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HYPOELLIPTIC OPERATORS OF PRINCIPAL TYPE WITH INFINITE DEGENERACY

Dedicated to Professor Hiroki Tanabe on his sixtieth birthday in 1992

By

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Introduction.

Let P be a classical pseudodifferential operator of order m. We assume P is of principal type, that is, the Hamilton vector field H_p of the principal symbol p of P is not parallel to the radial direction where the principal symbol p vanishes. In this paper we study the microhypoellipticity for P, under the following $(\bar{\Psi})$ condition given by Nirenberg-Treves [15];

$$(\bar{\Psi})\qquad \begin{cases} \text{the imaginary part } p_2 \text{ of the principal symbol } p \\ \text{does not change sign from } + \text{ to } - \text{ along any oriented} \\ \text{(null-) bicharacteristic of the real part } p_1 \text{ of } p. \end{cases}$$

Let us recall that $(\bar{\Psi})$ is necessary for adjoint operator P^* of P to be locally solvable (see Hörmander [6; Theorem 26.4.7], cf. Moyer [14]). Since it follows from the hypoellipticity of P that P^* is locally solvable, it is reasonable to assume the condition $(\bar{\Psi})$.

By supplying the missing arguments of Egorov [2], Hörmander [5] (see also [6; Chapter 27]) showed that a pseudodifferential operator P of principal type is subelliptic (and hence hypoelliptic) if and only if the principal symbol p of P satisfies $(\bar{\Psi})$ and a finite type assumption ((27.1.8) in [6]). Without the finite type assumption, the problem of hypoellipticity seems to be difficult. For example, consider a first-order pseudodifferential operator of Egorov type as follows:

$$P_0=D_t+i(t^s D_{x_1}+t^k x_1^m|D|) \qquad \text{in } \boldsymbol{R}_t\times\boldsymbol{R}_x^n,\ |D|^2=D_t^2+|D_x|^2,$$

where s, k, m are nonnegative integers. For P_0, condition $(\bar{\Psi})$ and the finite type assumption are expressed as

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$$s, m \text{ even}, \quad k \text{ odd}, \quad s<k .$$

Then P_0 is subelliptic with loss of $r/(r+1)$ derivatives $(r=k+m(s+1))$ and hence hypoelliptic. If t^s, t^k, x_1^m of P_0 are replaced by functions infinitely vanishing then the hypoellipticity of P_0 is unknown. The aim of the present paper is to solve this particular problem, but we shall reply only for special cases, unfortunately, because we do not know even whether L^2 *a priori-estimate* holds for this modified P_0, in general. Actually, a remarkable counter-example given by Lerner [10] shows that we can not always expect L^2 *a priori-estimate* for operators satisfying $(\bar{\Psi})$.

To end the introduction, we state a few historical remarks: As a perfection of the preceding results of Nirenberg-Treves [15] in the analytic case or the finite type case, Beals-Fefferman [1] proved L^2 *a priori-estimate* (and hence local solvability) for pseudodifferential operators of principal type, under condition (P) (i.e. the imaginary part p_2 of the principal symbol of P does not change sign along the bicharacterisitic of the real part p_1, which is equivalent to $(\bar{\Psi})$ for differential operators.) Furthermore, Hörmander [6; Chapter 26] extented the local existence result of [1] to the semi-global one and fully studied the regularities of solutions for operators, of principal type, satisfying condition (P). Under condition $(\bar{\Psi})$, L^2 *a priori-estimate* for operators in 2-dimension space was proved by Lerner [8], whoes method also plays an important role in the present paper.

1. Main results

Let P be a classical pseudodifferential operator on $\boldsymbol{R}^{n+1}$, of order m, of principal type, which satisfies the condition $(\bar{\Psi})$. We are interested in the microhypoellipticity of P; that is, for $\rho_0 \in T^*(\boldsymbol{R}^{n+1})\backslash 0$, we shall see whether

$$(1.1) \qquad \rho_0 \notin \mathrm{WF}(Pu) \text{ implies } \rho_0 \notin \mathrm{WF}(u) \qquad \text{for } \forall u \in \mathcal{D}'(\boldsymbol{R}^{n+1}).$$

We assume $\rho_0 \in \mathrm{Char}\, P$ because (1.1) is trivial, otherwise, where Char P denotes the set of characteristic points. Let $p=p_1+ip_2$ (p_1, p_2 real-valued) be the principal symbol of P and let Γ be a subset of Char P where the Poisson bracket $\{p_1, p_2\}$ vanishes. It is known by Hömander's classical theorem [4] (and also Egorov-Hörmander Theorem [6; Theorem 27.1.11]) that (1.1) is true if $\rho_0 \notin \Gamma$, because we have a subelliptic estimate with loss of 1/2 derivatives. In what follows we consider the case where $\rho_0 \in \Gamma$. We assume that in a conic neighborhood of ρ_0

$$(1.2)\qquad \begin{cases} \Gamma \text{ is contained in a } C^\infty\text{-hypersurface in } T^*(\boldsymbol{R}^{n+1})\backslash 0 \\ \text{to which the Hamilton vector field } H_1 \text{ of } p_1 \text{ is transversal.} \end{cases}$$

After the multiplication by an elliptic factor, we may assume P is of first order. Furthermore, by homogeneous canonical transformation and Malgrange preparation theorem we may assume that $\rho_0=(0,(0,\xi_0))\in T^*(\boldsymbol{R}_t\times\boldsymbol{R}_x^n)\backslash 0$, $(|\xi_0|=1)$, and the principal symbol p of P is expressed as, in a small conic neighborhood V of ρ_0,

$$(1.3)\qquad p=p(t,x,\tau,\xi)=\tau+iq(t,x,\xi),$$

where $q(t,x,\xi)\in C^\infty(\boldsymbol{R}_t\times\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n)$ is real valued, positively homogeneous of degree one for $|\xi|\geqq 1/2$; in particular q satisfies:

$$(1.4)\qquad q(t,x,\xi)=\lambda q(t,x,\xi/\lambda), \qquad \text{if } |\xi|\geqq 1/2 \text{ and } 0<\lambda\leqq 1,$$

and

$$(1.5)\qquad |(D_t^k D_x^\alpha D_\xi^\beta q)(t,x,\xi)|\leqq C_{\alpha,\beta,k}(1+|\xi|)^{1-|\beta|}.$$

We may also assume that lower order terms p_0, p_{-1}, $\cdots$ in the symbol of P are independent of τ in a conic neighborhood V of ρ_0 (see the paragragh after [6; Theorem 26.4.7′]). Hence we can write

$$(1.3)'\qquad P=D_t+iQ(t,x,D_x) \qquad \text{in } V,$$

where the principal symbol of Q is $q(t,x,\xi)$. In that frame work, condition $(\overline{\Psi})$ is expressed as

$$(1.6)\qquad q(t,x,\xi)>0 \text{ and } s>t \text{ imply } q(s,x,\xi)\geqq 0.$$

Moreover, the set Γ is defined by

$$\{(t,x,0,\xi)\in T^*(\boldsymbol{R}^{n+1})\backslash 0\,;\,\partial_t q(t,x,\xi)=q(t,x,\xi)=0\}$$

and it follows from assumption (1.2) that

$$(1.7)\qquad \begin{cases} \text{for any } \mu>0 \text{ there exists a } \delta_\mu>0 \text{ such that} \\ \left\{(t,x,0,\xi)\,;\,\mu\leqq|t|\leqq 2\mu,\quad |x|+\left|\dfrac{\xi}{|\xi|}-\xi_0\right|<\delta_\mu\right\}\cap\Gamma=\emptyset. \end{cases}$$

because $\rho_0=(0,(0,\xi_0))\in\Gamma$.

In order to state a sufficient condition for (1.1), we define a microlocalized operator of P at ρ_0 as follows: Let $h(x)$ be a $C_0^\infty(\boldsymbol{R}^n)$ function such that $0\leqq h\leqq 1$, $h(x)=1$ for $|x|\leqq 1/5$ and $h(x)=0$ for $|x|\geqq 7/24$. For a $\delta>0$ we set $h_\delta(x)=h(x/\delta)$ and $H_\delta(x,\xi\,;\lambda)=h_\delta(x)h_\delta(\lambda\xi-\xi_0)$, where $0<\lambda\leqq 1$ is a parameter. Let δ_1 be a small positive such that the projection of V into $\boldsymbol{R}_t\times\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n$ contains $\{|t|\leqq 2\delta_1\}\times\operatorname{supp} h_{2\delta_1}(x)h_{2\delta_1}(\lambda\xi-\xi_0)$. For a parameter $0<\lambda\leqq 1$, we set

$$P_\lambda = D_t + ih_{\delta_1}(x)Q(t, x, D_x)h_{\delta_1}(\lambda D_x - \xi_0) \equiv D_t + iQ_\lambda(t, x, D_x).$$

THEOREM 1. *Let Γ be the above set in* Char P *and assume* (1.2). *Let* $\rho_0 = (0, (0, \xi_0)) \in \Gamma$ *and let P be a pseudodifferential operator of the form* (1.3)′ *in a conic neighborhood V of ρ_0. Let δ be a small positive such that* $100\delta < \delta_1$ *for the above δ_1. Assume that for each δ there exist non-negative symbols* $\varphi(x, \xi; \lambda) \in S^0_{1,0}$ *and* $\alpha(t, x, \xi) \in C^\infty(\boldsymbol{R}_t; S^0_{1,0})$ *such that* $\{\varphi(x, \xi; \lambda); 0 < \lambda \leqq 1\}$ *is a bounded set of* $S^0_{1,0}$ *and we have*

$$(1.8) \qquad \begin{cases} \varphi \geqq 1 & \text{outside of } \operatorname{supp} H_{5\delta}(x, \xi; \lambda) \\ \varphi = 0 & \text{on } \operatorname{supp} H_\delta(x, \xi; \lambda), \end{cases}$$

$$(1.9) \qquad |(H_q\varphi)(t, x, \xi; \lambda)| \leqq \alpha(t, x, \xi) \quad \text{on } \{|t| \leqq \delta_1\} \times \operatorname{supp} H_{100\delta}(x, \xi; \lambda)$$

and the following estimate: For any $\varepsilon > 0$ *there exists a* $C_\varepsilon > 0$ *independent of* $0 < \lambda \leqq 1$ *such that*

$$(1.10) \qquad \begin{aligned} &\|u\|^2 + (\log \lambda)^2 \|\alpha(t, x, D_x)u\|^2 \\ &\leqq \varepsilon \|P_\lambda u\|^2 + C_\varepsilon(\lambda \|u\|^2 + \lambda^{-2} \|(1 - H_{20\delta}(x, D_x; \lambda))u\|^2) \end{aligned}$$

if $u \in C_0^\infty([-\delta_1, \delta_1]; \mathcal{S}(\boldsymbol{R}^n_x))$. *Then we have* (1.1).

COROLLARY. *The same conclusion of Theorem* 1 *follows if we replace* (1.9) *and* (1.10), *respectively, by*

$$(1.9)' \qquad |(H_q\varphi)(t, x, \xi; \lambda)|^2 \leqq \alpha(t, x, \xi) \quad \text{on } \{|t| \leqq \delta_1\} \times \operatorname{supp} H_{100\delta}(x, \xi; \lambda)$$

and

$$(1.10)' \qquad \begin{aligned} &\|u\|^2 + (\log \lambda)^2 \operatorname{Re}(\alpha(t, x, D_x)u, u) \\ &\leqq \varepsilon \|P_\lambda u\|^2 + C_\varepsilon(\lambda \|u\|^2 + \lambda^{-2} \|(1 - H_{20\delta}(x, D_x; \lambda))u\|^2). \end{aligned}$$

REMARK 1. The function $h(x)$ in Theorem 1 and Corollary is not necessary to be homogeneous spatially. For example, we can replace it by $h(x_1/\nu)h(x')$ for any $\nu > 0$ and for $h(x_1)$, $h(x')$ similar as $h(x)$.

REMARK 2. As criteria of hypoellipticity, logarithmic regularity up estimates were used in Morimoto [11-13]. The simple proof of Theorem 1 in the present paper is inspired by Kajitani-Wakabayashi [7; Theorem 1.2] (see also [16]) and Hörmander [6; Lemma 26.9.3].

As an application of Theorem 1, we consider a pseudodifferential operator of principal type which has the following form:

$$(1.11) \qquad P = D_t + ia(t, x, D_x)(D_{x_1} + f(t)|D_x|),$$

in a conic neighborhood V of $\rho_0=(0, (0, \xi_0))$, where $a(t, x, \xi)\in C^\infty(\boldsymbol{R}_t ; S^0_{1,0})$, $f(t) \in C^\infty$ satisfy

$$(1.12) \qquad a(t, x, \xi)\geqq 0, \quad f'(t)>0 \ (t\neq 0), \quad f'(0)=0.$$

It follows from (1.12) that P satisfies $(\overline{\Psi})$.

THEOREM 2. *Let* ρ_0 *be* $(0, (0, \xi_0))\in T^*(\boldsymbol{R}^{n+1})$. *Let* P *be of the form* (1.11) *in a conic neigborhood of* V *of* ρ_0 *and satisfy* (1.12). *Then we have* (1.1) *if the following conditions are fulfilled*;

$$(1.13) \qquad \exists\alpha(t),\ \beta(t)\in C^\infty ;\ \beta(t)>0 \ (t\neq 0),\ t\alpha'(t)\geqq 0,$$

$$(1.14) \qquad \beta(t)\leqq a(t, x, \xi)\leqq\alpha(t) \quad \textit{in } V,$$

$$(1.15) \qquad |\nabla_x a(t, x, \xi)|+|\nabla_\xi a(t, x, \xi)|\xi||\leqq a(t) \quad \textit{in } V,$$

$$(1.16) \qquad \lim_{t\to 0} t\alpha(t) \log f'(t)=0 \quad \textit{and}$$

$$(1.17) \qquad \lim_{t\to 0} t\alpha(t) \log \beta(t)=0 .$$

It follows from (1.13) and (1.14) that $a(t, x, \xi)>0$ $(t\neq 0)$ and hence we see, together with (1.12),

$$\Gamma\subset\{\tau=t=0\}$$

Consequently we have (1.2) (and hence (1.7)). The operator of the form (1.11) is infinitely degenerate model corresponding to the case of $m=0$ in the operator P_0 of Egorov type stated in the introduction. We do not know the microhypoellipticity for a simple operator with $f(t)$ in (1.11) replaced by $f(t)x_1^2$, because of the difficulty in deriving L^2 *a priori estimate*.†

However, if $a(t, x, \xi)$ in (1.12) does not vanish we can treat infinitely degenerate model of P_0 a little more generally. This case is geometrically stated as

(1.18) H_1, H_2 and the radial direction are linearly independent in $V\cap\mathrm{Char}\, P$,

which is invariant condition under the multiplication of elliptic factors. If (1.18) is valid then it follows from condition $(\overline{\Psi})$ that we have the maximal hypoelliptic estimate, in a sense of Helffer-Nourrigat [3], as follows;

$$\|D_t u\|^2+\|Q_\lambda(t, x, D_x)u\|^2\leqq C(\|P_\lambda u\|^2+\|u\|^2) \quad \text{(cf., (4.9))}.$$

By means of this estimate, the problem of hypoellipticity for D_t+iQ can be reduced to the similar one for second operator $D_t^2+Q^2$ as in [13]. From now

† Some special cases will be studied in the forthcoming paper [17].

on we shall consider the case corresponding to [13; Theorem 4]. Let Γ be a C^∞ submanifold of codimension 2 in Char P and symplectic, that is,

$$T\Gamma \cap T\Gamma^{\perp}=0 \text{ at every point of } \Gamma. \tag{1.19}$$

It follows from (1.19) that both H_1 and H_2 are transversal to Γ because $H_1, H_2 \in T(\text{Char } P)^{\perp} \subset T\Gamma^{\perp}$. Hence (1.2) holds and so (1.7). In order to state the additional condition we have to fix a special coordinate. By a symplectic linear transformation, it follows from (1.18) that $q(t, x, \xi)$ of (1.3) satisfies $\partial_{\xi_1} q(t, x, \xi) \neq 0$. It follows from the implicit function theorem that there exist $a(t, x, \xi) \in C^\infty(\boldsymbol{R}_t; S^0_{1,0})$ and $b(t, x, \xi') \in C(\boldsymbol{R}_t; S^1_{1,0})$ such that

$$q(t, x, \xi)=a(t, x, \xi)(\xi_1+b(t, x, \xi')), \quad a(t, x, \xi)\neq 0, \quad \text{in } V, \tag{1.20}$$

where $\xi'=(\xi_2, \cdots, \xi_n)$. Let $\gamma_\rho(t)$ be the bicharacteristic of $p_1=\tau$ through $\rho=(0, x, 0, \xi_1, \xi') \in V \cap \text{Char } P$, that is, $\gamma_\rho(t)=\{(t, x, 0, \xi_1, \xi')\}$. By setting $\xi_1=-b(t, x, \xi')$ we define a projection $\pi\gamma_\rho(t)$ into Char P of the bicharacteristic. We assume that

$$\left\{\begin{array}{l}\text{there exist a } \delta_0>0 \text{ and a } 0\neq e(t, x, \xi)\in C^\infty(\boldsymbol{R}_t; S^0_{1,0}) \\ \text{such that for any } \rho=(0, x, 0, \xi)\in V\cap \text{Char } P,\ F_\rho(t)\equiv \\ (e\partial_t q)|_{\text{Char } P}(\pi\gamma_\rho(t)) \text{ has a unique extremum at } t=s(\rho) \\ \text{in } (-\delta_0, \delta_0), \text{ and } s(\rho) \text{ belongs to } C^\infty \text{ with respect to } \rho.\end{array}\right. \tag{1.21}$$

THEOREM 3. *Let ρ_0 be $(0, (0, \xi_0))\in\Gamma$ and let P be the form* (1.3)$'$ *and satisfy* (1.18) *in a conic neighborhood V of ρ_0. Assume that Γ is a C^∞-symplectic submanifold and of codimension* 2 *in* Char P. *Then we have* (1.1) *if the condition* (1.21) *holds with $q(t, x, \xi)$ expressed as* (1.20).

As a typical example of Theorem 3 we have the following:

$$p(t, x, \tau, \xi)=\tau+i\left\{\xi_1+\int_0^t \exp-(s^2+x_1^2)^{-\delta/2}ds|\xi|\right\}, \quad \delta>0.$$

2. Proof of Theorem 1

Let $\chi(t)$ be a $C_0^\infty(\boldsymbol{R}_t)$ function such that $0\leqq\chi(t)\leqq 1$, $\chi(t)=1$ for $|t|\leqq 1$, $\chi(t)=0$ for $|t|\geqq 2$. Set $\Phi(\tau, \xi; \mu)=\chi(|\tau|/\mu|\xi|)(1-\chi)(|\xi|)$ for a small $\mu>0$. For cutting $\boldsymbol{R}^n_\xi$ we define the following:

DEFINITION 1. For $\delta>0$ and $\xi_0\in\boldsymbol{R}^n$ ($|\xi_0|=1$) we say that a function $\phi(\xi)\in C^\infty(\boldsymbol{R}^n)$ belongs to Ψ_{δ,ξ_0} if $0\leqq\phi\leqq 1$ satisfies

$$\begin{cases} \psi(\xi)=1 & \text{for } |\xi/|\xi|-\xi_0|\leqq\delta/12 \text{ and } |\xi|\geqq 2/3, \\ \psi(\xi)=0 & \text{for } |\xi/|\xi|-\xi_0|\geqq\delta/10 \text{ or } |\xi|\leqq 1/2, \\ \psi(\xi)=\psi(\xi/\lambda) & \text{for } 0<\lambda\leqq 1 \text{ and } |\xi|\geqq 1. \end{cases}$$

In the proof of the theorem we may assume $u\in\mathcal{E}'$ and hence u belongs to H_{-N} for an integer $N>0$. Suppose that $\rho_0\notin\mathrm{WF}(Pu)$. Then for a sufficiently small $\mu>0$ we have

$$\chi(t/2\mu)\Phi(D_t, D_x; 2\mu)\psi_\mu(D_x)h_\mu(x)Pu\in H_s$$

for any real s, where $\psi_\mu(\xi)\in\Psi_{\mu,\xi_0}$. If we set $v=\chi(t/\mu)\Phi(D_t, D_x; \mu)u$ then it follows from (1.7) that $\psi_{\delta_\mu}(D_x)h_{\delta_\mu}(x)Pv\in H_s$ for a $\psi_{\delta_\mu}(\xi)\in\Psi_{\delta_\mu,\xi_0}$ because P is microhypoelliptic on the intersection of $\mathrm{supp}\, h_{\delta_\mu}(x)\psi_{\delta_\mu}(\xi)$ and the support of derivatives of $\chi(t/\mu)\Phi(\tau, \xi; \mu)$. Fix a positive δ such that $100\delta<\min(\delta_\mu, \delta_1)$. We shall show $\psi_\delta(D_x)h_\delta(x)v\in H_s$, which will yield (1.1).

For the above δ we take $\varphi(x, \xi; \lambda)$ in the assumption of the theorem. For an integer $l>s+N+1$ we denote a pseudodifferential operator with a symbol $\lambda^{l\varphi(x,\xi;\lambda)}$ by $K(x, D_x; \lambda)$. If λ varies $0<\lambda\leqq 1$ then $K(x, D_x; \lambda)H_{10\delta}(x, D_x; \lambda)$ belongs to a bounded set of S^0_{1,ε_1} for any small $\varepsilon_1>0$. For any real a, $[K(x, D_x; \lambda), H_{10\delta}(x, D_x; \lambda)]\lambda^{-a}$ belongs to a bounded set of $S^{-(l-a-\varepsilon_1)}_{1,0}$ because of (1.8). Furthermore, $h_{10\delta}(x)$, $h_{10\delta}(\lambda D_x-\xi_0)$ and $K(x, D_x; \lambda)$ are commutative, each other, as a product of three factors, neglecting term in $\lambda^a\times S^{-(l-a-\varepsilon_1)}_{1,0}$.

Let $w\in\mathcal{S}$ satisfy

$$\mathrm{supp}\, w\subset\{|t|\leqq 2\mu\} \tag{2.1}$$

and substitute $K(x, D_x; \lambda)H_{10\delta}(x, D_x; \lambda)w$ into (1.10) in place of u. Then

$$\begin{aligned} &\|KH_{10\delta}w\|^2+(\log\lambda)^2\|\alpha(t, x, D_x)KH_{10\delta}w\|^2 \\ &\leqq 2\varepsilon\{\|h_{10\delta}(\lambda D_x-\xi_0)h_{10\delta}(x)KPw\|^2+\|H_{10\delta}[Q(t, x, D_x), K]w\|^2\} \\ &\quad +C_\varepsilon(\lambda\|KH_{10\delta}w\|^2+\lambda^{2s+1}\|w\|^2_{(0,-N)}) \end{aligned} \tag{2.2}$$

because the same commutative argument as above follows for $H_{10\delta}$ and KQ by means of (1.8). Here for real a we have set $\|w\|_{(0,a)}=\|(1+\Lambda)^a w\|$, $\Lambda^2=1+|D_x|^2$ and by this norm we define the space $H_{(0,a)}$. Note that the principal symbol of $[Q, K]$ is equal to

$$-il(\log\lambda)(H_q\varphi)\lambda^{l\varphi(x,\xi;\lambda)}$$

and symbols of lower orders are a sum of $\lambda^{1/2+l\varphi(x,\xi;\lambda)}$ multiplied by symbols in a bounded set of $S^0_{1,0}$ uniformly with respect to $0<\lambda\leqq 1$. It follows from (1.9) that

$$\|H_{10\delta}[Q(t, x, D_x), K]w\|^2 \leqq l^2(\log\lambda)^2\|\alpha(t, x, D_x)KH_{10\delta}w\|^2 + C_l(\lambda\|KH_{10\delta}w\|^2+\lambda^{2s+1}\|w\|^2_{(0,-N)}). \tag{2.3}$$

Choose $2\varepsilon l^2<1$, then for a constant C'_l we have

$$(1-2\varepsilon\lambda C_l)\|KH_{10\delta}w\|^2$$
$$\leqq 2\varepsilon\|h_{10\delta}(\lambda D_x-\xi_0)h_{10\delta}(x)KPw\|^2+C'_l\lambda^{2s+1}\|w\|^2_{(0,-N)}.$$

It follows from (1.8) that $\|h_\delta(\lambda D_x-\xi_0)h_\delta(x)w\|^2 \leqq \|KH_{10\delta}w\|^2+\tilde{C}_l\lambda^{2s+1}\|w\|^2_{(0,-N)}$. Take a λ_0 satisfying $\lambda_0(2\varepsilon C_l+C_\varepsilon)<1/4$. Then for $0<\lambda\leqq\lambda_0$ we have

$$\|h_\delta(\lambda D_x-\xi_0)h_\delta(x)w\|^2$$
$$\leqq 4\varepsilon\{\|h_{10\delta}(\lambda D_x-\xi_0)h_{10\delta}(x)Pw\|^2+C''_l\lambda^{2s+1}\|w\|^2_{(0,-N)}).$$

Multiplying $\lambda^{-2s}(1+\kappa\lambda^{-1})^{-2(l+1)}$ with a parameter $\kappa>0$ by both sides, for $0<\lambda\leqq\lambda_0$ we have

$$\|h_\delta(\lambda D_x-\xi_0)(1+\kappa\Lambda)^{-(l+1)}h_\delta(x)w\|^2_s$$
$$\leqq 4\varepsilon(\|h_{10\delta}(\lambda D_x-\xi_0)(1+\kappa\Lambda)^{-(l+1)}h_{10}(x)Pw\|^2_s+C''_l\lambda\|w\|^2_{(0,-N)})$$

because λ^{-1} is equivalent to $|\xi|$ on supp $h(\lambda\xi-\xi_0)$. Integrate λ from 0 to λ_0 after dividing both sides by λ. Then by means of [12; Proposition 1.7] we have for suitable $\phi_\delta(\xi)\in\Psi_{\delta,\xi_0}$ and $\tilde{\phi}_\delta(\xi)\in\Psi_{70\delta,\xi_0}$,

$$\|(1+\kappa\Lambda)^{-(l+1)}\phi_\delta(D_x)h_\delta(x)w\|^2_{(0,s)}$$
$$\leqq C(\|(1+\kappa\Lambda)^{-(l+1)}\tilde{\phi}_\delta(D_x)h_{10\delta}(x)Pw\|^2_{(0,s)}+\|w\|^2_{(0,-N)}).$$

It follows from $u\in H_{-N}$ that one can find a sequence $\{u_j\}$ in $\mathcal{S}$ satisfying $u_j\to u\in H_{-N}$. If $w_j=\chi(t/\mu)\Phi(D_t, D_x;\mu)u_j$ then $w_j\to v$ in $H_{(0,-N)}$ and $Pw_j\to Pv$ in $H_{(0,-(N+1))}$. Letting $j\to\infty$ in the above estimate with $w=w_j$, in view of $Pv\in H_s$ we get for $\kappa>0$

$$\|(1+\kappa\Lambda)^{-(l+1)}\phi_\delta(D_x)h_\delta(x)v\|^2_{(0,s)}\leqq C(\|\tilde{\phi}_\delta(D_x)h_{10\delta}(x)Pv\|^2_s+\|u\|^2_{-N}).$$

Making $\kappa\to0$ we see $\phi_\delta(D_x)h_\delta(x)v\in H_s$ because $v=\chi(t/\mu)\Phi(D_t, D_x;\mu)u$. Thus we have proved that $Pu\in H_s$ at ρ_0 implies $u\in H_s$ at ρ_0.

The proof of Corollary is obvious if we replace the term $\|\alpha(t, x, D_x)KH_{10\delta}w\|^2$ in (2.2) and (2.3) by Re$(\alpha(t, x, D_x)KH_{10\delta}w, KH_{10\delta}w)$.

3. Proof of Theorem 2

If $\xi_0\notin\Sigma=\{\xi_1+f(0)|\xi|=0\}$ then the theorem is obvious because $q(t, x, \xi)=a(\xi_1+f(t)|\xi|)$ is semi-definite in a small conic neighborhood of $\rho_0=(0, (0, \xi_0))$ and we can apply the result about the propagation of regularities (Hörmander

[6; Proposition 26.6.1]). In what follows we assume $\xi_0 \in \Sigma$ (though we will not use this condition). We apply Theorem 1 by setting

$$\varphi(x, \xi) = (1 - h_{5\delta}(x)) + (1 - h_{5\delta}(\lambda\xi - \xi_0)).$$

Then we have (1.8) and it follows from (1.14) and (1.15) that (1.9) holds with $\alpha(t, x, \xi) = C\alpha(t)$ for a suitable $C > 0$ if δ is small enough. The proof of Theorem 2 would be completed if we could show (1.10). Set

$$a_\lambda(t, x, \xi) = h_{\delta_1}(x) a(t, x, D_x) h_{2\delta_1}(\lambda D_x - \xi_0).$$

Let $A(t) = a_\lambda^w(t, x, D_x)$ denote a pseudodifferential operator with Weyl symbol $a_\lambda(t, x, \xi)$. Setting $B(t) = (D_{x_1} + f(t)|D_x|) h_{\delta_1}(\lambda D_x - \xi_0)$ moreover, we consider $A(t)$, $B(t)$ as a real operator on Hilbert space $\mathcal{H} = L^2(\boldsymbol{R}_x^n)$. Note that for a fixed $\lambda > 0$ $B(t)$ is bounded operator in $\mathcal{H}$. If $\Omega_+(t) = \{\xi ; \xi_1 + f(t)|\xi| > 0\}$ and if

$$S_+(t)v(x) = \frac{1}{(2\pi)^n} \int e^{ix\xi} 1_{\Omega_+(t)}(\xi) \hat{v}(\xi) d\xi, \quad v \in \mathcal{H}, \quad S_-(t) = Id - S_+(t)$$

then we can define the sign $M(t)$ of $B(t)$, $M(t) = S_+(t) - S_-(t)$ and it follows from (1.12) that

$$(3.1) \qquad (M(t_1) - M(t_2))(t_1 - t_2) \geqq 0 \quad \text{on } \mathcal{H}.$$

From this condition we have the following lemma given by Lerner [8; § 2]:

LEMMA (Lerner [8, 9]). *There exists a* $\delta' > 0$ *independent of* $0 < \lambda \leqq 1$ *such that for any* $u(t) \in C_0^1(\boldsymbol{R}_t ; \mathcal{H})$ *we have*

$$(3.2) \qquad 2\int |P_\lambda u(t)|_{\mathcal{H}} dt \geqq \sup |u(t)|_{\mathcal{H}} \quad \textit{if } \operatorname{supp} u \subset \{|t| \leqq \delta'\},$$

where $|\cdot|_{\mathcal{H}} = \|\cdot\|_{L^2(\boldsymbol{R}_x^n)}$.

PROOF. By means of [8; Lemma 2.3.1], it follows from (3.1) that

$$(3.3) \qquad \operatorname{Re} \int (\dot{u}(t), M(t)u(t))_{\mathcal{H}} dt \leqq 0, \quad \dot{u}(t) = \frac{du}{dt}(t).$$

If $H(t)$ denotes Heaviside function then for any T we have

$$-\operatorname{Re} \int (\dot{u}(t), \{H(t-T)S_+(t) - H(T-t)S_-(t)\} u(t))_{\mathcal{H}} dt$$

$$(3.4) \qquad = -\operatorname{Re} \int (\dot{u}(t), H(t-T)(M+S_-)u(t) + H(T-t)(M-S_+)u(t))_{\mathcal{H}} dt$$

$$\geqq -\operatorname{Re} \int (\dot{u}(t), \{H(t-T)S_-(t) - H(T-t)S_+(t)\} u(t))_{\mathcal{H}} dt,$$

where we have used (3.3) in the last inequality. Adding the left hand side of

(3.4) to both sides of (3.4), we have in view of $S_{+}+S_{-}=Id$

$$-2\mathrm{Re}\int(\dot{u}(t),\ \{H(t-T)S_{+}(t)-H(T-t)S_{-}(t)\}u(t))_{\mathscr{H}}dt$$

$$\geqq -\mathrm{Re}\int(\dot{u}(t),\ \{H(t-T)-H(T-t)\}u(t))_{\mathscr{H}}dt \tag{3.5}$$

$$=2|u(T)|^{2}_{\mathscr{H}}.$$

It follows from [8; Lemma 2.3.2] that

$$\mathrm{Re}\,(\pm S_{\pm}\,\mathrm{Re}\,(AB))\geqq-\frac{10}{3}\|A\|^{1/4}\|[A,\ B]\|^{1/2}\|[B,\ [B,\ A]]\|^{1/4}, \tag{3.6}$$

where $\|A\|$ denotes the operator norm of $A(t)$ in $\mathscr{H}$. Note that the right hand side of (3.6) has the bound independent of λ. Since the difference between P_{λ} and $D_{t}+i\,\mathrm{Re}\,(A(t)B(t))$ is bounded in $\mathscr{H}$ uniformly with respect to $0<\lambda\leqq 1$, in view of (3.5) and (3.6) there exists a $C>0$ independent of λ such that

$$\mathrm{Re}\int(P_{\lambda}u(t),\ i\{H(t-T)S_{+}(t)-H(T-t)S_{-}(t)\}u(t))_{\mathscr{H}}dt$$

$$\geqq|u(T)|^{2}_{\mathscr{H}}-C\int|u(t)|^{2}_{\mathscr{H}}dt\,.$$

If $\mathrm{supp}\,u\subset\{|t|\leqq\delta'\}$ then the second term of the right hand side is estimated above from $2C\delta'\sup|u(t)|^{2}_{\mathscr{H}}$, so that we have (3.2) for a small $\delta'>0$ satisfying $4C\delta'\leqq 1$.

By means of the Schwartz inequality it follows from (3.2) that

$$\|P_{\lambda}u\|\geqq(2\delta')^{-1}\|u\| \quad \text{if } \mathrm{supp}\,u\subset\{|t|\leqq\delta'\}. \tag{3.7}$$

It follows from (1.16) and (1.17) that for any $\varepsilon>0$ there exists a $\delta_{\varepsilon}>0$ such that

$$t\alpha(t)\{|\log f'(t)|+|\log\beta(t)|\}\leqq\varepsilon^{2} \quad \text{if } |t|\leqq\delta_{\varepsilon}\,. \tag{3.8}$$

For the sake of simplicity we assume $\alpha(t)$ is even function (the general case would be clear once we could prove this case), It follows from the monotoneness of $\alpha(t)$ that for a small parameter $\lambda>0$ there exists a unique $t_{\lambda}>0$ such that $t_{\lambda}\alpha(t_{\lambda})|\log\lambda|=2\varepsilon$. Similarly we choose $s_{\lambda}>0$ such that $s_{\lambda}\alpha(s_{\lambda})|\log\lambda|=\varepsilon$. For a while we assume λ is sufficently small such that $s_{\lambda}<\delta_{\varepsilon}$. If we set $\delta'=t_{\lambda}$ in (3.7) then

$$\begin{aligned}4\varepsilon\|P_{\lambda}u\|&\geqq\|\alpha(t_{\lambda})(\log\lambda)u\|\\ &\geqq\|\alpha(t)(\log\lambda)u\| \qquad \text{if } \mathrm{supp}\,u\subset\{|t|\leqq t_{\lambda}\}.\end{aligned} \tag{3.9}$$

If $s_{\lambda}\leqq|t|\leqq\delta_{\varepsilon}$ then it follows from (3.8) that

$$\frac{\varepsilon}{|\log \lambda|}\{|\log f'(t)|+|\log \beta(t)|\}\leqq \varepsilon^2,$$

so that if $0<\lambda\leqq\lambda_\varepsilon$ for a sufficiently small λ_ε then

$$f'(t),\ \beta(t)\geqq\lambda^\varepsilon \quad \text{on } s_\lambda<|t|\leqq\delta_1\,. \tag{3.10}$$

In fact, if λ_ε is small enough we have $f'(t)$, $\beta(t)\geqq(\lambda_\varepsilon)^\varepsilon$ for $\delta_\varepsilon<|t|\leqq\delta_1$ in view of (1.12) and (1.13). Note that

$$\begin{aligned}\|P_\lambda u\|^2=&\|D_t u\|^2+\|a_\lambda(t,\ x,\ D_x)Bu\|^2\\ &+2\,\mathrm{Re}\,((\partial_t a_\lambda)Bu,\ u)\\ &+2\,\mathrm{Re}\,(a_\lambda f'(t)|D_x|h_{\delta_1}(\lambda D_x-\xi_0)u,\ u)\end{aligned} \tag{3.11}$$

Since it follows from (3.10) and (1.14) that

$$a_\lambda(t,\ x,\ \xi)\geqq\lambda^\varepsilon \quad \text{on } \{s_\lambda\leqq|t|\leqq\delta_1\}\times\mathrm{supp}\,H_{\delta_1}(x,\ \xi\,;\ \lambda)$$

the second term of the right hand side of (3.11) is estimated above from

$$C(\lambda^{-\varepsilon}\|a_\lambda Bu\|\|u\|+\|u\|^2)\leqq\|a_\lambda Bu\|^2+C'\lambda^{-2\varepsilon}\|u\|^2\,.$$

By means of (3.10) again we have, if $\mathrm{supp}\,u\subset\{s_\lambda\leqq|t|\leqq\delta_1\}$,

$$2\,\mathrm{Re}\,(a_\lambda f'(t)|D_x|h_{\delta_1}(\lambda D_x-\xi_0)u,\ u)\geqq\lambda^{2\varepsilon-1}\|H_{20\delta}u\|^2-C\|u\|^2.$$

Therefore, if $\mathrm{supp}\,u\subset\{s_\lambda\leqq|t|\leqq\delta_1\}$ then

$$\|P_\lambda u\|^2\geqq\lambda^{2\varepsilon-1}\|H_{20\delta}u\|^2-C\lambda^{-2\varepsilon}\|u\|^2,$$

provided that $0<\lambda\leqq\lambda_\varepsilon$. If $\varepsilon<1/16$ and if $0<\lambda\leqq\min(\lambda_\varepsilon,\ \varepsilon^2)=\lambda'_\varepsilon$ we have

$$\begin{aligned}\varepsilon\|P_\lambda u\|^2\geqq\lambda^{2\varepsilon-1/2}\|u\|^2-C\lambda^{-2}\|(1-H_{20\delta})u\|^2&\\ \text{if } \mathrm{supp}\,u\subset\{s_\lambda\leqq|t|\leqq\delta_1\}\,.&\end{aligned} \tag{3.12}$$

Let $\chi_0(t)$ be $C^\infty(\boldsymbol{R})$ such that $\chi_0(t)=1$ for $t\leqq0$ and $\chi_0(t)=0$ for $t\geqq1$. Set $\phi_\pm(t)=\chi_0(\pm(t\pm s_\lambda)/(s_\lambda-t_\lambda))$ and $\phi(t)=\phi_+(t)\phi_-(t)$. The fact that $t_\lambda-s_\lambda\geqq c\varepsilon/|\log\lambda|$ for a suitable $c>0$ shows $|\phi^{(j)}(t)|\leqq C_\varepsilon|\log\lambda|^j$ $(j=1,\ 2,\ \cdots,\ )$. It follows from (3.12) that

$$\begin{aligned}&\|[P_\lambda,\ \phi]u\|^2=\|\phi' u\|^2\\ &\leqq C\lambda^{1/2-2\varepsilon}\{\|[P_\lambda,\ \phi']u\|^2+|\log\lambda|^2(\|P_\lambda u\|^2+\lambda^{-2}\|(1-H_{20\delta})u\|^2)\}\,.\end{aligned} \tag{3.13}$$

Since similar estimates hold with ϕ replaced by $\phi^{(j)}|\log\lambda|^{-j}$, $j=1,\ 2,\ \cdots$, in view of $u=\phi(t)u+(1-\phi(t))u$, it follows from (3.9) and (3.12) that

$$16\varepsilon\|P_\lambda u\|^2\geqq\|\alpha(t)(\log\lambda)u\|^2-C\lambda^{-2}\|(1-H_{20\delta})u\|^2. \tag{3.14}$$

if $0<\lambda\leqq\lambda'_\varepsilon$. From (3.14), (3.7) and (3.12) we have the desired estimate (1.10) because it is trivial for $\lambda'_\varepsilon<\lambda\leqq 1$ by taking a sufficiently large C_ε in the right hand side.

4. Proof of Theorem 3

Since $\rho_0=(0, (0, \xi_0))\in\Gamma$ it follows from (1.20) that $\xi_{01}+b(0, 0, \xi'_0)=0$. By taking the canonical transformation such that $\xi_1+b(0, 0, \xi')\rightarrow\xi_1$ and $\xi'\rightarrow\xi'$ we may assume that $\xi_0=(0, \xi'_0)$, $|\xi'_0|=1$. Because Γ is of codimension 2 in Char P it follows from (1.20) and (1.6) that $\partial_t b(t, x, \xi')$ has the definite sign. Note that

$$(4.1)\qquad (a^{-1}\partial_t q)|_{\mathrm{Char}\,P}(\pi\gamma_\rho(t))=\partial_t b(t, x, \xi').$$

For each $\rho=(0, x, (0, -b(t, x, \xi'), \xi'))\in\mathrm{Char}\,P\cap V$, let $t(x, \xi')$ denote the extremal point in the condition (1.21). Since it follows from (4.1) that $F_\rho(t)$ in (1.21) equals $(\tilde{e}\partial_t b)(t, x, \xi')$ for some $\tilde{e}(t, x, \xi')\in C^\infty(\boldsymbol{R}_t\times\boldsymbol{R}_{x_1}; S^0_{1,0})$, we have in a conic neighborhood of ρ_0

$$(4.2)\qquad \begin{aligned}&|(\tilde{e}\partial_t b)(t(x, \xi'), x, \xi')|<|(\tilde{e}\partial_t b)(s, x, \xi')|\leqq|(\tilde{e}\partial_t b)(t, x, \xi')|\\ &\qquad\qquad\text{if } 0<|s-t(x, \xi')|<|t-t(x, \xi')|.\end{aligned}$$

Set $\tilde{b}(t, x, \xi')=\int_{t(x,\xi')}^t \partial_t b(s, x, \xi')ds$ and take the canonical transformation in $T^*(\boldsymbol{R}^n_x)$, keeping x_1 variable, such that

$$\xi_1+b(t(x, \xi'), x, \xi')\longrightarrow\xi_1 \qquad (\text{and } (0, \xi'_0)\rightarrow(0, \xi'_0)).$$

Then $\xi_1+b(t, x, \xi')$ is transformed to $\xi_1+b_0(t, x, \xi')$ of the form:

(4.3) $\quad b_0(t, x, \xi')=\tilde{b}(t, x_1, \Phi(x, \xi'), \Psi(x, \xi'))$ in a small conic neiborhood of ρ_0,

where $\Phi(x, \xi')\in S^0_{10}$, $\Psi(x, \xi')\in S^1_{1,0}$. It follows from (4.2) that

$$(4.4)\qquad |\nabla_x b_0(t, x, \xi')|+|\nabla_{\xi'} b_0(t, x, \xi')|\,|\xi|\leqq C|\partial_t b_0(t, x, \xi')|.$$

In fact, for example, the direct calculation gives

$$(4.5)\qquad \begin{aligned}|\partial_{x_2} b_0(t, x, \xi')|&\leqq C_1|\partial_t b(t(x_1, x', \xi'), x', \xi')|_{(x',\xi')=(\Phi(x,\xi'),\Psi(x,\xi'))}|\\ &\quad+C_2\left|\int_{t(x,\xi')}^t|\partial_t\partial_{x_2} b(s, x_1, x', \xi')|_{(x',\xi')=(\Phi(x,\xi'),\Psi(x,\xi'))}|ds\right|.\end{aligned}$$

By means of (4,2), the first term of the right hand side is estimated above from $C|\partial_t b_0(t, x, \xi')|$. Because $\partial_t b$ is semi-definite we have $|\partial_t\partial_{x_2} b|\leqq C|\partial_t b|^{1/2}$ and the second term is estimated above from

$$C\left|\int_{t(x,\xi')}^t|(\tilde{e}\partial_t b)(s, x, \xi')|^{1/2}ds\right|\leqq C'|\partial_t b(t, x, \xi')|^{1/2}$$

with $(x', \xi')=(\Phi(x, \xi'), \Psi(x, \xi'))$. Here we have used (4.2) in the last inequality.

As stated in the section 1, it follows from (1.19) that Hamilton vector fields $H_1=\partial_t$ and $H_2=H_q$ are transversal to Γ. In view of (4.4), the fact that $\partial_t b_0(0, 0, \xi_0')=0$ shows that

$$(4.6)\quad \begin{cases} \text{for any small } \mu>0 \text{ there exists a } \delta_\mu>0 \text{ such that} \\ \left\{(t, x, 0, \xi);\ \mu\leqq \max(|t|, |x_1|)\leqq 2\mu,\ |x'|+\left|\frac{\xi}{|\xi|}-\xi_0\right|<\delta_\mu\right\}\cap\Gamma=\emptyset. \end{cases}$$

In the new variable we shall apply Corollary of Theorem 1, together with Remark 1. For the brevity we write b instead of b_0 in what follows. Set $\varphi(x,\xi)=(1-\chi(x_1/\mu))+(1-h_{2\delta}(x'))+(1-h_{5\delta}(\lambda\xi-\xi_0))$. Choosing $\nu=2\mu/\delta$ in Remark 1 of Corollary we have (1.8). Since $H_q\varphi=a(\partial_{x_1}\varphi+H_b\varphi)+(H_a\varphi)(\xi_1+b)$, in view of $a\neq 0$ it follows from (4.4) and (1.20) that

$$(4.7)\quad \begin{aligned} &|H_q\varphi|^2\leqq C((\chi'(x_1/\mu))^2+a\partial_t b/|\xi|+(q/|\xi|)^2) \\ &\qquad \text{on } \{|t|\leqq\delta_1\}\times\operatorname{supp} H_{100\delta}(x, \xi; \lambda) \end{aligned}$$

because the second term of the right hand side is non-negative by means of (4.1) and (1.6). Putting $\alpha(t, x, \xi)$ equal to the right hand side of (4.7), we shall check (1.10)′. It follows from (4.6) that

$$(4.8)\quad \lambda^{-1}\|\chi'(x_1/\mu)H_{20\delta}u\|^2\leqq C(\|P_\lambda u\|^2+\|u\|^2).$$

Setting $Q_\lambda(t, x, \xi)=Q(t, x, \xi)H_{\delta_1}(x, \xi; \lambda)$ we have

$$\|P_\lambda u\|^2=\|D_t u\|^2+\|Q_\lambda(t, x, D_x)u\|^2+2\operatorname{Re}(Op(\partial_t Q_\lambda(t, x, \xi))u, u),$$

where $Op(r)$ denotes the pseudodifferential operator with symbol r. Since the principal symbol of $\partial_t Q_\lambda(t, x, \xi)$ equals $(a\partial_t b+(\partial_t a/a)q)H_{\delta_1}(x, \xi; \lambda)$ it follows from the Schwartz inequality

$$(4.9)\quad \begin{aligned} \|P_\lambda u\|^2 &\geqq \|D_t u\|^2+\|Q_\lambda(t, x, D_x)u\|^2/2 \\ &\quad +2\operatorname{Re}(Op(a\partial_t b H_{\delta_1})u, u)-C\|u\|^2 \\ &\geqq \|D_t u\|^2+\|Q_\lambda(t, x, D_x)u\|^2/2-C'\|u\|^2. \end{aligned}$$

Noting that $(a\partial_t b/|\xi|+(q/|\xi|)^2)H_{20\delta}^2\leqq a\partial_t b H_{\delta_1}/\lambda+Q_\lambda^2/\lambda^2$, by means of the sharp Gårding inequality we have (1.10)′ from (4.8) and (4.9), because it follows from the Poincaré inequality that the term $\|u\|^2$ is absorbed by $\|D_t u\|^2$ if δ_1 is small enough.

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