike [9], extended this to QF-3 rings without chain conditions by showing that the reflexive modules over these rings are the modules of R-dominant dimension $\geqq 2$ that satisfy a suitable linear compactness condition.

Recall that a ring is left QF-3 when it has a minimal faithful left module and left QF-3′ when the injective envelope $E({}_RR)$ is torsionless. When R is left and right QF-3′, we will simply say that it is a QF-3′ ring (and a similar convention will be used for other classes of rings). QF-3′ rings have been studied by a number of authors and their relation with Morita duality and the properties of the double dual functors has been analyzed by Colby and Fuller in a series of papers (see, e.g., [1] and its references). One of the aims of this paper is to show that a characterization of reflexive modules similar to Masaike's one may be given for the much larger class of QF-3′ rings. In fact,

* Wark partially supported by the DGICYT (PB93-0515, Spain). The first author was also partially supported by the European Community (Contract CHRX-CT93-0091).
Received November 22, 1993. Revised December 7, 1994.

we obtain a more general module-theoretic result that embraces also the theorem of Müller mentioned above. As a further application of the techniques developed here, we study the interplay between R being right QF-3' and linear compactness conditions on the left, that leads to a necessary and sufficient condition for a left QF-3' ring to be right QF-3', and to a new one-sided characterization of QF-3 maximal quotient rings.

Throughout this paper, R denotes an associative ring with identity and R-Mod (resp. Mod-R) the category of left (resp. right) R-modules. If X and M are left R-modules, X is said to be finitely M-generated when it is a quotient of a finite direct sum of copies of M and X has M-dominant dimension $\geqq 2$ (M-dom. dim $X \geqq 2$) when there exists an exact sequence $0 \to X \to Y \to Z$, whith Y and Z isomorphic to direct products of copies of X.

We will call $\mathcal{T}_M$ to the localizing subcategory of R-Mod cogenerated by the injective envelope $E(M)$ of M. The corresponding quotient category of R-Mod will be denoted by $R\text{-Mod}/\mathcal{T}_M$ and its objects are precisely the modules of $E(M)$-dom. dim$\geqq 2$. The most important case of this construction arises for $M={}_RR$, and then $\mathcal{T}_M=\mathcal{L}$ is just the Lambek (or dense) localizing subcategory of R-Mod (see [15]).

2. Reflexive modules.

We will fix a module $M \in R$-Mod and call $S=\mathrm{End}({}_RM)$. The M-dual functors $\mathrm{Hom}_R(-, M)$ and $\mathrm{Hom}_S(-, M)$ will be denoted by $(\ )^*$, and their composition in either order by $(\ )^{**}$. For each $X \in R$-Mod there is a canonical (evaluation) morphism $\sigma_X : X \to X^{**}$; σ_X is a monomorphism precisely when X is M-cogenerated and when σ_X is an isomorphism, X is said to be M-reflexive (or just reflexive if we take $M={}_RR$).

We are interested in characterizing reflexive modules and, not surprisingly, a certain form of linear compactness plays a key role in this characterization. Recall from [3] that an object of a Grothendieck category $\mathcal{A}$ is said to be linearly compact when, for each inverse system $\{p_i : X \to X_i\}_I$ in $\mathcal{A}$ such that the p_i are epimorphisms, the induced morphism $\varprojlim p_i : X \to \varprojlim X_i$ is also an epimorphism (this just gives ordinary linear compactness when $\mathcal{A}=R$-Mod). We will also use the following related concept (introduced by Hoshino and Takashima in [5]): An R-module X will be called $\mathcal{T}_M$-linearly compact when, for each inverse system $\{p_i : X \to X_i\}_I$ in R-Mod such that the X_i are M-cogenerated and Coker $p_i \in \mathcal{T}_M$, Coker $(\varprojlim p_i) \in \mathcal{T}_M$. It is not difficult to show that when every finitely M-generated submodule of $E(M)$ is M-cogenerated and M is an object of R-

Mod/$\mathcal{T}_M$ (M rationally complete), then M is $\mathcal{T}_M$-linearly compact if and only if it is linearly compact in the category R-Mod/$\mathcal{T}_M$. When a module is $\mathcal{L}$-linearly compact, we will also say that it is Lambek linearly compact.

$\mathcal{T}_M$-linearly compact modules have the following useful property:

PROPOSITION 2.1. *Let M be a left R-module such that each finitely M-generated submodule of $E(M)$ is M-cogenerated. Then, for each $\mathcal{T}_M$-linearly compact R-module X, Coker $\sigma_X \in \mathcal{T}_M$.*

PROOF. The proof is essentially the same of [5, Corollary 2.2], where this is shown in the case $M={}_RR$. □

LEMMA 2.2. *Let $X \in R$-Mod, Y an M-reflexive module, and I a set. If $f: X \to Y^I$ is a homomorphism, then there exists a homomorphism $g: X^{**} \to Y^I$ such that $g \circ \sigma_X = f$.*

PROOF. Let, for each $i \in I$, $p_i: Y^I \to Y$ be the canonical projection and consider the homomorphism $g_i := \sigma_Y^{-1} \circ (p_i \circ f)^{**}: X^{**} \to Y$. Since $\sigma_Y \circ p_i \circ f = (p_i \circ f)^{**} \circ \sigma_X$ we see that $p_i \circ f = \sigma_Y^{-1} \circ (p_i \circ f)^{**} \circ \sigma_X = g_i \circ \sigma_X$ for each $i \in I$ and so, calling $g: X^{**} \to Y^I$ to the unique homomorphism such that $p_i \circ g = g_i$ $\forall i \in I$, we see that $p_i \circ f = p_i \circ g \circ \sigma_X$ $\forall i \in I$ and hence that $f = g \circ \sigma_X$. □

PROPOSITION 2.3. *Let $M \in R$-Mod be such that every finitely M-generated submodule of $E(M)$ is M-cogenerated and let $X \in R$-Mod a $\mathcal{T}_M$-linearly compact module. Then X is M-reflexive if and only if M-dom. dim $X \geqq 2$.*

PROOF. The necessity is clear, for if X is M-reflexive and $S^{(J)} \to S^{(I)} \to X^* \to 0$ is a free presentation of X^* in Mod-S, then applying ( )* we get an exact sequence in R-Mod: $0 \to X \cong X^{**} \to M^I \to M^J$ and so M-dom. dim $X \geqq 2$.

To prove the sufficiency, assume that X is $\mathcal{T}_M$-linearly compact and that there exists an exact sequence in R-Mod: $0 \to X \xrightarrow{u} M^I \xrightarrow{p} M^J$. By Proposition 2.1, Coker $\sigma_X \in \mathcal{T}_M$ and, as X^{**} is $\mathcal{T}_M$-torsionfree, it is clear that σ_X is an essential monomorphism. On the other hand, by Lemma 2.2 we see that there exists a homomorphism $g: X^{**} \to M$ such that $u = g \circ \sigma_X$ and, as σ_X is essential, g is a monomorphism. Therefore, Coker σ_X is a $\mathcal{T}_M$-torsion module which is isomorphic to a submodule of the M-cogenerated module Coker u and so Coker $\sigma_X = 0$. Thus σ_X is an isomorphism and X is M-reflexive. □

In the case $M=R$, the preceding result has been observed by Hoshino and

Takashima in [5, Remark, p. 9]. In the following proposition we denote by $\mathcal{T}'_M$ the localizing subcategory of Mod-S cogenerated by $E(M_S)$.

PROPOSITION 2.4. *Let $M \in R$-Mod. Then $E(_RM)$ is M-cogenerated if and only if, for every monomorphism g of R-Mod, Coker $g^* \in \mathcal{T}'_M$.*

PROOF. The proof can be easily adapted from that of [4, Theorem 1.1], where a similar result is proved in the case $M=R$. □

We can now give our main result characterizing M-reflexive modules. Recall that a bimodule $_RM_S$ is called faithfully balanced when $R=\mathrm{End}(M_S)$ and $S=\mathrm{End}(_RM)$.

THEOREM 2.5. *Let $_RM_S$ be a faithfully balanced bimodule such that both $E(_RM)$ and $E(M_S)$ are M-cogenerated, and let $X \in R$-Mod. Then X is M-reflexive if and only if it is $\mathcal{T}_M$-linearly compact and M-dom. dim $X \geqq 2$.*

PROOF. Applying Proposition 2.3, the only thing that remains to be proved is that any M-reflexive left R-module is $\mathcal{T}_M$-linearly compact. Assume then that X is M-reflexive and let $\{p_i : X \to X_i\}_I$ be an inverse system with X_i M-cogenerated and Coker $p_i \in \mathcal{T}_M$, for each $i \in I$. Since σ_X is an isomorphism, we can identify the inverse system $\{p_i^{**}\}_I$ with the inverse system $\{\sigma_{X_i} \circ p_i\}_I$ and we have:

$$\varprojlim \sigma_{X_i} \circ \varprojlim p_i = \varprojlim p_i^{**} = (\varinjlim p_i^*)^* .$$

Since Coker $p_i \in \mathcal{T}_M$, the p_i^* are monomorphisms and so is $\varinjlim p_i^*$. Now, since $E(M_S)$ is M-cogenerated and $R=\mathrm{End}(M_S)$, it follows from Proposition 2.4 that Coker $(\varprojlim p_i^{**}) \in \mathcal{T}_M$. But, on the other hand, as $\varprojlim$ is a left exact functor, we have that $\varprojlim \sigma_{X_i}$ is a monomorphism and so Coker $(\varprojlim p_i) \subseteq$ Coker $(\varprojlim p_i^{**})$. Thus Coker $(\varprojlim p_i) \in \mathcal{T}_M$ and so X is $\mathcal{T}_M$-linearly compact. □

Specializing Theorem 2.5 to the case $M=R$, we obtain the promised characterization of reflexive modules over QF-3′ rings.

COROLLARY 2.6. *Let R be a QF-3′ ring and $X \in R$-Mod. Then X is reflexive if and only if it is Lambek linearly compact and R-dom. dim $X \geqq 2$.*

As we have remarked after Proposition 2.3, the "if" part of Corollary 2.6 has been proved by Hoshino and Takashima in [5], assuming only that every finitely generated submodule of $E(R_R)$ is torsionless. The "only if" part, however, does not hold even in the case that R has this property on both sides.

An easy example is the following. Let $R=\boldsymbol{Z}$ be the ring of rational integers and X a countable direct sum of copies of $_RR$. Then it is clear that X is not Lambek linearly compact, but X is reflexive by a theorem of E. Specker [14].

3. Right QF-3′ rings.

It is easy to infer from the proof of Theorem 2.5 that a right QF-3′ ring is Lambek linearly compact on the left, and now we want to go in the opposite direction and, similarly to what is done in [9, Theorem 5] (see also [4, Theorem 2.2]) to give conditions on the left for a left QF-3′ ring to be QF-3′ (on both sides). Since the property of being QF-3′ does not pass well from the maximal quotient ring of R to R, we will assume that R is, furthermore, a left maximal quotient ring. We will also need a stronger linear compactness condition that appeared in [3]. Assuming that $R\in R\text{-Mod}/\mathcal{L}$, let $\sigma^f_{\mathcal{L}}[R]$ be the full subcategory of R-Mod/$\mathcal{L}$ consisting of the subobjects of quotients of finite direct sums of copies of R in this category (this is just the smallest finitely closed. i.e., closed under subobjects, quotient objects, and finite direct sums-subcategory of R-Mod/$\mathcal{L}$ containing R). We will say that $\sigma^f_{\mathcal{L}}[R]$ is a linearly compact subcategory of R-Mod/$\mathcal{L}$ if, for each inverse system $\{p_i: X_i\to Y_i\}_I$ in R-Mod/$\mathcal{L}$ with the p_i epimorphisms and $X_i\in\sigma^f_{\mathcal{L}}[R]$, the morphism $\varprojlim p_i$ is also an epimorphism of R-Mod/$\mathcal{L}$.

THEOREM 3.1. *Let R be a left maximal quotient ring. Then the following statements hold:*

i) *If $\sigma^f_{\mathcal{L}}[R]$ is a linearly compact subcategory of R-Mod/$\mathcal{L}$, then R is right QF-3′ if and only if every finitely generated submodule of $E(R_R)$ is torsionless.*

ii) *If every finitely generated submodule of $E(_RR)$ is torsionless, then R is right QF-3′ if and only if $\sigma^f_{\mathcal{L}}[R]$ is a linearly compact subcategory of R-Mod/$\mathcal{L}$.*

PROOF. i) Assume that each finitely generated submodule of $E(_RR)$ is torsionless. Then, using Proposition 2.4 and [4, Theorem 1.1], it is enough to prove that if $j: X\to Y$ is a monomorphism in Mod-R, then Coker $j^*\in\mathcal{L}$, assuming that the analogous property holds for monomorphisms in Mod-R that have finitely generated codomain. Thus, let $j: X\to Y$ be a monomorphism of Mod-R and write $Y=\varinjlim Y_i$, where $\{Y_i\}_I$ is the direct system of all the finitely generated submodules of Y. For each $i\in I$, set $X_i:=X\cap Y_i$, with inclusions $j_i: X_i\to Y_i$. Using AB5 we see that $j=\varinjlim j_i$ and, taking R-duals, that $j^*=(\varinjlim j_i)^*=\varprojlim j_i^*$. Since the Y_i are finitely generated right R-modules, we have that Coker $j_i^*\in\mathcal{L}$ for each $i\in I$ and, since R is a maximal quotient ring, the

X_i^* and Y_i^* are objects of R-Mod/$\mathcal{L}$, so that we have an inverse system of epimorphisms $j_i^*: Y_i^* \to X_i^*$ in R-Mod/$\mathcal{L}$, with $Y_i^* \in \sigma_{\mathcal{L}}^f[R]$. Now, as $\sigma_{\mathcal{L}}^f[R]$ is a linearly compact subcategory of R-Mod/$\mathcal{L}$, we see that $j^* = \varprojlim j_i^*$ is an epimorphism of R-Mod/$\mathcal{L}$ and so Coker $j^* \in \mathcal{L}$, completing the proof of i).

ii) Assume first that every finitely generated submodule of $E({}_RR)$ is torsionless and R is right QF-3′. Since R is, furthermore, a left maximal quotient ring, it follows from [4, Theorem 1.5] that every object of $\sigma_{\mathcal{L}}^f[R]$ is reflexive. Thus if we have an inverse system of epimorphisms $\{p_i: X \to X_i\}_I$ in R-Mod/$\mathcal{L}$ with $X_i \in \sigma_{\mathcal{L}}^f[R]$, we may identify each p_i with p_i^{**} and we have $\varprojlim p_i = (\varinjlim p_i^*)^*$. Since Coker $p_i \in \mathcal{L}$, each p_i^* is a monomorphism in Mod-R, and hence so is $\varinjlim p_i^*$. Now, as R is right QF-3′, we have by Proposition 2.4 Coker $(\varprojlim p_i) \in \mathcal{L}$ and so $\sigma_{\mathcal{L}}^f[R]$ is linearly compact. Finally, assume that every finitely generated submodule of $E({}_RR)$ is torsionless and $\sigma_{\mathcal{L}}^f[R]$ is linearly compact. Then R is a linearly compact object of R-Mod/$\mathcal{L}$ and by [4, Theorem 2.2], we have that every finitely generated submodule of $E(R_R)$ is torsionless, so that, applying i) we see that R is right QF-3′. □

Recall that a right R-module P_R is called dominant if it is a finitely generated faithful projective module such that if $T = \mathrm{End}(P_R)$, then ${}_TP$ cogenerates all the simple left T-modules [7]. Then, assuming again that R is a left maximal quotient ring, the existence of a dominant right module is equivalent to R-Mod/$\mathcal{L}$ being a module category by [7]. As it is well known, the left minimal faithful module over a left QF-3 ring is dominant [13] and so we may use the preceding theorem to characterize QF-3 maximal quotient rings. This is an important class of rings for, according to the Ringel-Tachikawa theorem [12], they correspond to Morita dualities. We next show that QF-3 maximal quotient rings can be characterized by conditions on the left that are similar to, but weaker than, those given by Masaike [9, Theorem 5] for QF-3 rings that are not necessarily maximal quotient rings.

COROLLARY 3.2. *Let R be a left maximal quotient ring. Then R is QF-3 if and only if the following conditions hold:*

i) *R is left QF-3′*

ii) *R is left Lambek linearly compact*

iii) *R-Mod/$\mathcal{L}$ is a module category (equivalently, R has a dominant right module).*

PROOF. It is clear from what we have already said that if R is QF-3, then all three conditions above hold. Conversely, if conditions ii) and iii) hold, then it follows from [6, Theorem 7.1] that $\sigma^{f}_{\mathcal{L}}[R]$ is a linearly compact subcategory of R-Mod/$\mathcal{L}$ and then, if i) also holds, we see from Theorem 3.1 that R is a QF-3′ ring. Now, using [2, Corollary 6], we see that R is a QF-3 ring. □

REMARKS. i) The hypothesis that R is a left maximal quotient ring cannot be dropped from Theorem 3.1 and Corollary 3.2. Indeed, the ring $R=\begin{pmatrix} \boldsymbol{Z} & \boldsymbol{Q} \\ 0 & \boldsymbol{Q} \end{pmatrix}$ satisfies i), ii) and iii) of Corollary 3.2 but is neither left QF-3 nor right QF-3′.

ii) Assume that R is a left maximal quotient ring which is linearly compact as an object of R-Mod/$\mathcal{L}$. Then, a sufficient condition for $\sigma^{f}_{\mathcal{L}}[R]$ to be a linearly compact subcategory of R-Mod/$\mathcal{L}$ is that R-Mod/$\mathcal{L}$ has a projective generator, as can be seen in the proof of [3, Corollary 7]. Thus an argument similar to the one used in the proof of Corollary 3.2 gives that if R is a left maximal quotient ring such that every finitely generated submodule of $E({}_RR)$ is torsionless, R-Mod/$\mathcal{L}$ has a projective generator, and R is Lambek linearly compact, then R is right QF-3′.

Acknowledgements.

We thank the referee for pointing out that the "if" part of Corollary 2.6 was already contained in [5], and also for suggesting the example given after this corollary.

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