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ON THE LONG-RANGE SCATTFRING FOR ONE- AND TWO-PARTICLE SCHRÖDINGER OPERATORS WITH CONSTANT MAGNETIC FIELDS*

By

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1. Introduction

In this paper we prove the asymptotic completeness for the following 1-particle Schrödinger operator with a constant magnetic field $B\in\boldsymbol{R}^3$, $B\neq 0$:

$$H=H_0+V=\frac{1}{2}\left(p-\frac{1}{2}B\times r\right)^2+V(r),$$

where $r\in\boldsymbol{R}^3$ and $p=-i\nabla_r$. The real-valued smooth function $V(r)$ is a long-range potential, that is, we impose the following decay condition on $V(r)$:

(V) *As* $|r|\rightarrow\infty$,

(1.1) $$|V(r)|+|r_{\|}\partial_{\|}V(r)|=o(1).$$

Moreover, for some $\delta_0>0$,

(1.2) $$|\partial_{\perp}V(r)|\leqq C\langle r_{\|}\rangle^{-1-\delta_0},$$

(1.3) $$|\partial_{\|}^{l}V(r)|\leqq C_l\langle r_{\|}\rangle^{-l-\delta_0} \quad \textit{for any integer } l\geqq 0.$$

Here $r_{\|}=r\cdot B/|B|$, and $r_{\perp}$ denotes the component of r perpendicular to B; $\partial_{\perp}$ and $\partial_{\|}$ denote the partial differentials with respect to the variables $r_{\perp}$ and $r_{\|}$, respectively. We use $\langle r\rangle$ for $(1+|r|^2)^{1/2}$ throughout this paper. As will be easily seen from below, $V(r)$ is allowed to include short-range parts with some local singularities.

In the absence of magnetic fields, scattering theory is quite well understood for 2-body Schrödinger operators with a large class of long-range potentials and, recently, the long-range scattering for constant electric fields have been intensively investigated by several authors (cf. [5, 8, 9, 16]). On the other hand, the asymptotic completeness for long-range Schrödinger operators with constant magnetic fields was first proved by Avron-Herbst-Simon in [1]; they

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treated only azimuthally symmetric potentials, though their results contains the case for potentials unbounded along the directions prependicular to the magnetic fields. They employed a general argument of Kuroda [11] (see also [4]) to reduce the completeness to the existence of modified wave operators. After we had completed this work, we learned the work of Łaba [13]. Her restriction on the long-range potential V_L: $|\partial^\alpha V_L(r)|\leqq C_\alpha\langle r\rangle^{-|\alpha|-\nu_0}$, $\nu_0>0$, is stronger than ours and the approach is based on the estimates of the growth of the angular momentum $B\times r\cdot p$, that is, on the fact that scattering states do not essentially propagate in the space-time region $|r_\perp|\geqq|t|^\nu$ with some $\nu<\nu_0$.

We shall now state our main result in this paper. For simplicity we write $z=r_\parallel$ and $p_z=-i\partial_\parallel$. Let

$$W(z)=V(r)|_{r_\perp=0}\,, \tag{1.4}$$

and let $\chi(s)\in C^\infty(\boldsymbol{R}^1)$ such that $0\leqq\chi\leqq 1$, $\chi(s)=1$ if $|s|\geqq 2$, and $=0$ if $|s|\leqq 1$. Define

$$\chi(t, z)=\chi\Big(\frac{\log\langle t\rangle}{\langle t\rangle}z\Big), \qquad W(t, z)=\chi(t, z)W(z).$$

Then the smooth function $W(t, z)$ satisfies

$$|\partial_t^l\partial_z^k W(t, z)|\leqq C_{lk}\langle(t, z)\rangle^{-l-k-\delta_1}\,,$$

for some positive $\delta_1<\delta_0$. Let $S(t, \xi)$ be a solution to the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t}(t, \xi)=\frac{1}{2}\xi^2+W\Big(t, \frac{\partial S}{\partial \xi}(t, \xi)\Big), \qquad \xi\in\boldsymbol{R}^1$$

(see [10, 12]). Let $H_{0\perp}=H_0-(1/2)p_z^2$.

THEOREM. *Assume that the condition (V) is obeyed. Then, the modified wave operators $\Omega_\pm$ defined by*

$$\Omega_\pm=s-\lim_{t\to\pm\infty} e^{itH}e^{-iS(t,p_z)-itH_{0\perp}}$$

exist and are complete.

The present work is new in the sense that all the results for the 1-particle operator obtained in this paper remain true for a 2-particle system $H(k)$ with the center-of-mass removed (see [2]):

$$\begin{aligned} H(k)&=H_0(k)+V(r-\beta)\\ &=\frac{1}{2\mu}\Big[p-\frac{m_2-m_1}{2(m_1+m_2)}B\times r\Big]^2+\frac{1}{2M}|B\times r|^2+\frac{1}{2M}k_\parallel^2+V(r-\beta), \end{aligned}$$

where m_1 and m_2 stand for the masses of the first and the second particle, respectively, μ the reduced mass, and M the total mass; we have denoted by r the relative coordinate of the two particles and by p the conjugate momentum. The parameter $k\in\boldsymbol{R}^3$ is the total pseudomomentum and $\beta=k\times B/|B|^2$. The Hamiltonian $H(k)$ is derived under the condition of the total charge zero (see also [3, 7]).

Note also that this paper covers such a potential that is not azimuthally symmetric and slowly decays as $|r|\to\infty$ along the direction perpendicular to B. For example, if $B=(0, 0, B)$ and $r=(x, y, z)$, then a function

$$V(r)=[\log\{(1+x^2+z^2)(1+y^2+z^2)\}]^{-1}\langle z\rangle^{-\delta_0}, \qquad \delta_0>0$$

satisfies the condition (V) and the property mentioned above.

Our strategy is similar to that of [13] and is to see the asymptotic completeness from the point of view of propagation and non-propagation estimates. We follow the idea of Sigal [14] for propagation estimates. We do not use, anywhere, the exponential decay property of the eigenfunctions for $H_{0\perp}$; both in the proof of the existence of wave opeartors and in that of inverse wave operators, we use the non-propagation estimate (see Theorem 2.4 and the remark following it) that is our essential result. Our non-propagation estimate is much simpler than that of [13] and, in the case of a 1-particle problem, is based on the mechanics of classical particles and the commuativity of two operators, the pseudomomentum and the free Hamiltonian:

$$\left[p+\frac{1}{2}B\times r, \left(p-\frac{1}{2}B\times r\right)^2\right]=0.$$

In the case of a two-particle problem, the estimate is a direct consequence of the property of the domain $D(H(k))$.

2. Nonpropagation estimates

For simplicity of the arguments below, we may restrict ourselves to the magnetic field $B=(0, 0, B)$, $B>0$. Then we write $r=(r_\perp, z)=(x, y, z)$, $p=(p_\perp, p_z)=(p_x, p_y, p_z)$, and

$$H_0=H_{0\perp}\otimes Id+Id\otimes H_{0\parallel} \qquad \text{on } L^2(\boldsymbol{R}^2_{x,y})\otimes L^2(\boldsymbol{R}^1_z),$$

$$H_{0\parallel}=\frac{1}{2}p_z^2,$$

$$H_{0\perp}=\frac{1}{2}\left(p_\perp-\frac{1}{2}B\times r\right)^2$$

$$=-\frac{1}{2}\Delta_{x,y}+\frac{B^2}{8}(x^2+y^2)-\frac{B}{2}(-yp_x+xp_y).$$

Let $\mathcal{T}_0$ be the set of pure point spectrum of $H_{0\perp}$ given by

$$\mathcal{T}_0=\left\{\frac{B}{2}(2n+1)\,|\,n=0,\,1,\,2,\,\cdots\right\}.$$

Define $A=(1/2)(z\cdot p_z+p_z\cdot z)$ and

$$d(\lambda)=\begin{cases}\operatorname{dist}(\lambda,\,(-\infty,\,\lambda]\cap\mathcal{T}_0) & \text{if } \lambda\geqq\dfrac{B}{2}\,,\\ 0 & \text{otherwise.}\end{cases}$$

Then, under the conditions (1.1) and (1.3) we have (cf. [7])

LEMMA 2.1. (1) *For every $\varepsilon>0$ and $\lambda\in\boldsymbol{R}^1$, there exist an open interval I, containing λ, and a compact operator K such that the Mourre estimate holds:*

$$E_I(H)i[H,\,A]E_I(H)\geqq 2(d(\lambda)-\varepsilon)E_I(H)+K\,, \tag{2.1}$$

where $E_I(H)$ is the spectral projection for H associated with interval I.

(2) *The set $\sigma_p(H)$ of point spectrum of H is discrete in $\boldsymbol{R}^1\backslash\mathcal{T}_0$.*

(3) *Define $\mathcal{T}=\sigma_p(H)\cup\mathcal{T}_0$. Let I be a compact interval in $\boldsymbol{R}^1\backslash\mathcal{T}$ and for any integer $n>0$, let $s>n-1/2$. Then there exists a constant $C>0$ such that*

$$\sup_{0<\kappa\leq 1,\,\lambda\in I}\|\langle z\rangle^{-s}\{(H-(\lambda\pm i\kappa))^{-1}\}^n\langle z\rangle^{-s}\|\leqq C\,.$$

A direct consequence of Lemma 2.1 is

LEMMA 2.2. *Let $f(\lambda)\in C_0^\infty(\boldsymbol{R}^1\backslash\mathcal{T})$. Then, for any s, s' with $0<s'<s$, there exists $C>0$ such that*

$$\|\langle z\rangle^{-s}e^{-itH}f(H)\langle z\rangle^{-s}\|\leqq C\langle t\rangle^{-s'}\,. \tag{2.2}$$

With the Mourre estimate (2.1) in mind we can apply the argument of Skibsted [15, Exs. 1, 2] to obtain the following minimal velocity estimates.

LEMMA 2.3. *Let $\lambda>B/2$ and $\lambda\notin\mathcal{T}$. Let I be a compact interval containing λ such that $I\cap\mathcal{T}=$empty. Then there exists a constant $m>0$ such that for any $f\in C^\infty(\boldsymbol{R}^1)$ supported in I and for any $s>s'>0$,*

$$F\Big(\frac{z^2}{t^2}\leqq m\Big)e^{-itH}f(H)\langle z\rangle^{-s}=O(t^{-s'})\qquad\text{as } t\to\infty\,, \tag{2.3}$$

where $F(S)$ denotes the projection onto the set S.

THEOREM 2.4. *Suppose that the condition (V) is obeyed. For any compact* *interval* $I\subset \boldsymbol{R}^1\backslash \mathcal{T}$, *let* $\phi=E_I(H)\phi\in L^2(\boldsymbol{R}^3)$ *and* $\phi\in D(|p|)\cap D(|r|^{1+\varepsilon})$, $\varepsilon>0$. *Then there exists a positive constant* C_ϕ, *independent of* t *such that for any* $t\in\boldsymbol{R}^1$,

$$\|r_\perp e^{-itH}\phi\|\leqq C_\phi\,. \tag{2.4}$$

PROOF. We first note that

$$\left[H_0,\ p_x-\frac{B}{2}y\right]=\left[H_0,\ p_y+\frac{B}{2}x\right]=0\,. \tag{2.5}$$

We will only show that $\|ye^{-itH}\phi\|$ is bounded in $t\geqq 0$. The case $t<0$ and the boundedness of $\|xe^{-itH}\phi\|$ can be verified in a similar manner. We set $\phi_t=e^{-itH}\phi$ and $I(t)=\|(p_x-(B/2)y)\phi_t\|^2$. Since $\|H_0\phi_t\|$ is bounded in t, we see that $\|(p_x+(B/2)y)\phi_t\|$ is also bounded. So, to prove the boundedness of $\|y\phi_t\|$, it suffices to show that $I(t)$ is bounded in t.

We use (2.5) to compute:

$$\begin{aligned}\frac{d}{dt}I(t)&=2\,\mathrm{Re}\left(i\left[H,\ p_x-\frac{B}{2}y\right]\phi_t,\ \left(p_x-\frac{B}{2}y\right)\phi_t\right)\\&=2\,\mathrm{Re}\left(i[V,\ p_x]\phi_t,\ \left(p_x-\frac{B}{2}y\right)\phi_t\right)\\&=2\,\mathrm{Re}\left(-V_x\phi_t,\ \left(p_x-\frac{B}{2}y\right)\phi_t\right),\end{aligned}$$

where we have put $V_x=\partial_x V$. The Schwartz inequality implies that

$$\frac{d}{dt}I(t)\leqq\|V_x\phi_t\|+\|V_x\phi_t\|I(t).$$

We can apply Gronwall's inequality to get

$$I(t)\leqq\exp\left(\int_0^t\|V_x\phi_s\|ds\right)\left(I(0)+\int_0^t\|V_x\phi_s\|ds\right).$$

The propagation estimate (2.2) is combined with the condition (1.2) and the assumption $\phi\in D(|r|^{1+\varepsilon})$ to imply that $\|V_x\phi_t\|=O(t^{-1-\delta})$ as $t\to\infty$ for some $\delta>0$. Hence we obtain with some positive constants C_ϕ and C'_ϕ,

$$I(t)\leqq C_\phi\left(\left\|\left(p_x-\frac{B}{2}y\right)\phi\right\|^2+C'_\phi\right),$$

which completes the proof.

REMARK. (1) In the case of two-particle problems, the estimate (2.4) is a direct consequence of the fact that $|B\times r|^2(H(k)+i)^{-1}$ is bounded.

(2) We consider a classical free one-particle Hamiltonian: $h_0(r,\xi)=(1/2)(\xi-(1/2)B\times r)^2$. Then the Hamilton flow $(r(t),\xi(t))$ for h_0 has the property:

$\xi(t)_\perp+(1/2)B\times r(t)=$constant. This is the motivation for introducing $I(t)$ in the proof of the theorem.

3. Asymptotic completeness

To make our contribution clear we split the proof of Theorem 1.1 into two steps by introducing the following intermediate Hamiltonian:

$$H_{0W}=H_0+W\,,$$

where the function $W=W(z)$ is given by (1.4). We note that the assertions of Lemma 2.3 and Theorem 2.4 also hold for H_{0W} if $\mathcal{T}$ is replaced by $\mathcal{T}_W:=\sigma_p(H_{0W})\cup\mathcal{T}_0$.

As the first step, we obtain

THEOREM 3.1. *The following inverse wave operators exist:*

$$s-\lim_{t\to\pm\infty} e^{itH_{0W}}e^{-itH}(I-E_p(H)), \tag{3.2$_\pm$}$$

where $E_p(H)$ denotes the projection onto the point spectral subspace of H.

PROOF. We shal prove the assertion only for '+' case. Let I be any compact interval in $\boldsymbol{R}^1\setminus\mathcal{T}$. If we show that the limit (3.2$_+$) exists for any $\phi\in L^2(\boldsymbol{R}^3)$ such that $E_I(H)\phi=\phi$ and $\phi\in D(|p|)\cap D(|r|^{1+\varepsilon})$, $\varepsilon>0$, then we can obtain the assertion by the density argument. With $V(t, r)=\chi(t, z)V(r)$, we compute the derivative:

$$\begin{aligned}\frac{d}{dt}e^{itH_{0W}}e^{-itH}\phi&=ie^{itH_{0W}}(H_{0W}-H)e^{-itH}\phi\\&=ie^{itH_{0W}}(W(z)-W(t, z))e^{itH}\phi+ie^{itH_{0W}}(W(t, z)-V(t, r))e^{-itH}\phi\\&\quad+ie^{itH_{0W}}(V(t, r)-V(r))e^{-itH}\phi\\&=T_1(t)+T_2(t)+T_2(t)\,.\end{aligned}\tag{3.3}$$

Since $(\log\langle t\rangle)^{-1}=o(1)$ as $t\to\infty$, we can use the minimal velocity estimate (2.3) to obtain

$$(W(z)-W(t, z))e^{-itH}\langle z\rangle^{-1-\varepsilon}=O(t^{-1-\delta}),$$
$$(V(t, r)-V(r))e^{-itH}\langle z\rangle^{-1-\varepsilon}=O(t^{-1-\delta})$$

for some $\delta>0$, and hence

$$\|T_1(t)\|=O(t^{-1-\delta})\quad\text{and}\quad\|T_3(t)\|=O(t^{-1-\delta})\,. \tag{3.4}$$

We examine $T_2(t)$. Note that

$$W(t, z)-V(t, r)=V(t, 0, z)-V(t, r_{\perp}, z)$$
$$=\int_0^1 \nabla_{\perp} V(t, \theta r_{\perp}, z) d\theta \cdot r_{\perp},$$

and that $\nabla_{\perp} V(t, \theta r_{\perp}, z)=O(t^{-1-\delta_1})$ by condition (1.2). Then we can apply Theorem 2.4 to get

$$\|(W(t, z)-V(t, r))e^{-itH}\phi\| \leqq C\langle t\rangle^{-1-\delta_1}\|r_{\perp}e^{-itH}\phi\| \leqq C'\langle t\rangle^{-1-\delta_1},$$

and conclude that $\|T_2(t)\|=O(t^{-1-\delta_1})$. Combining this with (3.4) implies that (3.3) is integrable over $[0, \infty)$, and therefore the limit (3.2_+) exists. Q.E.D.

The same argument as in the proof of Theorem 3.1 yields

THEOREM 3.2. *The wave operators*

$$s-\lim_{t\to\pm\infty} e^{itH}e^{-itH_{0W}}(I-E_p(H_{0W}))$$

exist.

Thus it remains to prove

THEOREM 3.3. *The modified wave operators*

$$s-\lim_{t\to\pm\infty} e^{itH_{0W}}e^{-iS(t, p_z)-itH_{0\perp}} \tag{3.5}$$

and the inverse operators

$$s-\lim_{t\to\pm\infty} e^{iS(t, p_z)+itH_{0\perp}}e^{-itH_{0W}}(I-E_p(H_{0W})), \tag{3.6}$$

exist.

PROOF. Since H_{0W} has a form of direct sum

$$H_{0W}=H_{0\perp}\otimes Id+Id\otimes(H_{0\parallel}+W) \quad \text{on } L^2(\boldsymbol{R}^2_{x,y})\otimes L^2(\boldsymbol{R}^1_z),$$

the existence of limit (3.5) can be reduced to the existence of modified were operators for 1-dimensional Schrödinger operator $H_{0\parallel}+W$ and it is a direct consequence of Hörmander [6]. The existence of limit (3.6) can be verified similary as in Sigal [14], by using sharp propagation estimates for $e^{-itH_{0W}}$ and we omit it. Q.E.D.

Now the assertion of Theorem 1.1 can be verified by combining Theorems 3.1-3.3.

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