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# ON SOME CLASSES OF ALMOST CONTACT METRIC MANIFOLDS

By

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## 1. Introduction

In [1] J. Berndt and L. Vanhecke introduced two classes ($\mathfrak{C}$- and $\mathfrak{B}$-spaces) of Riemannian manifolds which include the class of locally symmetric spaces using the properties of Jaoobi operators along geodesics. They provided some characterizations of $\mathfrak{C}$- and $\mathfrak{B}$-spaces and gave the classifications for dimensions two and three. For further developments on the two spaces, we refer to [2], [3] and [8]. Further, T. Takahashi ([19]) introduced the notion of a (Sasakian) locally $\varphi$-symmetric space which may be considered as the analogue in the almost contact metric case of locally Hermitian symmetric spaces. Also he gave examples and equivalent properties of Sasakian locally $\varphi$-symmetric spaces. For further results about the Sasakian locally $\varphi$-symmetric spaces, we refer to [5], [6].

In the present paper, we introduce in an analogous way as in [1] four classes of almost contact metric manifolds involving Sasakian locally $\varphi$-symmetric spaces. In section 2, we recall definitions and several elementary properties of an almost contact, a contact, a $K$-contact metric manifold and a Sasakian manifold. In sections 3 and 4 we give the definitions of a $\mathfrak{D}\mathfrak{C}$-*space*, a $\mathfrak{D}\mathfrak{P}$-*space*, a $\xi\mathfrak{C}$-*space* and a $\xi\mathfrak{P}$-*space* which are almost contact metric analogues of a $\mathfrak{C}$-space or a $\mathfrak{P}$-space in the Riemannian case. We may observe that a Sasakian manifold is a $\xi\mathfrak{C}$-space and at the same time a $\xi\mathfrak{P}$-space. Also we prove that a Sasakian manifold is locally $\varphi$-symmetric if and only if it is a $\mathfrak{D}\mathfrak{C}$-space and at the same time a $\mathfrak{D}\mathfrak{P}$-space. In section 5, we show that the tangent sphere bundle of a 2-dimensional Riemannian manifold is a $\xi\mathfrak{P}$-space if and only if the base manifold is flat or of constant curvature 1. Furthermore, we give some examples of almost contact metric $\mathfrak{D}\mathfrak{C}$-spaces and $\mathfrak{D}\mathfrak{P}$-spaces. In section 6, we consider real hypersurfaces of a complex projective space $CP^n$ with Fubini-Study metric and determine $\xi\mathfrak{P}$-hypersurfaces of $CP^n$. We also show that a homogeneous real hypersurface of $CP^n$ is a $\xi\mathfrak{C}$-space, and moreover, we give

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a characterization of homogeneous real hypersurfaces of two types which appeared in the classification given by R. Takagi ([18]). All manifolds in the present paper are assumed to be connected and of class $C^{\infty}$ unless otherwise specified.

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## 2. Preliminaries

In the present section, we recall definitions and elementary properties of an almost contact, a contact, a $K$-contact metric, and a Sasakian manifold. We refer to [4] for more details. A $(2n+1)$-dimensional differentiable manifold $M$ is called an almost contact manifold it it admits a (1, 1)-tensor field $\varphi$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$\eta(\xi)=1 \quad \text{and} \quad \varphi^2=-I+\eta\otimes\xi \tag{2.1}$$

where $I$ denotes the identity transformation. From (2.1) we get

$$\varphi\xi=0 \quad \text{and} \quad \eta\circ\varphi=0\,. \tag{2.2}$$

Moreover, it is easily observed that an almost contact manifold $M$ admits a Riemannian metric $g$ such that

$$g(\varphi X, \varphi Y)=g(X, Y)-\eta(X)\eta(Y) \tag{2.3}$$

for all vector fields $X$ and $Y$ tangent to $M$. Setting $Y=\xi$ in (2.3), we also see that $\eta(X)=g(X, \xi)$. A Riemannian manifold equipped with structure tensors $(\varphi, \xi, \eta, g)$ satisfying (2.1) and (2.3) is called an almost contact metric manifold and denoted by $(M, \varphi, \xi, \eta, g)$. For an almost contact metric manifold $M=(M, \varphi, \xi, \eta, g)$, one may define an almost complex structure $J$ on $M\times\boldsymbol{R}$ by $J(X, f(d/dt))=(\varphi X-f\xi, \eta(X)(d/dt))$, where $X$ is tangent to $M$, $f$ is a function on $M\times\boldsymbol{R}$ and $t$ the coordinate on $\boldsymbol{R}$. If the almost complex structure $J$ is integrable, $M$ is said to be normal. The integrability condition for the almost complex structure $J$ is the vanishing of the tensor field $[\varphi, \varphi]+2d\eta\otimes\xi$, where $[\varphi, \varphi]$ denotes the Nijenhuis torson of $\varphi$.

Also, for an almost contact metric manifold we define its fundamental 2-form $\Phi$ by

$$\Phi(X, Y)=g(X, \varphi Y)\,.$$

If $\Phi=d\eta$, $M=(M, \varphi, \xi, \eta, g)$ is called a contact metric manifold. In particular, we have $\eta\wedge(d\eta)^n\neq0$. If the characteristic vector field $\xi$ of a contact metric manifold $M$ is a Killing vector field with respect to $g$, then $M$ is called a $K$-contact metric manifold. We denote by $R$ the curvature tensor defined by $R(X, Y)Z=\nabla_X(\nabla_Y Z)-\nabla_Y(\nabla_X Z)-\nabla_{[X,Y]}Z$, where $\nabla$ is the Levi-Civita connection and $X$, $Y$, $Z$ are vector fields. It is known that the curvature tensor of a $K$-contact metric manifold satisfies

$$R(X, \xi)\xi=X-\eta(X)\xi . \tag{2.4}$$

A normal contact metric manifold is called a Sasakian manifold. We may see that the conditions of being normal and contact metric are equivalent to

$$(\nabla_X\varphi)Y=g(X, Y)\xi-\eta(Y)X . \tag{2.5}$$

We note that (2.5) implies

$$\nabla_X\xi=-\varphi X , \tag{2.6}$$

from which it follows that $\xi$ is a Killing vector field. The curvature tensor of a Sasakian manifold satisfies

$$R(X, Y)\xi=\eta(Y)X-\eta(X)Y , \tag{2.7}$$

$$R(X, \xi)Y=\eta(Y)X-g(X, Y)\xi . \tag{2.8}$$

### 3. $\mathfrak{DC}$-spaces and $\mathfrak{DP}$-spaces

In this section, we introduce two classes ($\mathfrak{DC}$- and $\mathfrak{DP}$-*spaces*) of almost contact metric manifolds which extend Sasakian locally $\varphi$-symmetric spaces. Let $M=(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Let $T$ be a tensor field of type (1, 2) defined by (cf. [17])

$$T_XY=-\frac{1}{2}\varphi(\nabla_X\varphi)Y-\frac{1}{2}\eta(Y)\nabla_X\xi-\eta(X)\varphi Y+(\nabla_X\eta)(Y)\xi ,$$

for all vector fields $X$ and $Y$. We define a linear connection on $M$ by

$$\bar{\nabla}_XY=\nabla_XY+T_XY . \tag{3.1}$$

The linear connection $\bar{\nabla}$ has the torsion tensor $T_XY-T_YX$. Also, using (2.1) and (2.2), we have

$$\bar{\nabla}\varphi=0, \quad \bar{\nabla}\xi=0, \quad \bar{\nabla}\eta=0, \quad \bar{\nabla}g=0. \tag{3.2}$$

We remark that the above connection $\bar{\nabla}$ coincides with the Tanaka connection (defined in [20]) on a strongly pseudo-convex integral $CR$-manifold whose structure is determined by a given contact metric structure (see Proposition 2.1 in [22]).

The tangent space $T_pM$ of $M$ at $p\in M$ decomposes as $T_pM=\mathfrak{D}_p\oplus\xi_p$ (direct sum), where we denote $\mathfrak{D}_p=\{v\in T_pM\,|\,\eta(v)=0\}$. Then $\mathfrak{D}: p\rightarrow\mathfrak{D}_p$ defines a distribution orthogonal to $\xi$. From (3.2) we see that a $\bar{\nabla}$-geodesic (not necessarily a ($\nabla$-)geodesic) which is initially tangent to $\mathfrak{D}$ remains tangent to $\mathfrak{D}$, where a $\bar{\nabla}$-geodesic means a geodesic with respect to the linear connection $\bar{\nabla}$. We call such a $\bar{\nabla}$-geodesic which is tangent to $\mathfrak{D}$ a *horizontal* $\bar{\nabla}$*-geodesic*. Let $\gamma$ be a horizontal $\bar{\nabla}$-geodesic parametrized by the arc-length parameter $s$. We denote $\dot{\gamma}=\gamma_*(d/ds)$ where $\gamma_*$ is the differential of $\gamma: I\rightarrow M$. Using the Jacobi operator $R_{\dot{\gamma}}=R(\cdot, \dot{\gamma})\dot{\gamma}$ along $\gamma$, we introduce two new classes $\mathfrak{D}\mathfrak{C}$ and $\mathfrak{D}\mathfrak{P}$ of almost contact metric manifolds as analogous concepts of the $\mathfrak{C}$- and $\mathfrak{P}$-classes (defined in [1]) of Riemannian manifolds. Namely, we denote by $\mathfrak{D}\mathfrak{C}$ the class of almost contact metric manifolds such that the eigenvalues of $R_{\dot{\gamma}}$ are constant along $\gamma$ and by $\mathfrak{D}\mathfrak{P}$ that of almost contact metric manifolds such that $R_{\dot{\gamma}}$ is diagonalizable by a parallel orthonormal frame field along $\gamma$ with respect to $\bar{\nabla}$, for any $\bar{\nabla}$-geodesic $\gamma$ whose tangent vectors belong to $\mathfrak{D}$. An almost contact metric manifold $M$ is said to be a $\mathfrak{D}\mathfrak{C}$*-space* (resp. $\mathfrak{D}\mathfrak{B}$*-space*) if $M$ belongs to $\mathfrak{D}\mathfrak{C}$ (resp. $\mathfrak{D}\mathfrak{P}$).

In particular, let $M=(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. Then by (2.5) and (2.6) we have

$$T_XY=g(X, \varphi Y)\xi-\eta(X)\varphi Y+\eta(Y)\varphi X$$

for all vector fields $X$ and $Y$ on $M$. Moreover, we have $T_XX=0$ and

$$\bar{\nabla}\varphi=0,\quad \bar{\nabla}\xi=0,\quad \bar{\nabla}\eta=0,\quad \bar{\nabla}g=0,\quad \bar{\nabla}T=0. \tag{3.3}$$

Also, we have

$$\begin{aligned}(\bar{\nabla}_VR)(X, Y)Z=&(\nabla_VR)(X, Y)Z+g(V, \varphi R(X, Y)Z)\xi-\eta(V)\varphi R(X, Y)Z\\&+\eta(R(X, Y)Z)\varphi V-g(V, \varphi X)R(\xi, Y)Z+\eta(V)R(\varphi X, Y)Z\\&-\eta(X)R(\varphi V, Y)Z-g(V, \varphi Y)R(X, \xi)Z+\eta(V)R(X, \varphi Y)Z\\&-\eta(Y)R(X, \varphi V)Z-g(V, \varphi Z)R(X, Y)\xi+\eta(V)R(X, Y)\varphi Z\\&-\eta(Z)R(X, Y)\varphi V\end{aligned} \tag{3.4}$$

for all vector fields $V$, $X$, $Y$, $Z$ on $M$. From (3.4), using (2.7) and (2.8) we have

$$g((\bar{\nabla}_VR)(X, Y)Z, \xi)=0\,, \tag{3.5}$$

$$g((\bar{\nabla}_VR)(X, Y)Z, W)=g((\nabla_VR)(X, Y)Z, W) \tag{3.6}$$

for all $V, X, Y, Z, W\in\mathfrak{D}$. Taking account of the fact $T_XX=0$ and from (3.3), we have

LEMMA 3.1. *Let $M$ be a Sasakian manifold. Then a $\bar{\nabla}$-geodesic coincides with a ($\nabla$-)geodesic, and a geodesic which is initially tangent to $\mathfrak{D}$ remains tangent to $\mathfrak{D}$.*

We recall the definition of a Sasakian locally $\varphi$-symmetric space ([19]).

DEFINITION 3.2. A Sasakian manifold $M=(M, \varphi, \xi, \eta, g)$ is said to be a *locally $\varphi$-symmetric space* if the curvature tensor $R$ satisfies $\varphi^2(\nabla_V R)(X, Y)Z=0$ for all $V, X, Y, Z\in\mathfrak{D}$.

Taking account of (2.1), we see that the condition $\varphi^2(\nabla_V R)(X, Y)Z=0$ is equivalent to $g((\nabla_V R)(X, Y)Z, W)=0$ for all $V, X, Y, Z, W\in\mathfrak{D}$.

Now we give a characterization of a Sasakian locally $\varphi$-symmetric space.

THEOREM 3.3. *Let $M$ be a Sasakian manifold. Then $M$ is locally $\varphi$-symmetric if and only if $M$ belongs to $\mathfrak{DC}\cap\mathfrak{DP}$, i.e., $M$ is a $\mathfrak{DC}$-space and at the same time a $\mathfrak{DP}$-space.*

PROOF. Let $M$ be a locally $\varphi$-symmetric space and $\gamma: I\to M$ be a geocesic parametrized by the arc-length parameter $s$ with $\dot{\gamma}(0)\in\mathfrak{D}_{\gamma(0)}$. Then from Lemma 3.1 we see that $\gamma$ is also a $\bar{\nabla}$-geodesic and $\dot{\gamma}(s)\in\mathfrak{D}$ for all $s\in I$. At first, for the vector field $\xi$, we see that $\bar{\nabla}_{\dot{\gamma}}\xi=0$ and $R_{\dot{\gamma}}\xi=\xi$ from (2.8). Thus it is sufficient to consider the Jacobi operator $R_{\dot{\gamma}}$ on $\mathfrak{D}$. Now we assume $R_{\dot{\gamma}}(s_0)v=\kappa v$ for some $s_0\in I$ and $v\in\mathfrak{D}_{\gamma(s_0)}$. Let $E_v$ be the parallel vector field with respect to $\bar{\nabla}$ along $\gamma$ with $E_v(s_0)=v$. Then since $M$ is locally $\varphi$-symmetric, from (3.5) and (3.6) we see that $R_{\dot{\gamma}}E_v$ and $\kappa E_v$ are parallel vector fields alongs $\gamma$ with respect to $\bar{\nabla}$. Thus we have $R_{\dot{\gamma}}E_v=\kappa E_v$. Therefore we have the conclusion.

Conversely, let us assume that $M$ is a $\mathfrak{DC}$-space and at the same time a $\mathfrak{DP}$-space. Then by definition we may assume that $R_{\dot{\gamma}}E_i=\kappa_i E_i$, $i=1, 2, \cdots, 2n+1$, where $\kappa_i$ are constant along $\gamma$ and $\{E_i\}$ is an orthonormal parallel frame field along $\gamma$ with respect to $\bar{\nabla}$. By covariantly differentiating both sides of the above equations with respect to $\bar{\nabla}$ along $\gamma$ (as a $\bar{\nabla}$-geodesic), we get $(\bar{\nabla}_{\dot{\gamma}}R)(\cdot, \dot{\gamma})\dot{\gamma}=0$, which implies $(\bar{\nabla}_v R)(\cdot, v)v=0$ for any $v\in\mathfrak{D}_p$ and $p\in M$. Thus with (3.6) we have $g((\bar{\nabla}_V R)(X, V)V, W)=g((\nabla_V R)(X, V)V, W)=0$ for all $V, X, W\in\mathfrak{D}$. By polarization of the above equation and using the first and the second Bianchi identities, we have $g((\nabla_V R)(X, Y)Z, W)=0$ for all $V, X, Y, Z, W\in\mathfrak{D}$ (cf. [9], [23]). Therefore from Definition 3.2 we see that $M$ is locally $\varphi$-symmetric. (Q.E.D.)

REMARK 3.4. In particular, let $M$ be a 3-dimensional Sasakian manifold. It is well-known that the curvature tensor $R$ of a 3-dimensional Riemannian manifold is expressed by

$$R(X, Y)Z=\rho(Y, Z)X-\rho(X, Z)Y+g(Y, Z)QX-g(X, Z)QY - \frac{1}{2}\tau\{g(Y, Z)X-g(X, Z)Y\} \tag{3.7}$$

for all vector fields $X, Y, Z$, where $Q$ is the Ricci (1, 1)-tensor determined by $\rho(X, Y)=g(QX, Y)$ and $\tau$ is the scalar curvature of the manifold. Let $\gamma$ be a geodesic parametrized by the arc-length parameter $s$ with $\dot\gamma(s)\in\mathfrak{D}_{\gamma(s)}$ (see Lemma 3.1). From (3.3) we see that $\{\dot\gamma, \varphi\dot\gamma, \xi\}$ is a parallel orthonormal frame field along $\gamma$ with respect to $\bar\nabla$. From (2.8) and (3.7), we have $R(\xi, \dot\gamma)\dot\gamma=R(\xi, \varphi\dot\gamma)\varphi\dot\gamma=\xi$ and $R(\varphi\dot\gamma, \dot\gamma)\dot\gamma=\{(1/2)\tau-\rho(\xi, \xi)\}\varphi\dot\gamma$. Thus we see that *a 3-dimensional Sasakian manifold is a $\mathfrak{D}\mathfrak{P}$-space*. Applying Theorem 3.3 to the 3-dimensional case, we see that *a 3-dimensional Sasakian manifold is locally $\varphi$-symmetric if and only if the scalar curvature is constant for all directions orthogonal to $\xi$*. This gives another proof of Theorem 4.1 in [24].

Returning to the general case, we characterize an almost contact metric $\mathfrak{D}\mathfrak{C}$-space and $\mathfrak{D}\mathfrak{P}$-space in a similar way as in [1]. We prove

PROPOSITION 3.5. *An almost contact metric manifold $M$ is a $\mathfrak{D}\mathfrak{C}$-space if and only if for each $p\in M$ and $v\in\mathfrak{D}_p$, there exists an endomorphism $S_v$ of $T_pM$ such that $R'_v=R_v\circ S_v-S_v\circ R_v$ where we denote $R'_v=(\bar\nabla_vR)(\cdot, v)v$.*

PROOF. Let $M$ be a $\mathfrak{D}\mathfrak{C}$-space and $\gamma$ be a horizontal $\bar\nabla$-geodesic in $M$ which is parametrized by the arc-length parameter $s$ and $\gamma(0)=p$ and $\dot\gamma(0)=v$ for any $p\in M$ and $v\in\mathfrak{D}_p$. Let $\tau^\gamma_{0,s}$ be the parallel translation along $\gamma$ from $\gamma(0)$ to $\gamma(s)$ with respect to $\bar\nabla$. Then from the property $\bar\nabla g=0$, we see that $\tau^\gamma$ is an isometry along $\gamma$. Now we put $A(s)=\tau^\gamma_{s,0}\circ R_{\dot\gamma}\circ\tau^\gamma_{0,s}$, then $A(s)$ is a family of self-adjoint endomorphisms of $T_pM$ and the eigenvalues of $A(s)$ are constant. Thus applying Lemma 4 in [1], there exists a family of endomorphisms $S(s)$ of $T_pM$ such that $A'(s)=A(s)\circ S(s)-S(s)\circ A(s)$. This implies $A'(0)=A(0)\circ S(0)-S(0)\circ A(0)$. Thus we have $R'_{\dot\gamma}(0)=R_{\dot\gamma}(0)\circ S(0)-S(0)\circ R_{\dot\gamma}(0)$, and hence $R'_v=R_v\circ S_v-S_v\circ R_v$ where $S_v=S(0)$. In order to prove the converse, let $\gamma: I\to M$ be a horizontal $\bar\nabla$-geodesic parametrized by the arc-length parameter $s$ with $\gamma(s_0)=p$, $s_0\in I$. Let $A(s)=\tau^\gamma_{s,s_0}\circ R_{\dot\gamma}(s)\circ\tau^\gamma_{s_0,s}$ and $S(s)=\tau^\gamma_{s,s_0}\circ S_{\dot\gamma(s)}\circ\tau^\gamma_{s_0,s}$. Then we see that $A(s)$ and $S(s)$ are families of endomorphisms of $T_pM$ and by a calculation we have

$$A'(s)=\tau^{\gamma}_{s,s_0}\circ R'_{\dot\gamma}\circ\tau^{\gamma}_{s_0,s}$$
$$=\tau^{\gamma}_{s,s_0}\circ(R_{\dot\gamma}\circ S_{\dot\gamma}-S_{\dot\gamma}\circ R_{\dot\gamma})\circ\tau^{\gamma}_{s_0,s} \text{ (by the assumption)}$$
$$=A(s)\circ S(s)-S(s)\circ A(s),$$

i.e., there exists a family of endomorphisms $S(s)$ of $T_pM$ such that $A'(s)=A(s)\circ S(s)-S(s)\circ A(s)$. Thus by Lemma 4 in [1], we see that the eigenvalues of the endomorphism $A$, and therefore also of $R_{\dot\gamma}$ are constant. (Q.E.D.)

On the other hand, as a characterization of an almost contact metric $\mathfrak{D}\mathfrak{P}$-space, we have

PROPOSITION 3.6. *If $M$ is a $\mathfrak{D}\mathfrak{P}$-space, then $R_v\circ R'_v=R'_v\circ R_v$ for all $v\in\mathcal{D}_p$, $p\in M$, where $R'_v=(\nabla_vR)(\cdot, v)v$. Moreover, if $M$ is real analytic, then also the converse holds.*

We refer to Lemma 5 in [1] for the proof of the above Proposition 3.6.

#### 4. $\xi\mathfrak{C}$-spaces and $\xi\mathfrak{P}$-spaces

In this section, we study local symmetry in the direction $\xi$. All almost contact metric manifolds do not satisfy the following condition: (*) each trajectory of $\xi$ is a geodesic. However some special cases of almost contact metric manifold do satisfy it. For example, the tangent sphere bundle of a Riemannian manifold as a hypersurface of the tangent bundle with an almost Kähler structure inherits an almost contact metric structure and satisfies (*) (cf. chapter 7 in [4]). Another example is a homogeneous real hypersurface of an $n$-dimensional complex projective space $CP^n$ with Fubini-Study metric (cf. [11]). We may also observe that every contact metric manifold satisfies the condition (*) (cf. [4]). Moreover, from (2.4) and (2.7), we see that a $K$-contact metric manifold and a Sasakian manifold satisfy in addition $(\nabla_\xi R)(\cdot, \xi)\xi=0$.

DEFINITION 4.1. An almost contact metric manifold $M$ with a structure $(\varphi, \xi, \eta, g)$ is said to be a *locally $\xi$-symmetric space* if $M$ satisfies (*) (i.e., $\nabla_\xi\xi=0$) and $(\nabla_\xi R)(\cdot, \xi)\xi=0$.

We remark that a contact metric manifold whose characteristic vector field $\xi$ belongs to the $k$-nullity distribution (see [21]) is a locally $\xi$-symmetric space. We may characterize a locally $\xi$-symmetric space using the Jacobi operator $R_\xi=R(\cdot, \xi)\xi$ associated with the vector field $\xi$ in a similar way as in Theorem 1 in [1]. Namely, an almost contact metric manifold $M$ satisfying the condition $(*)$ is locally $\xi$-symmetric if and only if $M$ satisfies the following two conditions: $(c)$ the eigenvalues of $R_\xi$ are constant along each trajectory of $\xi$ and $(p) R_\xi$ is diagonalizable by a parallel orthonormal frame field along each trajectory of $\xi$. We denote by $\xi\mathfrak{C}$ the class of almost contact metric manifolds with $(*)$ and $(c)$, and by $\xi\mathfrak{P}$ that of almost contact metric manifolds with $(*)$ and $(p)$. An almost contact metric manifold $M$ is said to be a *$\xi\mathfrak{C}$-space* (resp. *$\xi\mathfrak{P}$-space*) if $M$ belongs to $\xi\mathfrak{C}$ (resp. $\xi\mathfrak{P}$).

From Theorem 2 (resp. Theorem 5) in [1], we immediately have the following Remark 4.2 (resp. Remark 4.3) as a characterization of a $\xi\mathfrak{C}$-(resp. $\xi\mathfrak{P}$-) space.

REMARK 4.2. An almost contact metric manifold $M$ is a $\xi\mathfrak{C}$-space if and only if $M$ satisfies $(*)$ and there exists a skew-symmetric (1, 1)-tensor field $B_\xi$ such that $\dot{R}_\xi = R_\xi \circ B_\xi - B_\xi \circ R_\xi$ where we denote $\dot{R}_\xi = (\nabla_\xi R)(\cdot, \xi)\xi$.

REMARK 4.3. If an almost contact metric manifold $M$ is a $\xi\mathfrak{P}$-space, then we have $R_\xi \circ \dot{R}_\xi = \dot{R}_\xi \circ R_\xi$ and moreover, if $M$ satisfies $(*)$ and is real analytic, then the converse holds.

Also, we have some interesting equivalent properties of a $\xi\mathfrak{P}$-space related to the geometry of Jacobi vector fields and the geometry of geodesic spheres along geodesic trajectories of $\xi$. For more details concerning that, we refer to [1] and [2].

## 5. Tangent sphere bundle of a surface

Let $M$ be a 2-dimensional Riemannian manifold and $T_1M$ the tangent sphere bundle of $M$ (i.e., the set of all unit tangent vectors of $M$) with the projection map $\pi : T_1M \to M$. As we stated in the first part of section 4, it is known that the tangent bundle $TM$ admits an almost Kähler structure $(J, \bar{g})$ (cf. chapter 7 in [4]). Let $(x^1, x^2)$ be an isothermal local coordinate system on $M$ such that the Riemannian metric is of the form

$$\rho^2((dx^1)^2 + (dx^2)^2)$$

where $\rho$ is a function on $M$. Then by a calculation we see that the Gauss curvature $\kappa$ of $M$ is $-(\Delta_0 \log \rho / \rho^2)$ where $\Delta_0$ is the Laplacian with respect to Euclidean metric. Let $(u^1, u^2, y^1, y^2)$ be a local coordinate system around a point $p$ of $T_1M$ in $TM$ sucn that $u^i = x^i \circ \pi$ and $\rho^2((y^1)^2 + (y^2)^2) = 1$. The vector field $N = y^1(\partial/\partial y^1) + y^2(\partial/\partial y^2)$ is a unit normal and the position vector for the point $p$ of $T_1M$. Denote by $g$ the metric of $T_1M$ induced from $\bar{g}$ on $TM$.

Define $\varphi$, $\xi$, $\eta$ by

$$JN=-\xi, \qquad JX=\varphi X+\eta(X)N.$$

Then we see that $(\varphi, \xi, \eta, g)$ is an almost contact metric structure of $T_1M$ and we have a local orthonormal frame field $\{e_1, e_2, e_3\}$ as follows:

(5.1)
$$e_3=\xi=\sum_{ijk}\left(y^i\frac{\partial}{\partial u^i}-\left\{\begin{smallmatrix} \tilde{i} \\ j \quad k\end{smallmatrix}\right\}y^jy^k\frac{\partial}{\partial y^i}\right).$$
$$e_1=\sum_i z^i\frac{\partial}{\partial y^i},$$
$$e_2=-\varphi e_1=\sum_{ijk}\left(z^i\frac{\partial}{\partial u^i}-\left\{\begin{smallmatrix} \tilde{i} \\ j \quad k\end{smallmatrix}\right\}y^jz^k\frac{\partial}{\partial y^i}\right)$$

for $i, j, k=1, 2$ where we denote $(z^1, z^2)=(-y^2, y^1)$, $\left\{\begin{smallmatrix} \tilde{i} \\ j \quad k\end{smallmatrix}\right\}=\left\{\begin{smallmatrix} i \\ j \quad k\end{smallmatrix}\right\}\circ\pi$ and where $\left\{\begin{smallmatrix} i \\ j \quad k\end{smallmatrix}\right\}$ are the Christoffel symbols of the Riemannian connection of $M$.

For the local orthonormal frame field we have

(5.2)
$$[e_1, e_2]=-e_3, \quad [e_2, e_3]=-\tilde{\kappa}e_1, \quad [e_3, e_1]=-e_2,$$

where $\tilde{\kappa}=\kappa\circ\pi$. Put

$$\Gamma_{ijk}=g(\nabla_{e_i}e_j, e_k) \qquad \text{for } i, j, k=1, 2, 3.$$

Then we have $\Gamma_{ijk}=-\Gamma_{ikj}$. We recall the formula

$$2g(\nabla_XY, Z)=Xg(Y, Z)+Yg(Z, X)-Zg(X, Y)+g(Y, [Z, X])$$
$$+g(Z, [X, Y])-g(X, [Y, Z])$$

for all vector fields $X, Y, Z$ on $T_1M$. Using this formula, we obtain

(5.3)
$$\Gamma_{123}=\frac{1}{2}(\tilde{\kappa}-2), \quad \Gamma_{213}=\Gamma_{321}=\frac{\tilde{\kappa}}{2}, \quad \text{all other } \Gamma_{ijk} \text{ being zero.}$$

From (5.3) we see that $e_1$, $e_2$, $e_3$ are all geodesic vector fields, i.e., self-parallel vector fields and from (5.2) and (5.3) we get

(5.4)
$$R(e_1, e_3)e_3=\frac{1}{4}\tilde{\kappa}^2e_1+\frac{1}{2}(e_3\tilde{\kappa})e_2,$$
$$R(e_2, e_3)e_3=\frac{1}{2}(e_3\tilde{\kappa})e_1-\left(\frac{3}{4}\tilde{\kappa}_2-\tilde{\kappa}\right)e_2,$$

(5.5)
$$R(e_2, e_1)e_1=\frac{1}{4}\tilde{\kappa}^2e_2,$$
$$R(e_3, e_1)e_1=\frac{1}{4}\tilde{\kappa}^2e_3,$$

$$R(e_1, e_2)e_2=\frac{1}{4}\tilde{\kappa}^2 e_1-\frac{1}{2}(e_2\tilde{\kappa})e_3\,,$$

$$R(e_3, e_2)e_2=-\frac{1}{2}(e_2\tilde{\kappa})e_1-\Big(\frac{3}{4}\tilde{\kappa}^2-\tilde{\kappa}\Big)e_3\,.$$

Moreover, we have

$$(5.6)\qquad (\nabla_{e_3}R)(e_1, e_3)e_3=\tilde{\kappa}(e_3\tilde{\kappa})e_1+\frac{1}{2}\{e_3(e_3\tilde{\kappa})-\tilde{\kappa}^3+\tilde{\kappa}^2\}e_2$$

$$(\nabla_{e_3}R)(e_2, e_3)e_3=\frac{1}{2}\{e_3(e_3\tilde{\kappa})-\tilde{\kappa}^3+\tilde{\kappa}^2\}e_1+\{e_3\tilde{\kappa}-2\tilde{\kappa}(e_3\tilde{\kappa})\}e_2\,.$$

PROPOSITION 5.1. *The tangent sphere bundle $T_1M$ of a 2-dimensional Riemannian manifold $M$ is a $\xi\mathfrak{C}$-space if and only if the Gauss curvature of $M$ is constant.*

PROOF. From (5.4) we have the following matrix representation of $R_\xi$ with respect to $\{e_1, e_2, e_3\}$ :

$$R_\xi=\begin{pmatrix} \frac{1}{4}\tilde{\kappa}^2 & \frac{1}{2}(e_3\tilde{\kappa}) & 0 \\ \frac{1}{2}(e_3\tilde{\kappa}) & -\frac{3}{4}\tilde{\kappa}^2+\tilde{\kappa} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues $\lambda_i$, $i=1, 2$, $(\lambda_3=0)$ of $R_\xi$ are

$$\lambda_1=\frac{-\frac{1}{2}\tilde{\kappa}^2+\tilde{\kappa}+\sqrt{\tilde{\kappa}^2(\tilde{\kappa}-1)^2+(e_3\tilde{\kappa})^2}}{2}$$

$$\lambda_2=\frac{-\frac{1}{2}\tilde{\kappa}^2+\tilde{\kappa}-\sqrt{\tilde{\kappa}^2(\tilde{\kappa}-1)^2+(e_3\tilde{\kappa})^2}}{2}\,.$$

Now we assume that the tangent sphere bundle $T_1M$ of a 2-dimensional Riemannian manifold $M$ is a $\xi\mathfrak{C}$-space, that is, the eigenvalues $\lambda_i$ $(i=1, 2)$ of $R_\xi$ are constant along each trajectory of $\xi$. Let $W=\{p\in T_1M\,|\,\lambda_1(p)\neq\lambda_2(p)\}$. Then $W$ is an open and dense subset of $T_1M$. Thus we have $\xi(\lambda_1+\lambda_2)=0$ on $W$, which implies that $\xi\tilde{\kappa}=0$ on $W$. From the continuity of $\tilde{\kappa}$, we see that $\xi\tilde{\kappa}=0$ on $T_1M$ and from (5.1) we conclude that $\kappa$ is constant on $M$. Conversely, if $\kappa$ is constant on $M$, then $\tilde{\kappa}=\kappa\circ\pi$ is also constant on $T_1M$. Thus, from (5.4) and (5.6), we have

$$R_\xi=\begin{pmatrix} \frac{1}{4}\tilde{\kappa}^2 & 0 & 0 \\ 0 & -\frac{3}{4}\tilde{\kappa}^2+\tilde{\kappa} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \dot{R}_\nu=\begin{pmatrix} 0 & -\frac{1}{2}\tilde{\kappa}^3+\frac{1}{2}\tilde{\kappa}^2 & 0 \\ -\frac{1}{2}\tilde{\kappa}^3+\frac{1}{2}\tilde{\kappa}^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to $\{e_1, e_2, e_3\}$. Put

$$B_\xi=\begin{pmatrix} 0 & -\frac{1}{2}\tilde{\kappa} & 0 \\ \frac{1}{2}\tilde{\kappa} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we have $\dot{R}_\xi=R_\xi\circ B_\xi-B_\xi\circ R_\xi$. Thus from Remark 4.2 we see that the tangent sphere bundle $T_1M$ is a $\xi\mathfrak{C}$-space. (Q. E. D.)

THEOREM 5.2. *The tangent sphere bundle $T_1M$ of a 2-dimensional Riemannian manifold $M$ is a $\xi\mathfrak{P}$-space (or locally $\xi$-symmetric space) if and only if the Gauss curvature of $M$ is* 0 *or* 1.

PROOF. Assume that $T_1M$ is a $\xi\mathfrak{P}$-space. Then from Remark 4.3 we see that $T_1M$ satisfies $R_\xi\circ\dot{R}_\xi=\dot{R}_\xi\circ R_\xi$, where $\dot{R}_\xi=(\nabla_\xi R)(\cdot,\ \xi)\xi$. From (5.4) and (5.6), we calculate $R_\xi(\dot{R}_\xi(e_i))=\dot{R}_\xi(R_\xi(e_i))$ for $i=1, 2$. Then we have

$$\tilde{\kappa}^5-2\tilde{\kappa}^4+\tilde{\kappa}^3-(\xi(\xi\tilde{\kappa}))\tilde{\kappa}^2+\{3(\xi\tilde{\kappa})^2+\xi(\xi\tilde{\kappa})\}\,\tilde{\kappa}-(\xi\tilde{\kappa})^2=0\,.$$

From the above equation, we have $\tilde{\kappa}^5-2\tilde{\kappa}^4+\tilde{\kappa}^3=\tilde{\kappa}^3(\tilde{\kappa}^2-2\kappa+1)=0$. Thus we see that $\kappa=0$ or 1. Conversely, if $\kappa=0$ or 1, then from (5.4) we see that $T_1M$ is flat or a space of constant sectional curvature 1/4. Thus we see that $T_1M$ is of course a $\xi\mathfrak{P}$-space. We recall that a locally $\xi$-symmetric space is equivalently characterized as a $\xi\mathfrak{C}$- which is at the same time a $\xi\mathfrak{P}$-space. Thus from the result of Proposition 5.1 we see that $T_1M$ is a $\xi\mathfrak{P}$-space if and only if it is a locally $\xi$-symmetric space. (Q. E. D.)

We remark that ([13]) $T_1(S^2)$ is isometric to the elliptic space $\boldsymbol{RP}^3$ of constant curvature 1/4, where $S^2$ is the unit sphere in a Euclidean space $\boldsymbol{E}^3$ with the induced metric.

On the other hand, from (3.1), (3.2) and (5.3) we have

(5.7) $$\nabla_{e_i}\xi=0 \quad \text{and} \quad \nabla_{e_i}e_j=0 \qquad \text{for } i,\ j=1,\ 2$$

and moreover, we have

$$(5.8)\qquad \begin{aligned}&(\bar{\nabla}_{e_1}R)(e_2, e_1)e_1=0\,,\\ &(\bar{\nabla}_{e_1}R)(e_3, e_1)e_1=0\,,\\ &(\bar{\nabla}_{e_2}R)(e_1, e_2)e_2=\frac{1}{2}\tilde{\kappa}(e_2\tilde{\kappa})e_1-\frac{1}{2}e_2(e_2\tilde{\kappa})e_3\,,\\ &(\bar{\nabla}_{e_2}R)(e_3, e_2)e_2=-\frac{1}{2}e_2(e_2\tilde{\kappa})e_1-\frac{1}{2}\{3\tilde{\kappa}(e_2\tilde{\kappa})-2(e_2\tilde{\kappa})\}e_3\,.\end{aligned}$$

PROPOSITION 5.3. *The tangent sphere bundle $T_1M$ of a 2-dimensional Riemannian manifold $M$ is a $\mathfrak{DC}$-space if and only if the Gauss curvature of $M$ is constant.*

PROOF. Assume that the tangent sphere bundle $T_1M$ of a 2-dimensional manifold $M$ is a $\mathfrak{DC}$-space. Using a similar calculation and argument as in the proof of Proposition 5.1, we see that $\kappa$ is constant on $M$. Conversely, we assume that $\kappa$ is constant on $M$. Taking an endomorphism $S_v=0$ of $T_p(T_1M)$ for any $v\in\mathfrak{D}_p$ and $p\in T_1M$, then from (5.5), (5.8) and Proposition 3.5, we see that $T_1M$ is a $\mathfrak{DC}$-space. (Q.E.D.)

PROPOSITION 5.4. *The tangent sphere bundle $T_1M$ of a 2-dimensional Riemannian manifold is a $\mathfrak{DP}$-space if and only if the Gauss curvature of $M$ is constant.*

PROOF. Assume that $T_1M$ is a $\mathfrak{DP}$-space. Then from Proposition 3.6 we see that $T_1M$ satisfies $R_v\circ R'_v=R'_v\circ R_v$ for all $v\in\mathfrak{D}_p$, $p\in T_1M$, where $R'_v=(\bar{\nabla}_vR)\cdot(\cdot, v)v$. From (5.5) and (5.8) we calculate $R_{e_2}(R'_{e_2}(e_a))=R'_{e_2}(R_{e_2}(e_a))$ for $a=1, 3$. Then we get

$$(e_2\tilde{\kappa})^2(1-2\tilde{\kappa})+(e_2(e_2\tilde{\kappa}))\tilde{\kappa}(\tilde{\kappa}-1)=0\,.$$

From the above equation, we see that $\kappa$ is constant. Conversely, if $\kappa$ is constant, then with (5.8) taking account of (5.3) and (5.7), we have $(\bar{\nabla}_{e_i}R)(\cdot, e_j)e_k=0$ for $i, j, k=1, 2$. It may be observed that a $\mathfrak{DC}$- which is at the same time a $\mathfrak{DP}$-space is equivalently characterized by $(\bar{\nabla}_VR)(\cdot, V)V=0$ for any $V\in\mathfrak{D}$. Thus we see that $T_1M$ is a $\mathfrak{DP}$-space. (Q.E.D.)

## 6. Real hypersurfaces of $CP^n$

Let $(CP^n, g, J)$ be an $n$-dimensional complex projective space with Fubini-Study metric $g$ of constant holomorphic sectional curvature 4, and let $M$ be an oriented real hypersurface of $CP^n$. We denote by the same $g$ the induced metric on $M$. Let $N$ be a unit normal vector field of $M$ in $CP^n$. For any vector field $X$ tangent to $M$, we put

$$JX=\varphi X+\eta(X)N, \qquad JN=-\xi. \tag{6.1}$$

Then we may see that the structnre $(\varphi, \xi, \eta, g)$ is an almost contact metric structure on $M$. By $\tilde{\nabla}$ we denote the Riemannian connection on $CP^n$ and by $\nabla$ the one on $M$ determined by the induced metric. The the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y=\nabla_X Y+g(AX, Y)N, \qquad \tilde{\nabla}_X N=-AX$$

for any vector field $X$ and $Y$ tangent to $M$, where $A$ is the shape operator of $M$ in $CP^n$. An eigenvector (resp. eigenvalue) of the shape operator $A$ is called a principal curvature vector (resp. principal curvature). Also we denote by $V_\lambda$ the eigenspace of $A$ associated with an eigenvalue $\lambda$. From the fact $\tilde{\nabla}J=0$ and (6.1), making use of the Gauss and Weingarten formulas, we have

$$\nabla_X \xi=\varphi AX. \tag{6.2}$$

Let $R$ be the curvature tensor of $M$. Then we have following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z=&g(Y, Z)X-g(X, Z)Y+g(\varphi Y, Z)\varphi X-g(\varphi X, Z)\varphi Y \\ &+2g(X, \varphi Y)\varphi Z+g(AY, Z)AX-g(AX, Z)AY, \end{aligned} \tag{6.3}$$

$$(\nabla_X A)Y-(\nabla_Y A)X=\eta(X)\varphi Y-\eta(Y)\varphi X+2g(X, \varphi Y)\xi. \tag{6.4}$$

From (6.2), we have

LEMMA 6.1. *Each trajectory of $\xi$ is a geodesic if and only if $\xi$ is a principal curvature vector.*

Typical examples of real hypersurfaces in $CP^n$ on which the trajectory of $\xi$ is a geodesic are homogeneous ones which are classified by R. Takai ([18]). T.E. Cecil and P.J. Ryan ([7]) investigated real hypersurfaces of $CP^n$ on which $\xi$ is a principal curvature vector. They showed that if $\xi$ is a principal curvature vector and the corresponding focal map has constant rank, then $M$ lies on a tube of constant radius over a certain Kähler submanifold. Making use of this notion and the result of R. Takagi's classification, M. Kimura ([11]) proved the following

THEOREM 6.2. *Let $M$ be a real hypersurface of $CP^n$. $M$ has constant principal curvatures and $\xi$ is principal if and only if $M$ is locally isometric to a* *homogeneous real hypersurface i.e., a tube of radius r over one of the following Kähler submanifolds*:

($A_1$) *a hyperplane* $CP^{n-1}$, *where* $0<r<\pi/2$;

($A_2$) *a totally geodesic* $CP^k$ $(1\leqq k\leqq n-2)$, *where* $0<r<\pi/2$;

(B) *a complex quadric* $Q^{n-1}$, *where* $0<r<\pi/4$;

(C) *a* $CP^1\times CP^{(n-1/2)}$, *where* $0<r<\pi/4$ *and* $n$ $(\geqq 5)$ *is odd*;

(D) *a complex Grassmann* $G_{2,5}(C)$, *where* $0<r<\pi/4$, $n=9$;

(E) *a Hermitian symmetric space* $SO(10)/U(5)$, *where* $0<r<\pi/4$, $n=15$.

We note that the number of distinct eigenvalues of the above real hypersurfaces is 2, 3 or 5, and the principal curvature $\alpha$ corresponding to the vector field $\xi$ is $2\cot 2r$ with multiplicity 1. For more details, we refer to [11] and [18]. We only state two lemmas without proofs.

LEMMA 6.3 ([14]). *If $\xi$ is principal curvature vector, then the corresponding principal curvature $\alpha$ is constant.*

LEMMA 6.4 ([14]). *Assume* $A\xi=\alpha\xi$. *If* $AX=\lambda X$ *for* $X\perp\xi$, *then we have* $A\varphi X=(\alpha\lambda+2/2\lambda-\alpha)\varphi X$.

Now we give a characterization of real hypersurfaces of $CP^n$ in the class $\xi\mathfrak{P}$ introduced in section 4.

PROPOSITION 6.5. *Let* $M^{2n-1}$ *be a* $\xi\mathfrak{P}$-*hypersurface of* $CP^n$. *Suppose* $A\xi\neq 0$. *Then M is locally isometric to a homogeneous real hypersurface of type* ($A_1$) *or* ($A_2$). *Moreover, any real hypersurface of type* ($A_1$) *or* ($A_2$) *is a* $\xi\mathfrak{P}$-*space.*

PROOF. Assume $M$ is a $\xi\mathfrak{P}$-hypersurface of $CP^n$. We see from Lemma 6.1 that $\xi$ is a principal curvature vector and from Lemma 6.3 that the corresponding principal curvature $\alpha$ is constant. Thus from (6.3) we have

$$R_\xi X=X+\alpha AX-(1+\alpha^2)\eta(X)\xi \tag{6.5}$$

and

$$\begin{aligned}\dot{R}_\xi X&=(\nabla_\xi R)(X, \xi)\xi\\&=\alpha(\nabla_\xi A)X\end{aligned} \tag{6.6}$$

for any $X$ tangent to $M$.

From Remark 4.3, we have

$$(6.7) \qquad 0=(R_\xi \circ \dot{R}_\xi - \dot{R}_\xi \circ R_\xi)X$$
$$=\alpha^2\{A(\nabla_\xi A)X-(\nabla_\xi A)AX\}.$$

Since $\alpha \neq 0$ (the assumption), we have $A(\nabla_\xi A)X-(\nabla_\xi A)AX=0$, and hence taking account of Lemma 6.3, from (6.2), (6.4) and (6.7), we have

$$0=(\alpha A\varphi AX-A^2\varphi AX+A\varphi X)-(\alpha\varphi A^2X-A\varphi A^2X+\varphi AX)$$

for any $X\in\mathfrak{D}$. Assume $X\in V_\lambda$. Then from Lemma 6.4 we have

$$0=\Big(\alpha\lambda-\lambda\frac{\alpha\lambda+2}{2\lambda-\alpha}+1\Big)\Big(\frac{\alpha\lambda+2}{2\lambda-\alpha}-\lambda\Big)\varphi X.$$

Thus we have

$$\alpha\lambda-\lambda\frac{\alpha\lambda+2}{2\lambda-\alpha}+1=0 \quad \text{or} \quad \frac{\alpha\lambda+2}{2\lambda-\alpha}-\lambda=0,$$

which implies $\lambda^2-\alpha\lambda-1=0$ $(\alpha\neq 0)$, and hence $\lambda(2\lambda-\alpha)=\alpha\lambda+2$, that is, $\lambda=(\alpha\lambda+2/2\lambda-\alpha)$. From this we conclude that $\varphi V_\lambda=V_\lambda$ and our real hypersurface $M$ must be locally isometric to one of real hypersurface of type ($A_1$) and ($A_2$) (cf. [16]). Taking account of the fact that every homogeneous manifold admits an analytic structure (refer to p. 123 in [10]), from the Remark 4.3 and (6.7), we see that any real hypersurface of type ($A_1$) or ($A_2$) is a $\xi\mathfrak{P}$-space. (Q.E.D.)

The above Proposition 6.3 is an improvement of the result obtained by M. Kimura and S. Maeda ([12]). Also we remark that a homogeneous real hypersurface of type ($A_2$) is a locally $\xi$-symmetric space which is not a $K$-contact metric (and of course, not Sasakian) manifold. (cf. [15]).

We see from (6.5) that *homogeneous real hypersurfaces of* $\boldsymbol{C}P^n$ *are* $\xi\mathfrak{P}$*-spaces.* Applying Remark 4.2, then from (6.5) and (6.6) we have

PROPOSITION 6.6. *A homogeneous real hypersurface of* $\boldsymbol{C}P^n$ *admits a skew-symmetric* (1, 1)*-tensor field* $B_\xi$ *such that*

$$\alpha(\nabla_\xi A)X=\alpha(AB_\xi X-B_\xi AX)+(1+\alpha^2)\{g(X,\ B_\xi\xi)\xi-g(X,\ \xi)B_\xi\xi\}$$

*for any vector fields $X$ tangent to $M$.*

We note that in particular for a homogeneous one of type ($A_1$) and ($A_2$), there exists a skew-symmetric (1, 1)-tensor field $B_\xi=\varphi$ such that

$$\nabla_\xi A=A\circ\varphi-\varphi\circ A \ (=0).$$

(See [12] and [16]). Thus we are motivated to prove the following

PROPOSITION 6.7. *Let $M$ be a real hypersurface of* $\boldsymbol{C}P^n$. *Suppose that* $\nabla_\xi\xi$

$=0$ *and* $A\xi\neq -2$. *If* $\nabla_{\xi}A=A\circ\varphi-\varphi\circ A$, *then M is locally isometric to a homogeneous real hypersurface of type* $(A_1)$ *and* $(A_2)$.

PROOF. Using the same notations and similar calculations as in the proof of Proposition 6.5, from the rssumption we have

$$(\lambda^2-\alpha\lambda-1)(\alpha+2)=0\,.$$

A similar argument as in the proof of Proposition 6.5 then yields our assertion. (Q.E.D.)

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