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ON THE LAPLACIAN ON A SPACE OF WHITE NOISE FUNCTIONALS

By

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§ 1. Introduction.

We are greatly interested in the Laplacian on a space of white noise functionals. To have in mind aspects of application to mathematical physics, we can say that it is common in general to use the weak derivative D on a given basic Hilbert space, so as to define d_p which just corresponds to the de Rham exterior differential operator. In doing so, one of the remarkable characteristics of our work consists in adoption of the Hida differential ∂_t instead of D. This distinction from other related works does provide a framework of analysis equipped with the function for perception of the time t, with the result that it is converted into a more flexible and charming theory which enables us to treat time evolution directly. It can be said, therefore, that our work is successful in deepening works about the general theory done by Arai-Mitoma [2], not only on a qualitative basis but also from the applicatory point of view in direct description of operators in terms of time evolution.

The differential ∂_t has its adjoint operator ∂_t^* in Hida sense and it is called the Kubo operator. Indeed, ∂_t^* is realized by extending the functional space even into the widest one $(E)^*$, where a Gelfand triple $(E) \subsetneq (L^2) \subsetneq (E)^*$ is a fundamental setting in white noise analysis, in accordance with our more general choice of the basic Hilbert space H. On the contrary, we define the adjoint operator $d_p{}^*$ of d_p associated with ∂_t without extending the space up to that much. Consequently the Laplacian Δ_p constructed in such an associated manner with d_p (so that, with ∂_t) is realized as an operator having analytically nice properties, such as C^∞-invariance, etc. On the other hand, when we take the Kubo operator as its adjoint, then the so-called Hida Laplacian Δ_H is naturally derived. It is, however, well-known that Δ_H is an operator which maps (S) into $(S)^*$, or in our general setting from (E) into $(E)^*$, which means that it

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transforms a smooth class not into itself, but into the widest class of generalized white noise functionals (or the so-called Hida distributions). On this account the following problem is highly interesting in the standpoint of operator theory or infinite-dimensional analysis.

Let us choose the Hida differential ∂_t as the starting point of the theory. Then if we assume that the Laplacian Δ constructed according to the de Rham theory should possesse a nice property such that it maps a smooth class into itself (i.e. $\Delta: C^\infty \to C^\infty$), what on earth would its adjoint $d_p{}^*$ naturally corresponding to ∂_t (hence $d_p = d_p(\partial_t)$) be like? This is one of our motivations in this paper (cf. the beginning of §5). The followings are in fact outstanding features of our work and what have been acquired in connection with the aforementioned problem: (1) in regard to the adjoint operator $d_p{}^* \equiv d_p{}^*(\partial_t)$ of $d_p \equiv d_p(\partial_t)$, we have as a matter of fact succeeded in constructing it in such a satisfactory manner as to fit into our requirement; (2) as a consequence the constructed Laplacian Δ_p, which is associated with ∂_t, enjoys extremely nicer properties on analytical basis, i.e., Δ_p is a C^∞-invariant operator on a space of white noise functionals (cf. Theorem 7.7); (3) moreover, peculiar ideas of generalized functions totally released from smearing with respect to time t produces the corresponding higher version of theory in operators on functionals, which allows us, despite its implicity, to draw the description of time evolution; (4) our Δ_p primarily settled with the Hida derivative ∂_t is a Laplace-Beltrami type operator getting possession of such a nice property, and it is completely distinct from other Laplacians in white noise analysis, such as the Lévy Laplacian Δ_L, the Gross Laplacian Δ_G, and the Volterra Laplacian Δ_V; (5) the Laplacian is in a sense successfully constructed in concrete and satisfactory manner, simply corresponding to our more general choice of the basic Hilbert space H, and the explicit form $\Delta_p\omega$ of the Laplacian on $\omega \in \mathscr{P}$ (the space of polynomials) is also obtained (cf. Proposition 6.3); (6) as one of applications in terms of our Laplacians, this paper includes several versions of the so-called de Rham-Hodge-Kodaira decomposition theorem associated with Hida derivative in white noise calculus or Hida calculus (cf. Theorem 5.3, Theorem 7.1, and Theorem 7.8). To comment upon the above (4) in addition, it is therefore expected in a quite natural way that Δ_p should play a remarkable and proper role in white noise analysis, which is entirely different from those of the other Laplacians. It remains to be stimulating object in relation with other works [21, 28 & 30] on Laplacians, and it is highly interesting as well.

This paper is organized as follows:

§1. Introduction.

§2. Notation and preliminaries.
§3. Hida differentiation.
§4. De Rham complex.
§5. Laplacians $\Delta_p(\Theta, \partial_t)$ of de Rham complex $\{\bar{d}_p(\Theta, \partial_t)\}$.
§6. Explicit forms of the Laplacians $\Delta_p(\Theta, \partial_t)$.
§6. De Rham-Hodge-Kodaira decompositions associated with Hida derivative.
§8. Concluding remarks.

In §2 we shall introduce notations commonly used in this whole paper, and preliminary results are also stated in §2, some of which are generalizations [9-11] of the well-known results on basic and fundamental theorems in white noise analysis, having been obtained by many pioneers and forerunners [17, 24, 25 & 27]. §3 is devoted to general but brief explanations on the basic ideas, important concepts, and interpretations of Hida differentiation. This will be the key to understand the succeeding sections. There are contained some assertions, simply corresponding to our general setting (cf. [12-15]). §4-§7 are the main parts of our paper. In §4 we shall construct de Rham complexes. For a complex Hilbert space K, let $\Lambda^p K$ be the space of exterior product of order p. Consider a nonnegative selfadjoint operator A on a given normal Hilbert space H, and we denote by the symbol Θ the linear closed operator: $H_C \to K$, determined regarding A. Then the operator d_p from $\mathscr{P}(\Lambda^p K)$ into $\mathscr{P}(\Lambda^{p+1} K)$, depending on Θ, is able to be realized by making use of the Hida differential operator. In §5 we shall state a systematic construction of Laplacians Δ_p of $\{d_p\}$. The corresponding Laplace operator can be constructed theoretically and get into entity when we take advantage of the adjoint operator and have resort to functional analytical method (see Proposition 5.2). By virtue of closedness of the sequences of complexes we can obtain the de Rham-Hodge-Kodaira theorem (Theorem 5.3) in L^2-sense [16]. In §6 the explicit form of the Laplacian Δ_p will be obtained by a direct computation (see Proposition 6.3), where the leading idea is similar to [2], however, as stated before, the employed calculus and basic mathematical background are actually different, since we are totally based upon the white noise calculus or Hida calculus. In §7 we shall make mention of several versions of de Rham-Hodge-Kodaira type theorem associated with Hida derivative [8]. It is easy to see that such a type of decomposition holds for the space of smooth test functionals, induced by the Sobolev type space $H^{2,k}$ of functionals relative to the Laplacian (Theorem 7.1), namely,

$$H^{2,\infty}(\wedge_2^p(K)) = \mathrm{Im}\,[\Delta_p(\Theta) \upharpoonright H^{2,\infty}(\wedge_2^p(K))] \oplus \mathrm{Ker}\,\Delta_p(\Theta)\,.$$

On this account we may employ the Arai-Mitoma method (1991) to derive the similar decomposition theorem even for the category $(S)(\wedge^p K)$, just corresponding to the space of white noise test functionals (see Theorem 7.8). Basically, principal ideas for proofs are due to the spectral theory. However, some of statements include subtler precise estimates, for which we are definitely required to execute elaborate computation with some other results in orthodox probability theory and Malliavin calculus.

Finally it is quite interesting to note that this sort of result leads to the study of Dirac operators on the Boson-Fermion Fock space (cf. [1]), and also that our analysis could be another admissible key to the supersymmetric quantum field theory (e.g., [34]). We believe that this formalism proposed in this paper should be possibly regarded as a clue to open a new pass towards analysis of Dirac operators in quantum field theory through the framework of Hida calculus.

§ 2. Notation and preliminaries.

Let T be a separable topological space equipped with a σ-finite Borel measure $d\nu(t)$ on the topological Borel field $\mathscr{B}(T)$. Further suppose that ν be equivalent to the Lebesgue type measure dt. $H := L^2(T, d\nu; \boldsymbol{R})$ is the real separable Hilbert space of square integrable functions on T. Its norm and inner product will be denoted by $|\cdot|_0$ and $(\cdot, \cdot)_0$. Let A be a densely defined non-negative selfadjoint operator on H. We call A with domain $\mathrm{Dom}(A)$ *standard* if there exists a complete orthonormal basis $\{e_n\}_{n=0}^{\infty} \subset \mathrm{Dom}(A)$ such that

$$\text{(A.1)} \qquad Ae_n = \lambda_n e_n \quad \text{for } \lambda_n \in \boldsymbol{R},$$

$$\text{(A.2)} \qquad 1 < \lambda_0 \leqq \lambda_1 \leqq \cdots \longrightarrow \infty,$$

$$\text{(A.3)} \qquad \sum_{n=0}^{\infty} \lambda_n^{-2} < \infty \quad \text{holds (cf. [9, 10]).}$$

Obviously, A^{-1} is extended to an operator of Hilbert-Schmidt class. Put

$$\rho := \lambda_0^{-1} = \|A^{-1}\|_{\mathrm{op}},$$

and

$$\delta := \Big(\sum_{n=0}^{\infty} \lambda_n^{-2}\Big)^{1/2} = \|A^{-1}\|_{\mathrm{HS}}$$

where $\|\cdot\|_{\mathrm{op}}$ is the operator norm and $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm. We also note the following apparent inequalities:

$$0 < \rho < 1, \qquad \rho < \delta.$$

For a complex separable Hilbert space K, we further assume that

(A.4) There exists a densely defined, closed linear operator Θ from H_C into K such that $A=\Theta^*\Theta$,

where we define the complexification $H_C=H+iH$ as usual way, and Θ^* means the adjoint of Θ.

Given such a standard operator A on H, we can construct a Gelfand triple in the standard manner (see [22, p. 259], [27]). For $p\geqq 0$ let E_p be the completion of Dom(A^p) with respect to the Hilbertian norm $|\xi|_p:=|A^p\xi|_0$, $\xi\in$ Dom(A^p), where Dom$(A^p)=H$ for $p<0$. Then E_p becomes a Hilbert space with the norm $|\cdot|_p$. We thus obtain a chain of Hilbert spaces:

$$\cdots\subset E_p\subset\cdots\subset E_q\subset\cdots\subset H\subset\cdots$$
$$\cdots\subset E_{-q}\subset\cdots\subset E_{-p}\subset\cdots$$

for $0\leqq q\leqq p$. Equipped with the Hilbertian norms $\{|\cdot|_p\}_{p\geq 0}$,

$$E:=\bigcap_{p\geq 0}E_p$$

becomes a nuclear Fréchet space. E is topologized by the projective limit of Hilbert spaces $\{E_p\}_{p\in \boldsymbol{Z}}$ with inner products $(\xi, \eta)_p$ $(\xi, \eta\in E)$, and is called the space of test functions on T. The topological dual space E^* of E is obtained as

$$E^*:=\bigcup_{p\geq 0}E_{-p}\,,$$

i.e., the dual space E^* of E is the inductive limit of E_{-p} as $p\to\infty$. E^* is equipped with the inductive limit convex topology (e.g. [10, Eq. (3.1), § III]). The triplet $E\subset H\subset E^*$ is called a rigged Hilbert space [3] or a Gelfand triple. Then note that the dual space $E_C^*=(E_C)^*$ is equivalent to $(E^*)_C=E^*+iE^*$. It is known that the strong dual topology of E^* coincides with the inductive limit topology in our setting (see [35]). Let μ be the Gaussian probability measure on the measurable space $(E^*, \mathscr{B})$ whose characteristic functional is uniquely determined, by virtue of the Bochner-Minlos theorem, by

$$\int_{E^*}\exp(i\langle x, \xi\rangle)\mu(dx)=\exp\left(-\frac{1}{2}|\xi|_0^2\right), \qquad \xi\in E\,, \tag{2.1}$$

where $\mathscr{B}$ is the σ-algebra containing cylinder sets. For simplicity we denote only by $\langle\cdot, \cdot\rangle$ the canonical bilinear forms between any dual pairs unless it causes any confusion in the context. For instance, when $\langle\cdot, \cdot\rangle$ is a bilinear form on $E^*\times E$, then it is naturally extended to a $\boldsymbol{C}$-bilinear from on $E_C^*\times E_C$. We will denote the space $L^2(E^*, \mathscr{B}, \mu; \boldsymbol{C})$ briefly by (L^2) according to the notation in [17]. Let $\|\cdot\|_0$ denote its norm. Note htat (L^2) is a complex Hilbert space. We them assume the following three conditions (cf. [9-11]) which are suggested by Kubo-Takenaka [24].

(A.5) For every $\xi \in E$ there exists a unique continuous function $\tilde{\xi}$ on T which coincides with ξ up to ν-null functions.

(A.6) For each $t \in T$ the evaluation map $\delta_t : \xi \to \xi(t)$, $\xi \in E$, is continuous, i.e., $\delta_t \in E^*$.

(A.7) The map $t \to \delta_t$ is continuous from T into E^*.

By virtue of (A.5) we agree then that E consists of continuous functions. The symbol $E_C^{\otimes n}$ denotes the n-fold tensor product of the complexification of E. For $f \in E^{\otimes n}$ and $p \in \boldsymbol{R}$, define $|f|_{p,\otimes n} := |(A^p)^{\otimes n} f|_0$. Let $(E_p)_C^{\hat{\otimes} n}$ be the n-fold symmetric tensor product of $(E_p)_C$. $E_C^{\hat{\otimes} n}$ denotes the projective limit of $(E_p)_C^{\hat{\otimes} n}$ and $(E_C^*)^{\hat{\otimes} n}$ the inductive limit of $(E_{-p})_C^{\hat{\otimes} n}$ as p tends to infinity. In the following we shall consider all the time the inductive limit space together with the inductive limit convex topology.

REMARK 2.1. Note that the measure ν is supposed to be rotation invariant in the setting of white noise calculus. T is often thought of as time parameter space. In the above we have in mind the harmonic oscillator Hamiltonian [19, p. 148] as a concrete model of A (cf. Example 2.1 given later in § 2), which is typical in Hida calculus (see [7, 27]).

By the Wiener-Itô decomposition theorem we have

$$(L^2) = \sum_{n=0}^{\infty} \oplus K_n , \tag{2.2}$$

where K_n is the space of n-fold Wiener integrals $I_n(f_n)$, $f_n \in H_C^{\hat{\otimes} n}$ (cf. [24, 1981] or [9, Remark 1.2, § I]). $H_C^{\hat{\otimes} n}$ is the n-fold symmetric completed Hilbert space tensor product of the complexification of H, hence $H_C^{\hat{\otimes} n}$ is again a Hilbert space. It is a fact that (L^2) is canonically isomorphic to the Fock space over H_C, that is,

$$(L^2) \cong \sum_{n=0}^{\infty} \oplus H_C^{\hat{\otimes} n} . \tag{2.3}$$

For each $\varphi \in (L^2)$ there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$, $f_n \in H_C^{\hat{\otimes} n}$ such that

$$\|\varphi\|_0^2 = \sum_{n=0}^{\infty} n!\, |f_n|_{0,\otimes n}^2 , \tag{2.4}$$

and

$$\varphi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle , \qquad \mu\text{-a.e. } x \in E^* . \tag{2.5}$$

where the right hand side is an orthogonal direct sum of functions in (L^2) (e.g. [9, Theorem 2.3]; see also [20]). The symbol $:x^{\otimes n}:$ is the Wick ordering of the distribution $x^{\otimes n}\in(E^{\hat{\otimes} n})^*$, which is defined inductively as follows:

$$:x^{\otimes 0}:=1\,,\qquad :x^{\otimes 1}:=x\,,$$

$$:x^{\otimes n}:=x\hat{\otimes}:x^{\otimes(n-1)}:-(n-1)\tau\hat{\otimes}:x^{\otimes(n-2)}:\,,\qquad (n\geqq 2)$$

where $\tau\in(E\hat{\otimes}E)^*$ is the distribution defined by

$$\langle\tau,\ \xi\otimes\eta\rangle=\langle\xi,\ \eta\rangle\,,\qquad \xi,\ \eta\in E\,. \tag{2.6}$$

Note that τ is also expressed as

$$\tau=\int_T \delta_t^{\otimes 2}d\nu(t)=\sum_{j=0}^{\infty}e_j\otimes e_j\,. \tag{2.7}$$

When we define S-transform as

$$S\varphi(\xi)=\int_{E^*}\varphi(x)F(\xi\,;\,x)\mu(dx)\,, \tag{2.8}$$

then we have $(SI_n(f_n))(\xi)=\langle f_n,\ \xi^{\hat{\otimes} n}\rangle$, where

$$F(\xi\,;\,x)=:\exp\langle x,\ \xi\rangle:=\exp\left(\langle x,\ \xi\rangle-\frac{1}{2}|\xi|_0^2\right)$$

(see [24]; also [9, § I]). Based upon the result in (2.4) and (2.5) we may introduce a second quantized operator $\Gamma(A)$ on (L^2). Let $\mathrm{Dom}\,(\Gamma(A))$ be the subspace of $\varphi\in(L^2)$ given as in Eq. (2.5) such that

(i) $f_n=0$ except finitely many n;

(ii) $f_n\in\mathrm{Dom}\,(A)\otimes_{alg}\cdots\otimes_{alg}\mathrm{Dom}\,(A)$ (n-times).

Then for $\varphi\in\mathrm{Dom}\,(\Gamma(A))$ we put

$$(\Gamma(A)\varphi)(x)=\sum_{n=0}^{\infty}I_n(A^{\otimes n}f_n)(x)\,. \tag{2.9}$$

Let (E_p) be the completion of $\mathrm{Dom}\,(\Gamma(A)^p)$ with respect to the Hilbertian norm

$$\|\varphi\|_p^2:=\|\Gamma(A)^p\varphi\|_0^2=\sum_{n=0}^{\infty}n\,!\,|f_n|_{p,\otimes n}^2$$

$$=\sum_{n=0}^{\infty}n\,!\,|(A^p)^{\otimes n}f_n|_{0,\otimes n}^2\,,$$

where $f_n\in(E_p)_C^{\hat{\otimes} n}$. Equipped with the norm $\{\|\cdot\|\}_{p\geq 0}$,

$$(E):=\bigcap_{p\geq 0}(E_p)$$

becomes a nuclear Fréchet space. Let $(E)^*$ be the dual space of (E). For any $\varphi\in(E)$, φ has a continuous version $\tilde{\varphi}$, and it is bounded on each bounded set of E^*, moreover the evaluation map $\delta_x:\varphi\rightarrow\tilde{\varphi}(x)$ is a continuous linear functional on (E), i.e., $\delta_x \in (E)^*$ for any $x \in E^*$ (cf. [25]; see also [10, 11]). By the above fact we always regard (E), as a space of continuous functions on E^*. An element in (E) (resp. $(E)^*$) is called a test (resp. generalized) white noise functional. We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical $\boldsymbol{C}$-bilinear form on $(E)^* \times (E)$.

Lastly we introduce an example, which is enough to show that our general setting stated above is not unsubstantial.

EXAMPLE 2.1. When $T=\boldsymbol{R}$, $d\nu(t)=dt$, and when we choose $A=1+t^2-(d/dt)^2$, then Θ is given by $d/dt-t$ $(t \in \boldsymbol{R})$ with $H_C=K=L^2(\boldsymbol{R})$, and we have $(E)^*=(S)^*$, $(E)=(S)$ with Gelfand triple

$$(S) \subset (L^2) \subset (S)^* .$$

This is a typical model of white noise spaces in Hida calculus, originally introduced by T. Hida [17, 18] and developed by others [19, 22, 24 & 29] (see also [7, 26] for its applications).

§ 3. Hida differentiation.

We now introduce a differential operator ∂_t which plays a fundamental and important role in white noise calculus. We call ∂_t the Hida differential and $\partial_t \varphi(x)$ a Hida derivative. Originally the operator ∂_t is written as

$$\partial / \partial x(t) = \partial / \partial \dot{B}(t)$$

under the framework of choice $H=L^2(\boldsymbol{R}; dt)$, where $\dot{B}(t)$ indicates the formal time derivative of one-dimensional Brownian motion $B(t)$, $t \in \boldsymbol{R}$ (cf. [17, 18]). Because the causal calculus or Hida calculus is the analysis on white noise functionals and its basic idea is to take a white noise $\dot{B}(t)$ to be the system of variables of white noise functionals, it is quite natural to consider $\partial_t = \partial / \partial \dot{B}(t)$ as its coordinate differentiation. It is needless to say that T. Hida's original idea was a farsighted choice of coordinate system fitting for the causal calculus, if one sees its rapid exciting development and progress in white noise analysis (WNA) for the last few decades (cf. [19, 20 & 22]).

For $\varphi \in (E)$ and $\delta_t \in E^*$ we put

$$\tilde{\partial}_t \varphi(x) := (D_{\delta_t} \varphi)(x) = \sum_{n=1}^{\infty} n \langle : x^{\otimes(n-1)} :, \delta_t * f_n \rangle , \tag{3.1}$$

where $f_n \in E_C^{\hat{\otimes} n}$. Note that $\tilde{\partial}_t = D_{\delta_t}$ is a continuous linear operator on (E) [12]. It is known that

$$(\tilde{\partial}_t\varphi)(x)=\lim_{\theta\to 0}\theta^{-1}\{\varphi(x+\theta\cdot\delta_t)-\varphi(x)\},$$

for $\varphi\in(E)$. For $\Phi\in(E)^*$, its generalized U-functional $U(\xi)=U_\Phi(\xi)$ is defined to be

$$U[\Phi](\xi):=\langle\!\langle\Phi,\ :e^{\langle\cdot,\xi\rangle}:\rangle\!\rangle,\qquad \xi\in E.$$

where $:\exp\langle\cdot,\xi\rangle::=\exp\langle\cdot,\xi\rangle\times\exp(-(1/2)|\xi|_0^2)\in(E)$ (see [29] for its characterization). We can rephrase the above definition as follows: $(S\Phi)(\xi)=U[\Phi](\xi)$. In white noise calculus the collection $\{\dot{B}(t);t\in\boldsymbol{R}\}$ is taken as a coordinate system. Thus we need to define the coordinate differentiation with respect to this system. This can be done directly through the U-functional. Let Φ be in $(E)^*$. Suppose that the U-functional F of Φ has the Fréchet functional derivative $F'(\xi;u)\equiv\delta F(\xi)/\delta\xi(u)$. If the function $F'(\cdot\,;t)$ is a U-functional, then the Hida derivative $\partial_t\Phi$ of Φ is the element in $(E)^*$ with U-functional $F'(\cdot\,;t)$, i.e., $U[\partial_t\Phi](\xi)=F'(\xi;t)$. Note that in general $\partial_t\Phi$ is a distribution as a function of t. In other words, according to Kubo-Takenaka [24] we have

$$\partial_t\Phi(x)=S^{-1}\frac{\delta}{\delta\xi(t)}S\Phi(x),\tag{3.2}$$

(cf. [12–15]). Let $\mathcal{P}$ be the set of polynomials in E^*, and its element $P\in\mathcal{P}$ is expressed as

$$P(x)=\sum_{n=0}^{k}\langle :x^{\otimes n}:,\ f_n\rangle,\qquad f_n\in E_C^{\hat{\otimes}n}.$$

We know that, for $t\in T$, ∂_t and the Gâteaux derivative in direction δ_t coincide on $\mathcal{P}$ (see [14, Lemma 2.2]).

If $\varphi\in(E)$ has chaos expansion $\{f_n;n\in\boldsymbol{N}_0\}$, then denoting by $\tilde{\varphi}$ and $\tilde{f}_n$, $n\in\boldsymbol{N}_0$ their corresponding continuous versions (cf. [9, Remark 3.4], [10, Th. 3.1], and [11, Th. 2.1]), we have

$$\tilde{\partial}_t\tilde{\varphi}(x)=\sum_{n=1}^{\infty}n\langle :x^{\otimes(n-1)}:,\ \tilde{f}_n(t,\cdot)\rangle,\qquad t\in T,$$

where $\tilde{f}_n(t,\cdot)=\delta_t*\tilde{f}_n=\langle\delta_t,\tilde{f}_n\rangle$ (see [14, Remark 3.2]). We always identify $\varphi\in(E)$ with its continuous version on E^*, so that, in the following we shall suppress the distinction between them on a notational basis. The number operator N is defined by

$$N\Big(\sum_{n=0}^{\infty}\langle :x^{\otimes n}:,\ f_n\rangle\Big)=\sum_{n=1}^{\infty}n\langle :x^{\otimes n}:,\ f_n\rangle.\tag{3.3}$$

By [14, Theorem 3.5], generally, for any $y\in E^*$, D_y extends from $\mathcal{P}$ to a continuous linear map from (E) into itself. In particular, (E) is infinitely Gâteaux differentiable in every direction of E^*, moreover, for any $\varphi\in(E)$ the function

$y \to D_y\varphi$ is strongly continuous from E^* into (E). Therefore, in particular, the function $t \to \partial_t\varphi$ is continuous from T to (E) (see also [12, 13]). The followings are verified by employing reflexiveness of (E) (Lemma 4.1 in [14]) with the celebrated Schwartz kernel theorem: namely, for $\varphi \in (E)$, $\nabla\varphi \in E \otimes (E)$ holds, and for every $y \in E^*$,

$$D_y\varphi = \langle y, \nabla\varphi \rangle, \qquad \mu\text{—a.e.,} \tag{3.4}$$

suggesting that $\nabla\varphi$ is the Fréchet derivative of $\varphi \in (E)$. In particular, if $h \in H$, then

$$D_h\varphi = \int_T h(t)\partial_t\varphi d\nu(t), \qquad \mu\text{—a.e.,} \tag{3.5}$$

where the integral should be interpreted as a Bochner integral. Furthermore, every $\varphi \in (E)$ is infinitely Fréchet differentiable and the k-th Féchet derivative of φ is given by $\nabla^k\varphi \in E^{\hat{\otimes} k} \otimes (E)$ (cf. Theorem 4.3 and Theorem 4.4 in [14]). Moreover, the gradient ∇ extends from $\mathcal{P}$ to a continuous linear operator from $\mathrm{Dom}(\sqrt{N})$ into $L^2(T \to (L^2); d\nu)$ (see [15]), where $(\nabla\varphi)(t, x) = \partial_t\varphi(x)$.

§4. De Rham complex.

First of all we start on a notation. $\mathcal{P}$ is the whole space of $\boldsymbol{C}$-valued polynomials on E^* as described in §3. Note that $\mathcal{P}$ is dense in (L^2). For $p \in \boldsymbol{N}_+$, the p-fold exterior product space $\Lambda^p K$ is defined by $\Lambda^p K := \{\omega \in \otimes^p K : \sigma(\omega) = \mathrm{sgn}(\sigma) \cdot \omega, \ \forall \sigma \in \mathcal{G}_p\}$, where $\mathcal{G}_p$ is the symmetric group of order p. We introduce the following metric in $\wedge^p K$: i.e., for any $\omega, \gamma \in \wedge^p K$ such that $\omega = f_1 \wedge \cdots \wedge f_p$, $\gamma = g_1 \wedge \cdots \wedge g_p$, $f_k \in K$, $g_k \in K$ (for any $k = 1, 2, \cdots, p$), the inner product between ω and γ is given by

$$\langle \omega, \gamma \rangle^{\wedge^p K} := \sum_{\sigma \in \mathcal{G}_p} \mathrm{sgn}(\sigma) \cdot \prod_{k=1}^{p} \langle f_k, g_{\sigma(k)} \rangle_K .$$

$\Lambda^p K^c$ denotes the completion of $\Lambda^p K$ by the above metric $\langle \cdot, \cdot \rangle^{\wedge^p K}$, with $\Lambda^0 K^c = \boldsymbol{C}$. Its element is called a p-fold skew symmetric tensor, and A_p is an alternating operator from $\otimes^p K$ into $\Lambda^p K$. When $B := \Theta\Theta^*$, then $D^\infty(B) := \bigcap_{m \in \boldsymbol{N}} \mathrm{Dom}(B^m)$. We denote by $\mathcal{P}(\Lambda^p K^c)$ the whole space of $\Lambda^p K^c$-valued polynomials on E^*, whose element is expressed by

$$\omega(x) = \sum_{n=1}^{k} \tilde{P}_n(x) \cdot \xi_n, \qquad x \in E^*, \tag{4.1}$$

where $\tilde{P}_n \in \mathcal{P}$, $\xi_n \in A_p(\otimes^p D^\infty(B)) \subset \Lambda^p K^c$. Notice that $\mathcal{P}(\Lambda^p K^c)$ is dense in $\Lambda_2^p(K)$, and $\Lambda_2^p(K)$ is defined to be $(L^2) \otimes \Lambda^p K^c$ which is identified with

$L^2(E^* \to \Lambda^p K^c; d\mu)$ in a usual manner [32].

Now we will introduce a linear operator d_p from $\mathscr{P}(\Lambda^p K^c)$ into $\mathscr{P}(\Lambda^{p+1} K^c)$ for each $p \in \boldsymbol{N}_+$. Actually, for any $\omega \in \mathscr{P}(\Lambda^p K^c)$ especially of the form (4.1), the operator d_p $(\equiv d_p(\Theta) = d_p(\Theta, \partial_t))$ is defined as

$$d_p \omega(x) \equiv (p+1) \sum_{n=1}^{k} A_{p+1}(\Theta \cdot \partial_t \tilde{P}_n(x) \otimes \xi_n), \tag{4.2}$$

where ∂_t is the Hida differential (see § 3). We have $\tilde{P}_n$ in our standard representation of element in (L^2):

$$\check{P}_n(x) = \sum_{l=1}^{N(n)} \langle : x^{\otimes l} :, f_l \rangle ,$$

where f_l is the element in $E_C{}^{\hat{\otimes} l}$ given by

$$f_l = \sum_{\alpha \in N^l} b_\alpha \eta_{\alpha_1, l} \hat{\otimes} \cdots \hat{\otimes} \eta_{\alpha_l, l}, \ b_\alpha \in \boldsymbol{C}, \ \eta_{\alpha_j, l} \in E_C ,$$

Note that all representations of $\tilde{P}_n$ are everywhere defined, continuous functions on E^*. Therefore, the U-functional of $d_p(\Theta)\omega$ is given by

$$\begin{aligned} &U[d_p(\Theta)\omega](\zeta) \\ &= \sum_{n=1}^{k} \left\{ \sum_{l=1}^{N(n)} \sum_{\alpha \in N^l} b_\alpha \sum_{m=1}^{l} (\eta_{\alpha_1, l}, \zeta) \cdots (\eta_{\alpha_m, l}{}^{\vee}, \zeta) \cdots (\eta_{\alpha_l, l}, \zeta) \right\} \\ &\quad \cdot \Theta(\eta_{\alpha_\mu, l}(t)) \wedge w_1{}^{(n)} \wedge \cdots \wedge w_p{}^{(n)} , \qquad \zeta \in E , \end{aligned} \tag{4.3}$$

where the symbol $\vee$ means omission of the term. For each $p \in \boldsymbol{N}_+$, $d_p(\Theta)$ is densely defined linear operator in $\Lambda_2{}^p(K)$, and, it is easy to see that

$$d_{p+1}(\Theta, \partial_t) \circ d_p(\Theta, \partial_t) = 0 \qquad \text{on } \mathscr{P}(\Lambda^p K^c) . \tag{4.4}$$

Its adjoint operator $d_p{}^*(\Theta) \equiv d_p{}^*(\Theta, \partial_t)$ from $\Lambda_2{}^{p+1}(K)$ into $\Lambda_2{}^p(K)$ is defined by

$$\langle d_p(\Theta)\omega, \gamma \rangle_{\Lambda_2^{p+1}(K)} = \langle \omega, d_p{}^*(\Theta)\gamma \rangle_{\Lambda_2^p(K)}$$

for $\omega \in \Lambda_2^p(K)$, $\gamma \in \Lambda_2^{p+1}(K)$.

REMARK 4.1. Note that the U-functional representation of $d_p{}^*(\Theta)\omega$ is given by

$$\begin{aligned} U[d_p{}^*(\Theta)\omega](\zeta) = \sum_{n=1}^{k} \Bigg[\sum_{l=1}^{p+1} (-1)^{l-1} \Bigg\{ & \sum_{j=1}^{N(n)} \sum_{\alpha \in N^j} b_\alpha \prod_{i=1}^{j} (\eta_{\alpha_i, j}, \zeta)_{H_C} \\ & \times (\Theta^* \overline{w_1{}^{(n)}}, \zeta)_{H_C} - \Bigg(\sum_{j=1}^{N(n)} \sum_{\alpha \in N^j} b_\alpha \sum_{m=1}^{j} \eta_{\alpha_m, j}(t) \\ & \cdot (\eta_{\alpha_1, j}, \zeta) \cdots (\eta_{\alpha_m, j}{}^{\vee}, \zeta) \cdots (\eta_{\alpha_j, j}, \zeta), \Theta^* w_1{}^{(n)} \Bigg)_{H_C} \Bigg\} \\ & \times w_1{}^{(n)} \wedge \cdots \wedge \check{w}_1{}^{(n)} \wedge \cdots \wedge w_{p+1}{}^{(n)} \Bigg] , \end{aligned} \tag{4.5}$$

for $\zeta \in E$ (cf. Lemma 6.2). □

It follows immediately from (4.4) that

$$d_p^*(\Theta, \partial_t) \circ d_{p+1}^*(\Theta, \partial_t) = 0 \quad \text{on } \mathrm{Dom}\,(d_{p+1}^*(\Theta)). \tag{4.6}$$

It can be deduced from denseness and the adjoint argument that $d_p(\Theta)$ becomes closable for each $p \in \boldsymbol{N}_+$. We write its extension $\bar{d}_p$ of d_p, and we put $\Lambda_2^* := \sum_{p=0}^{\infty} \Lambda_2^p(K)$. Then the sequence $(\Lambda_2^*(K), \{\bar{d}_p(\Theta, \partial_t)\})$ forms a de Rham complex.

REMARK 4.2. For $\zeta \in E$, $\omega \in \mathcal{P}(\Lambda^p K^c)$, we have

$$(S\omega)(\zeta) = \sum_{n=1}^{k} \Big(\sum_{l=1}^{N(n)} \sum_{\alpha \in N^l} b_\alpha \prod_{i=1}^{l} (\eta_{\alpha_i, l}, \zeta)_{H_C} \Big) w_1^{(n)} \wedge \cdots \wedge w_p^{(n)}.$$

Recall Eq. (3.2) in §3, then (4.3) is obvious.

REMARK 4.3. In general, the operator $d_p(\Theta, \partial_t)$ constructed in such a way is not necessarily closable. The closability of $d_p(\Theta, \partial_t)$ depends on the structure of the measure μ on E^*. This is a very touchy problem indeed. However, fortunately in our case $d_p(\Theta, \partial_t)$ is well-defined for the Gaussian white noise measure μ defined in (2.1).

§5. Laplacians $\Delta_p(\Theta, \partial_t)$ of de Rham complex $\{\bar{d}_p(\Theta, \partial_t)\}$.

As we have stated in §1, it is clear why we stick to the Hida differentiation, for we are aiming at opening a new pass toward analysis in mathematical physics through the framework of Hida calculus. On the other hand, when we say that an operator is called to be smooth if it transforms the space of smooth elements into itself, there is the fact that the Hida Laplacian (cf. §1) is not smooth any longer in the above sense. That is why we would like to know what the desired Laplacian should be like, which is one of our motivations. One may find an answer to the matter in this section (see also Theorem 7.7 in §7).

Thanks to the fact that $\mathrm{Im}\,(\bar{d}_{p-1}(\Theta))$ and $\mathrm{Im}\,(d_p^*(\Theta))$ are closed for $p \in \boldsymbol{N}_+$ in our case, by making use of the sesquilinear form and elaborate functional analysis methods we can define a unique nonnegative selfadjoint operator acting in $\Lambda_2^p(K)$. This is nothing but the desired Laplacian corresponding to the de Rham complex $\{\bar{d}_p(\Theta, \partial_t)\}$. In the last we shall give a primictive version of the de Rham-Hodge-Kodaira type decomposition for the p-forms in the L^2-sense.

We first consider the bilinear function J_p on $\mathrm{Dom}\,(J_p(\Theta)) := \mathrm{Dom}\,(\bar{d}_p(\Theta)) \cap$

Dom $(d_{p-1}^*(\Theta))$, which is dense in $\Lambda_2{}^p(K)$. For $p\in \boldsymbol{N}_+$, $J_p(\Theta)\equiv J_p(\Theta, \partial_t)$ is defined to be

$$J_p(\Theta)(\omega, \gamma):=\langle \tilde{d}_p(\Theta)\omega, \tilde{d}_p(\Theta)\gamma\rangle_{\Lambda_2{}^{p+1}(K)} + \langle d_{p-1}^*(\Theta)\omega, d_{p-1}^*(\Theta)\gamma\rangle_{\Lambda_2{}^{p-1}(K)} \tag{5.1}$$

for any $\omega, \gamma\in$ Dom $(J_p(\Theta))$. This J_p turns to be a sesquilinear form on $\Lambda_2{}^p(K) \times\Lambda_2{}^p(K)$. Note that this formalism indicates the Laplacian Δ_p to be roughly given by $d_p{}^*\tilde{d}_p+\tilde{d}_{p-1}d_{p-1}{}^*$ as usual. As a matter of fact, it is easy to see that the form $J_p(\Theta)$ is a nonnegative, densely defined, closed form on Dom $(J_p(\Theta))$. On this account, we obtain the following representation of Friedrichs type.

PROPOSITION 5.1 [16]. *Let $J_p(\Theta, \partial_t)$ be a nonnegative closed sesquilinear form with the dense domain* Dom $(J_p(\Theta))$. *Then there exists a unique nonnegative selfadjoint operator $\Delta_p(\Theta)\equiv\Delta_p(\Theta, \partial_t)$ acting in $\Lambda_2{}^p(K)$ such that*

$$\langle\omega, \Delta_p(\Theta)\gamma\rangle_{\Lambda_2{}^p(K)}=J_p(\Theta)(\omega, \gamma), \tag{5.2}$$

for $\omega\in$ Dom $(J_p(\Theta))$, *$\gamma\in$* Dom $(\Delta_p(\Theta))$, *$p\in \boldsymbol{N}_+$.*

REMARK 5.1. In the above assertion, Dom $(\Delta_p(\Theta))$ is dense in Dom $(J_p(\Theta))$ in the sense of $J_p(\Theta)$-form norm, as a consequence Dom $(\Delta_p(\Theta))$ is also naturally dense in $\Lambda_2{}^p(K)$. For the proof, see Theorem 2.2 and § III in [16]. □

Proposition 5.1 and the second representation theorem [23, VI. 2] immediately gives:

PROPOSITION 5.2. *There exists a unique nonnegative selfadjoint operator $\Delta_p(\Theta, \partial_t)$ in $\Lambda_2{}^p(K)$ such that the equality*

$$\langle\Delta_p{}^{1/2}(\Theta)\omega, \Delta_p{}^{1/2}(\Theta)\gamma\rangle_{\Lambda_2{}^p(K)}=J_p(\Theta)(\omega, \gamma) \tag{5.3}$$

holds for every $\omega, \gamma\in$ Dom $(\Delta_p{}^{1/2}(\Theta))=$ Dom $(J_p(\Theta))$, *$p\in \boldsymbol{N}_+$* (see also [16, Theorem 2.3]).

REMARK 5.2. Proposition 5.1 is unsatisfactory in that it is not valid for all $u, v\in$ Dom (J_p), which is furnished by Proposition 5.2. What is essential in (5.3) is that $\Delta_p{}^{1/2}(\Theta)$ is selfadjoint, nonnegative, $(\Delta_p{}^{1/2}(\Theta))^2=\Delta_p(\Theta, \partial_t)$, and that Dom $(\Delta_p(\Theta))$ is a core of $\Delta_p{}^{1/2}(\Theta)$. □

For the case $p=0$, we need to define the operator $\Delta_0(\Theta)$ properly. The answer will be given by a version of the well-known von Neumann type theorem [32, II]. Hence we can define $\Delta_0(\Theta)\equiv\Delta_0(\Theta, \partial_t)$ by

$$\Delta_0(\Theta):=(d_0{}^*\tilde{d}_0)(\Theta).$$

Thus we attain that $\{\Delta_p(\Theta)\}_{p=0}^{\infty}$ is the Laplacians associated with the de Rham complex $\{\tilde{d}_p(\Theta)\}_{p=0}^{\infty}$. Now we are in a position to state a decomposition theorem of de Rham-Hodge-Kodaira type for the sapace $\Lambda_2{}^p(K)$ in L^2-sense [16, Th. 2.5].

THEOREM 5.3 (Decomposition of de Rham-Hodge-Kodaira type for the space $\Lambda_2{}^p(K)$). *For all $p\in\boldsymbol{N}_+$, the space $\Lambda_2{}^p(K)$ admits the following orthogonal decomposition*:

$$\Lambda_2{}^p(K)=\overline{\mathrm{Im}\,(\tilde{d}_{p-1}(\Theta))}\oplus\overline{\mathrm{Im}\,(d_p{}^*(\Theta))}\oplus\mathrm{Ker}\,\Delta_p(\Theta). \tag{5.4}$$

N. B. Notice that the above decomposition assertion (5.4) is valid even for $p=0$ with $\tilde{d}_{-1}(\Theta)=0$ for convension.

§6. Explicit forms of the Laplacians $\Delta_p(\Theta, \partial_t)$.

Here we shall give an explicit form of the Laplacians $\{\Delta_p(\Theta, \partial_t)\}_p$ on $\{\mathcal{P}(\Lambda^p K^c)\}_p$, which is extremely important on a basis of the fundamental properties of our Laplacians. We first consider the element $\omega\in\mathcal{P}(\Lambda^p K^c)$ of the form:

$$\begin{aligned}\omega(x)&=\sum_{n=1}^{k}\tilde{P}_n(x)\cdot\xi_n \qquad (x\in E^*,\ \tilde{P}_n\in\mathcal{P})\\ &=\sum_{n=1}^{k}\langle :x^{\otimes n}:,\ f_n\rangle\cdot w_1^{(n)}\wedge\cdots\wedge w_p^{(n)},\end{aligned}$$

where $f_n\in E_C{}^{\hat{\otimes} n}$, $\xi_n\in A_p(\otimes^p D^\infty(B))$. Then, recalling Eq. (4.2) we have

$$\begin{aligned}d_p(\Theta)\omega(x)=\sum_{n=1}^{k}\sum_{\alpha\in\boldsymbol{N}^n}b_\alpha\sum_{l=1}^{n}\langle :x^{\otimes(n-1)}:,\ \Xi^{\hat{\otimes}(n-1)}(\eta_*;l)\rangle\\ \times\Theta(\eta_{\alpha(l),n}(t))\wedge w_1{}^{(n)}\wedge\cdots\wedge w_p{}^{(n)},\end{aligned} \tag{6.1}$$

where we put

$$\Xi^{\hat{\otimes}(n-1)}(\eta_*;k):=\eta_{\alpha(1),n}\hat{\otimes}\cdots\overset{k}{\vee}\cdots\hat{\otimes}\eta_{\alpha(n),n}$$

and employed a formula for exterior products. Notice that

$$\delta_t{}^*f_n=\sum_{\alpha\in\boldsymbol{N}^n}b_\alpha/n\cdot\sum_{k=1}^{n}\eta_{\alpha(k),n}(t)\cdot\Xi^{\hat{\otimes}(n-1)}(\eta_*;k).$$

Then its U-functional (cf. (4.3)) is given by

$$U[d_p(\Theta)\omega](\xi)=\sum_{n=1}^{k}\sum_{\alpha\in N^n}b_\alpha\sum_{i=1}^{n}(\eta_{\alpha(1),n},\xi)\cdots\overset{\vee}{(\eta_{\alpha(i),n},\xi)}\cdots(\eta_{\alpha(n),n},\xi)$$
$$\times\tilde{w}_1^{(n)}(i,t;\Theta)\wedge\tilde{w}_2^{(n)}\wedge\cdots\wedge\tilde{w}_{p+1}^{(n)},\qquad(\xi\in E),$$

where we set $\tilde{w}_j^{(n)}:=w_{j-1}^{(n)}$ for $j=2,3,\cdots,p+1$, and $\tilde{w}_1^{(n)}(i,t;\Theta):=\Theta(\eta_{\alpha(i),n}(t))$.

LEMMA 6.1. *For any $\gamma\in\mathcal{P}(\Lambda^{p+1}K^c)$ with the form $\sum_{l=1}^{k}\tilde{Q}_l(x)\eta_l$, $d_p^*(\Theta)\gamma(x)$ is given by*

$$d_p^*(\Theta)\gamma(x)=\sum_{j=1}^{p+1}(-1)^{j-1}\sum_{l=1}^{k}\{\tilde{Q}_l(x)\cdot\langle x(t),\Theta^*(v_j^{(l)})\rangle$$
$$-(\overline{\Theta^*v_j^{(l)}},\partial_t\tilde{Q}_l(x))_{H_C}\}\cdot v_1^{(l)}\wedge\cdots\wedge\overset{\vee}{v_j^{(l)}}\wedge\cdots\wedge v_{p+1}^{(l)}.\tag{6.2}$$

PROOF. By the isomorphism in $\Lambda_2^{p+1}(K)$ we get

$$\langle\tilde{d}_p(\Theta)\omega,\gamma\rangle_{\Lambda_2^{p+1}(K)}\tag{6.3}$$
$$=\sum_{n=1}^{k}\sum_{l=1}^{k}\sum_{\sigma\in\mathcal{G}_{p+1}}\operatorname{sgn}(\sigma)\sum_{\alpha}b_\alpha\sum_{j=1}^{p+1}\prod_{i=1}\langle\tilde{w}_i^{(n)},v_{\sigma(i)}^{(l)}\rangle_K$$
$$\times\int_{E^*}\langle:x^{\otimes(n-1)}:,\Xi^{\hat{\otimes}(n-1)}(\eta_*;j)\rangle\langle:x^{\otimes l}:,g_l\rangle\mu(dx)$$

where note that only $\tilde{w}_1^{(n)}$ depends on the parameter j. By employing a direct result derived from the coordinate multiplication operator formula in WNA (cf. Remark 6.3 below), we may apply Lemma 2.2 [14] for (6.3) to obtain

$$\sum_{n=1}^{k}\sum_{l=1}^{k}\sum_{\sigma\in\mathcal{G}_{p+1}}\operatorname{sgn}(\sigma)\prod_{i=2}^{p+1}\langle\tilde{w}_i^{(n)},v_{\sigma(i)}^{(l)}\rangle_K$$
$$\times\int_{E^*}\langle x(t),\Theta^*v_{\sigma(1)}^{(l)}\rangle\cdot\langle:x^{\otimes n}:,f_n\rangle\cdot\langle:x^{\otimes 1}:,g_l\rangle\mu(dx)$$
$$-\sum_{n=1}^{k}\sum_{l=1}^{k}\sum_{\sigma\in\mathcal{G}_{p+1}}\operatorname{sgn}(\sigma)\prod_{i=2}^{p+1}\langle\tilde{w}_i^{(n)},v_{\sigma(i)}^{(l)}\rangle_K$$
$$\times\int_{E^*}\langle:x^{\otimes n}:,f_n\rangle\cdot(\Theta^*v_{\sigma(1)}^{(l)},\partial_t\langle:x^{\otimes l}:,g_l\rangle)_{H_C}\mu(dx)$$
$$=:I_1+I_2,$$

because we used above the Fubini type theorem relative to $d\mu$ and $d\nu$. Note that the relation

(6.4) $$l_1\wedge\cdots\wedge l_n=\sum_{k=1}^{n}(-1)^{k-1}l_k\otimes(l_1\wedge\cdots\wedge\overset{\vee}{l_k}\wedge\cdots<l_n).$$

By making use of (6.4) we can rewrite

$$I_2=-\sum_{l=1}^{k}\sum_{j=1}^{p+1}(-1)^{j-1}\langle\omega,(\Theta^*v_j^{(l)},\partial_t\tilde{Q}_l(x))_{H_C}$$
$$\times v_1^{(l)}\wedge\cdots\wedge\overset{\vee}{v_j^{(l)}}\wedge\cdots\wedge v_{p+1}^{(l)}\rangle_{\Lambda_2^p(K)},$$

where

$$\partial_t\tilde{Q}_l(x)=\langle :x^{\otimes(l-1)}:,\sum_{\beta\in N^1}b_\beta\sum_{i=1}^{l}\eta_{\beta(i),l}(t)\cdot\tilde{\Xi}^{\hat{\otimes}(l-1)}(\eta_*;i)\rangle$$

when g_l is given by $\sum_{\beta\in N^1}b_\beta\cdot\eta_{\beta(1),l}\hat{\otimes}\cdots\hat{\otimes}\eta_{\beta(l),l}$. Likewise as to the I_1 term, we conclude the assertion. q.e.d.

REMARK 6.1. We need to explain how to interpret the term $\langle x(t),\Theta^*v_{\sigma(1)}^{(l)}\rangle$. The element $\Theta^*v_{\sigma(1)}^{(l)}$ in H_C is well approximated by a sequence $\{y_k\}_{k=1}^{\infty}\subset E_C$ under our abstract setting. So we can define it by a limiting procedure.

REMARK 6.2. As a technical merit of computation in white noise calculus (cf. Remark 1.1 and Lemma 2.4 in [9]), we have

$$:\langle x,f_1\rangle\cdots\langle x,f_n\rangle:=\prod_{i=1}^{n}\frac{d}{d\lambda_i}:e^{\langle x,\sum_j\lambda_jf_j\rangle}:|_{\lambda_1=\cdots=\lambda_n=0}.\qquad\square$$

In fact, the operation of d_p^* on $\mathcal{P}(\Lambda^{p+1}K^c)$ is also described evidently by the U-functional (cf. Remark 4.1).

LEMMA 6.2. *The U-functional of* $d_p^*(\Theta)\gamma(x)$ $(x\in E^*)$ *is given by*

(6.5) $$U[d_p^*(\Theta)\gamma](\xi)=\sum_{j=1}^{p+1}(-1)^{j-1}\sum_{l=1}^{k}\Big\{\sum_\beta b_\beta\prod_{i=1}^{l}(\eta_{\beta(i),l},\xi)$$
$$\times(\Theta(v_j^{(l)},\xi)+\sum_\beta b_\beta\sum_{i=1}^{l-1}(\Theta^*(v_j^{(l)}),\eta_{\beta(i),l})$$
$$\times\prod_{\substack{k=1\\k\neq i}}^{l-1}(\eta_{\beta(k),l},\xi)-\sum_{\beta\in N^1}b_\beta\sum_{i=1}^{l}\sum_{\substack{k=1\\k\neq i}}(\eta_{\beta(k),l},\xi)$$
$$\times(\overline{\Theta^*v_j^{(l)}},\eta_{\beta(i),l}(t))\Big\}$$
$$\times v_1^{(l)}\wedge v_2^{(l)}\wedge\cdots\wedge\overset{\vee}{v_j^{(l)}}\wedge\cdots\wedge v_{p+1}^{(l)},\quad(\xi\in E).$$

PROOF. By Lemma 6.1 we immediately obtain

(6.6) $$U[d_p^*(\Theta)\gamma](\xi)=S(d_p^*(\Theta)\gamma)(\xi)$$
$$=\sum_{j=1}^{p+1}(-1)^{j-1}\sum_{l=1}^{k}\{S(\tilde{Q}_l(x)\cdot\langle x(t),\ \Theta^*(v_j^{(l)})\rangle)(\xi)$$
$$-(\overline{\Theta^*v_j^{(l)}},\ S(\partial_t\tilde{Q}_l(x))(\xi))_{H_C}\}$$
$$\times v_1^{(l)}\wedge v_2^{(l)}\wedge\cdots\wedge\overset{\vee}{v_j^{(l)}}\wedge\cdots\wedge v_{p+1}^{(l)}.$$

While, we easily get

(6.7) $$S(\partial_t\tilde{Q}_l(\cdot))(\xi)=\frac{\delta}{\delta\xi(t)}S(\tilde{Q}_l)(\xi)$$
$$=\sum_{\beta\in N^1}b_\beta\sum_{i=1}^{l}\eta_{\beta(i),l}(t)\cdot(\eta_{\beta(1),l},\ \xi)_{H_C}\cdots$$
$$\cdots\overset{\vee}{(\eta_{\beta(i),l},\ \xi)_{H_C}}\cdots(\eta_{\beta(l),l},\ \xi)_{H_C},\qquad \xi\in E.$$

To compute $S(\tilde{Q}_l(\cdot)\cdot\langle x,\ \Theta^*(v_j^{(l)})\rangle)(\xi)$ $(\xi\in E)$, we may utilyze similar type equalities as in Remark 6.2 (cf. Lemma 2.5 and § IV in [9]) to obtain

(6.8) $$S(:\prod_{i=1}^{l}\langle\cdot,\ \eta_{\beta(i),l}\rangle:\langle(\cdot)(t),\ \Theta^*(v_j^{(l)})\rangle)(\xi)$$
$$=\frac{1}{(l+1)!}\sum_{i=1}^{l+1}(-1)^{l-i+1}\sum_{j_1<j_2<\cdots<j_i}(\tilde{\eta}_{\beta(j_1),l}+\tilde{\eta}_{\beta(j_2),l}+\cdots+\tilde{\eta}_{\beta(j_k),l},\ \xi)^{l+1}$$
$$+\sum_{i=1}^{l-1}(\Theta^*(v_j^{(l)}),\ \eta_{\beta(i),l})\frac{1}{(l-1)!}\sum_{i=1}^{l-1}(-1)^{l-k-1}\sum_{j_1<j_2<\cdots<j_k}$$
$$\times(\eta_{\beta(j_1),l}+\cdots+\overset{\vee}{\eta_{\beta(j_i),l}}+\cdots+\eta_{\beta(j_k),l},\ \xi)^{l-1}$$

where we put
$$\tilde{\eta}_{\beta(k),l}:=\eta_{\beta(k),l},\qquad(\text{for } k=1,2,\cdots,l),$$
$$\tilde{\eta}_{\beta(l+1),l}:=\Theta^*(v_j^{(l)}),\qquad\text{for } k=l+1).$$

In connection with Remark 6.1, commutativity between the S-transform and the limiting procedure with $k\to\infty$ is required in the above computation. However, it is verified with the Lebesgue type bounded convergence theorem with respect to the Gaussian white noise measure. To complete the proof it is sufficient to substitute (6.7) and (6.8) for (6.6), paying attention to the fact that
$$S(\langle:x^{\otimes l}:,\ g_l\rangle\cdot\langle x(t),\ \Theta^*(v_j^{(l)})\rangle)(\xi)$$
$$=\sum_{\beta\in N^1}b_\beta\cdot S\Big(:\prod_{k=1}^{l}\langle x,\ \eta_{\beta(k),l}\rangle:\langle x(t),\ \Theta^*(v_j^{(l)})\rangle\Big)(\xi).$$

q. e. d.

REMARK 6.3. When we observe carefully the computation of the term

$\langle d_p(\Theta)\omega, \gamma\rangle_{\Lambda_2^{p+1}(K)}$ in the proof of Lemma 6.1, then we may regard that it is roughly equal to

$$C(p)A_{p+1}\Theta\langle\partial_t\tilde{P}_n, \tilde{Q}_l\rangle^\mu ,$$

where $C(p)$ is some constant depending on $p\in\boldsymbol{N}_+$. Then

$$\begin{aligned}\langle\partial_t\tilde{P}_n, \tilde{Q}_l\rangle^\mu &= \langle\tilde{P}_n, [x]\tilde{Q}_l\rangle^\mu - \langle\tilde{P}_n, \partial_t\tilde{Q}_l\rangle^\mu \\ &= \langle\tilde{P}_n, (\partial_t+\partial_t^*)\tilde{Q}_l\rangle^\mu - \langle\tilde{P}_n, \partial_t\tilde{Q}_l\rangle^\mu ,\end{aligned}$$

where we used the significant discovery on the coordinate multiplication operator by $x(t)$ in WNA (cf. [26]). The above computation means roughly that the adjoint ∂_t^* is employed in order to determine $d_p^*(\Theta)$, but in Θ-dependent manner. It is interesting to note that our discussion in Lemma 6.1 and Lemma 6.2 provides a subtle framework to construct a nicer Laplacian $\Delta_p(\Theta, \partial_t)$ by making use of the operator Θ. We would be able to take much advantage of it to apply our theory later for the problems arizing in quantum physics (see §7 or [8]). □

Now we are in a positon to express the explicit form of our Laplacian $\Delta_p(\Theta, \partial_t)$ on $\mathcal{P}(\Lambda^pK^c)$. By the discussion in §5, we have only to compute $d_p^*(\Theta)\tilde{d}_p(\Theta)\omega(x)$ and $\tilde{d}_{p-1}d_{p-1}^*\omega(x)$ respectively. To take (6.1) and (6.2) into consideration, it is easily checked that

$$\begin{aligned}(6.9)\qquad d_p^*(\Theta)\tilde{d}_p(\Theta)\omega(x) = \sum_{n=1}\sum_{\alpha} b_\alpha \sum_{l}\Bigg[\sum_{j=1}^{p+1}(-1)^{j-1} \\ \cdot\{\langle :x^{\otimes(n-1)}:, \Xi^{\hat{\otimes}(n-1)}(\eta_*; l)\rangle\cdot\langle x(t), \Theta^*(\tilde{w}_j^{(n)})\rangle \\ -(\overline{\Theta^*\tilde{w}_j^{(n)}}, \partial_t\langle :x^{\otimes(n-1)}:, \Xi^{\hat{\otimes}(n-1)}(\eta_*; l)\rangle)_{H_C}\} \\ \times\tilde{w}_1^{(n)}(i, t; \Theta)\wedge\tilde{w}_2^{(n)}\wedge\cdots\wedge\check{\tilde{w}}_j^{(n)}\wedge\cdots\wedge\tilde{w}_{p1}^{(n)}\Bigg].\end{aligned}$$

Next we consider the other part: in fact,

$$\begin{aligned}\tilde{d}_{p-1}(\Theta)d_{p-1}^*(\Theta)\omega(x) &= \tilde{d}_{p-1}(\Theta)(d_{p-1}^*(\Theta)\omega(x)) \\ &= \sum_{j=1}^{p}(-1)^{j-1}\sum_{n=1}^{k}[d_{p-1}\{\tilde{P}_n(x)\langle x(t), \Theta^*(w_j^{(n)})\rangle \\ &\quad\times w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}\}] \\ &\quad-\sum_{j=1}^{p}(-1)^{j-1}\sum_{n=1}^{k}[d_{p-1}\{(\overline{\Theta^*w_j^{(n)}}, \partial_t\tilde{P}_n(x))_{H_C} \\ &\quad\times w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}\}] \\ &=: J_1+J_2 .\end{aligned}$$

As to J_1-part computation, it is verified with ease that

$$d_{p-1}[\tilde{P}_n(x)\langle x(t),\ \Theta^*(w_j^{(n)})\rangle\cdot w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}]$$

$$=\sum_\alpha b_\alpha\sum_l\langle :x^{\otimes(n-1)}:,\ \Xi^{\hat{\otimes}(n-1)}(\eta_*;\ l)\rangle\cdot\langle x(t),\ \Theta^*(w_j^{(n)})\rangle$$

$$\times\Theta(\eta_{\alpha(l),n}(t))\wedge w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}$$

$$+\tilde{P}_n(x)\cdot\Theta[(\Theta^*w_j^{(n)})(t)]\wedge w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}\,.$$

As to J_2-part computation, it goes almost similarly. Indeed,

$$\tilde{d}_{p-1}\{(\Theta^*w_j^{(n)},\ \partial_t\tilde{P}_n(x))_{H_C}\cdot w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}\}$$

$$=\sum_\alpha b_\alpha\sum_l^n\sum_{\substack{k\\k\neq l}}^n(\Theta^*w_j^{(n)},\ \eta_{\alpha(l),n}(t))_{H_C}\cdot$$

$$\cdot\langle :x^{\otimes(n-2)}:,\ \Xi^{\hat{\otimes}(n-2)}[\Xi^{\hat{\otimes}(n-1)}(\eta_*;\ l)](k)\rangle$$

$$\times\tilde{w}_1^{(n)}(k,\ t;\ \Theta)\wedge w_1^{(n)}\wedge\cdots\wedge\check{w}_j^{(n)}\wedge\cdots\wedge w_p^{(n)}\,.$$

Finally we attain the principal result in this section.

PROPOSITION 6.3. *For $p\in\boldsymbol{N}_+$, we have*

$$\Delta_p(\Theta,\ \partial_t)\omega(x)=\sum_{n=1}^k\{\sum_\alpha b_\alpha\sum_m\langle x(t),\ \Theta^*\Theta(\eta_{\alpha(m),n}(t))\rangle$$

$$\cdot\langle :x^{\otimes(n-1)}:,\ \Xi^{\hat{\otimes}(n-1)}(\eta_*;\ m)\rangle-\sum_\alpha b_\alpha\sum_l\sum_{\substack{m\\m\neq l}}$$

$$\cdot(\eta_{\alpha(m),n}(t),\ \Theta^*\Theta(\eta_{\alpha(l),n}(t)))_{H_C}$$

$$\cdot\langle :x^{\otimes(n-2)}:,\ \Xi^{\hat{\otimes}(n-2)}[\Xi^{\hat{\otimes}(n-1)}(\eta_*;\ l)](m)\rangle\}$$

$$\times w_1^{(n)}\wedge w_2^{(n)}\wedge\cdots\wedge w_p^{(n)}$$

$$+\sum_{j=1}^p\sum_{n=1}^k\tilde{P}_n(x)\cdot w_1^{(n)}\wedge w_2^{(n)}\wedge\cdots\wedge\Theta\Theta^*w_j^{(n)}\wedge\cdots\wedge w_p^{(n)}.$$

§7. De Rham-Hodge-Kodaira decompositions associated with Hida derivative.

The purpose of this section is to introduce two distinct decomposition theorems of de Rham-Hodge-Kodaira type [8] (R-H-K type for short). Similar results in infinite dimensional analysis or stochastic analysis may be found in [2] & [31]. It is quite natural to employ the weak derivative in some sense in order to define the exterior differentials on forms, instead we do adopt the Hida differential to realize it. This is only our unique point, compared with other related works. Our decompositions being supplying with interesting and stimulating objects in mathematical physics, namely, with those especially oriented to analysis of Dirac operators in quantum physics, are naturally derived as one of applications in terms of our Laplacians constructed in the previous sections, which can be said to be the R-H-K type theorems associated with Hida derivative in WNA.

For $p\in\boldsymbol{N}_+$ we define

$$D^{\infty}(\Delta_p(\Theta)):=\bigcap_{m\in\boldsymbol{N}}\operatorname{Dom}(\Delta_p(\Theta)^m),$$

Moreover, for $\omega\in D^{\infty}(\Delta_p(\Theta))$, we define

$$\|\omega\|_k:=\left\{\sum_{l=0}^{k}\|(I+\Delta_p(\Theta))^l\omega\|^2_{L^2(E^*\to\Lambda^pK^c;d\mu)}\right\}^{1/2}$$

and denote by $H^{2,k}(\Lambda_2{}^p(K))$ the completion of $D^{\infty}(\Delta_p(\Theta))$ with respect to the norm $\|\cdot\|_k$. When we set

$$H^{2,\infty}(\Lambda_2{}^p(K)):=\bigcap_{k=0}^{\infty}H^{2,k}(\Lambda_2{}^p(K)),\tag{7.1}$$

then $(H^{2,\infty}(\Lambda_2{}^p(K)),\|\cdot\|_k)$ is a complete, countably normed space. We denote the spectrum of operator A by the symbol $\sigma(A)$. The second quantization operator $d\Gamma_1(A)$ for a selfadjoint operator A in H_C is defined by

$$(d\Gamma_1(A)\omega)(x)=\sum_{k=1}^{n}\langle :x^{\otimes n}:,\ A^{\otimes I}[k]f_n\rangle,\qquad \omega\in\mathcal{P}$$

where $A^{\otimes I}[k]:=I\otimes\cdots\otimes\underset{k}{\hat{A}}\otimes\cdots\otimes I$ $(k\leqq n)$. Then $d\Gamma_1(A)$ is a uniquely determined, selfadjoint operator acting in (L^2). We define the operator $d\Gamma_2(B)$ by

$$d\Gamma_2{}^{(p)}(B):=\sum_{k=1}^{p}B^{\otimes I}[k],$$

which is a nonnegative selfadjoint operator acting in Λ^pK^c. Recall that the operator B is given by $\Theta\Theta^*$ (cf. §4). So let us write the operator acting in $\Lambda_2{}^p(K)$ as

$$\mathcal{L}_p(\Theta):=d\Gamma_1(A)\otimes I_f+I_b\otimes d\Gamma_2{}^{(p)}(B)\tag{7.2}$$

with identities: $I_b:=I_{(L^2)}$, $I_f:=I_{\Lambda^pK^c}$. Further we define the unique nonnegative selfadjoint operator $\Gamma_1(A)$ acting in (L^2) by

$$\Gamma_1(A):=S^{-1}\Big(\sum_{n=0}^{\infty}A^{\otimes n}\Big)S,$$

where S is the S-transform (see (2.8)). Then it holds that

$$\Gamma_1(A)\omega(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:,\ A^{\otimes n}f_n\rangle$$

for $\omega\in(L^2)$, with $\Gamma_1(A)\mathbf{1}=\mathbf{1}$ (see (2.9)). The nonnegative selfadjoint operator

$\Gamma_2(B)$ in $\Lambda^p K^c$ is defined by

$$\Gamma_2^{(p)}(B) := \otimes^p B\,, \qquad (p \geqq 0)\,.$$

Let

$$\Gamma_p(\Theta) := \Gamma_1(A) \otimes \Gamma_2^{(p)}(B)$$

acting in $\Lambda_2^p(K)$. For $\omega \in \mathrm{Dom}\,(\Gamma_p(\Theta)^k)$ $(k \geqq 1)$, we define the norm

$$|||\omega|||_k := \|(I + \Gamma_p(\Theta))^k \omega\|_{\Lambda_2^p(K)}\,,$$

and denote by $(S)_k(\Lambda^p K)$ the completion of $\mathrm{Dom}\,(\Gamma_p(\Theta)^k)$ with respect to the inner product induced by the norm $|||\cdot|||_k$. Then $(S)_k(\Lambda^p K)$ becomes a Hilbert space. Set

$$(S)(\Lambda^p K) := \bigcap_{k=1}^{\infty} (S)_k(\Lambda^p K)\,. \tag{7.3}$$

$((S)(\Lambda^p K),\ |||\cdot|||_k)$ is a complete, countably normed space.

Now we shall state the first decomposition theorem:

THEOREM 7.1 ([8], 1992). *Suppose that* $\inf \sigma(\Theta^*\Theta) \setminus \{0\} > 0$. *Then the decomposition of R-H-K type*

$$H^{2,\infty}(\Lambda_2^p(K)) = \mathrm{Im}\,[\Delta_p(\Theta) \restriction H^{2,\infty}(\Lambda_2^p(K))] \oplus \mathrm{Ker}\,\Delta_p(\Theta) \tag{7.4}$$

holds for all $p \in \boldsymbol{N}_+$.

We need the following lemma:

LEMMA 7.2. *For all* $p \in \boldsymbol{N}_+$, *we have*

$$\Delta_p(\Theta) = \mathcal{L}_p(\Theta) \qquad (\textit{in } \Lambda_2^p(K)) \tag{7.5}$$

holds in operator equality sense.

PROOF OF LEMMA 7.2. We put

$$\hat{\Xi}_m^{\hat{\otimes} n}[\eta_*\,;\,\Theta^*\Theta] := \eta_{\alpha(1),n} \hat{\otimes} \cdots \hat{\otimes} \Theta^*\Theta(\eta_{\alpha(m),n}(t)) \hat{\otimes} \cdots \hat{\otimes} \eta_{\alpha(n),n}\,.$$

A simple computation with Proposition 6.3 and the recursive relation of the Wick ordering (cf. §2) gives

$$\begin{aligned}\Delta_p(\Theta)\omega(x) = &\sum_{m=1}^{n}\Big(\sum_{n=1}^{k} \langle :x^{\otimes n}:,\ \sum_{\alpha} b_\alpha \cdot \hat{\Xi}_m^{\hat{\otimes} n}[\eta_*\,;\,\Theta^*\Theta]\rangle\Big) \\ &\cdot w_1^{(n)} \wedge \cdots \wedge w_p^{(n)} \\ &+ \sum_{j=1}^{p}\Big(\sum_{n=1}^{k} \tilde{P}_n(x) \cdot w_1^{(n)} \wedge \cdots \wedge \Theta\Theta^* w_j^{(n)} \wedge \cdots \wedge w_p^{(n)}\Big) \\ = &(d\Gamma_1(\Theta^*\Theta) \otimes I_f)\omega(x) + (I_b \otimes d\Gamma_2^{(p)}(\Theta\Theta^*))\omega(x)\,,\end{aligned}$$

which implies that (7.6) holds on $\mathscr{P}(\Lambda^p K^c)$. Clearly $\mathcal{L}_p(\Theta)$ is essentially selfadjoint on $\mathscr{P}(\Lambda^p K^c)$, since so is $d\Gamma_1(A)$ (resp. $d\Gamma_2^{(p)}(B)$) on $\mathscr{P}$ (resp. $\mathscr{P}(\Lambda^p K^c)$). Therefore the closedness verifies the assertion. q.e.d.

PROOF. By virtue of the spectral property of the second quantization operators and the Deift theorem [4] for commutation formulae of operators, it follows immediately from Lemma 7.2 that $\inf \sigma(\Delta_p(\Theta))\setminus\{0\}>0$. Obviously we have

$$\Lambda_2^p(K)=\mathrm{Im}\,(\Delta_p(\Theta))\oplus \mathrm{Ker}\,\Delta_p(\Theta)\,.$$

Roughly speaking, the matter is whether $\Lambda_2^p(K)$ should be replaced with $H^{2,\infty}(\Lambda_2^p(K))$ when we put restriction on the domain of $\Delta_p(\Theta)$ to $H^{2,\infty}(\Lambda_2^p(K))$ in the right hand side. However, clearly this turns to be true. An application of the spectral representation theorem leads to

$$D^\infty(\Delta_p(\Theta))=\mathrm{Im}\,[\Delta_p(\Theta)\restriction D^\infty(\Delta_p(\Theta))]\otimes \mathrm{Ker}\,\Delta_p(\Theta)\,.$$

To complete the proof we have only to note that $H^{2,\infty}(\Lambda_2^p(K))$ is isomorphic to $D^\infty(\Delta_p(\Theta))$ as a vector space. q.e.d.

REMARK 7.1. In Lemma 7.2, when $p=0$ then we have $\mathcal{L}_0(\Theta)=d_0^*(\Theta)\cdot\bar{d}_0(\Theta)$, which is, of course, a nonnegative and selfadjoint operator. This is due to von Neumann theorem.

REMARK 7.2. It is generally right that the heat equation method is even effective for the proof of decomposition theorem on the space of the type like $H^{2,\infty}(\Lambda_2^p(K))$. In fact, similar works on R-H-K type decompositions by Shigekawa [31] and Arai-Mitoma [2] are greatly due to the heat equation method. □

Finally we shall introduce our second decomposition theorem for the space $(S)(\Lambda^p K)$ (see Theorem 7.8). However, since the structure of $(S)(\Lambda^p K)$ is different from that of $H^{2,\infty}(\Lambda_2^p(K))$, the heat equation method is not applicable any more to the case. So necessity will occur that we have resort to the Arai-Mitoma method. Their method is principally due to a comparison theorem, which is derived by a series of finer estimates based on precise computation of weighted norms. There the spectral theory plays again an essential role in reduction of the problem, representation of the operators, and precise estimates. Before mentioning the decomposition theorem we need to prepare for the basic estimates whereby the nice property of our Laplacians reveals itself, namely, our Laplacians do serve as desired operators which map the space of smooth p-forms into itself (see Theorem 7.7 below).

LEMMA 7.3. *Suppose that*

$$\Theta^*\Theta \geqq (1+\varepsilon) I_{H_C} \tag{7.6}$$

holds with a positive constant ε. *Then for each* $s>0$, *all* $p \in \boldsymbol{N}_+$ *and* $k \in \boldsymbol{N}_+$, *there exists a positive constant* $C_0(\varepsilon, k)$ *and there can be found a proper positive integer* k_0 *such that the inequality*

$$\|\tilde{T}_s^{-1}(I+\Delta_p(\Theta))^k \omega\|_{\Lambda_2^p(K)} \leqq C_0(\varepsilon, k) \cdot \|(I+\Gamma_p(\Theta))^{k_0} \omega\|_{\Lambda_2^p(K)} \tag{7.7}$$

holds for every $\omega \in \mathrm{Dom}(\Gamma_p(\Theta)^{k_0})$, *where* $\tilde{T}_s^{-1} := \Gamma_1(\mathrm{e}^s) \otimes I_f$. □

The proof is an easy exercise. It follows from the spectral theory and the fundamental properties of Ornstein-Uhlenbeck semigroups.

REMARK 7.3. We write the Ornstein-Uhlenbeck semigroup (e.g. [33]) on (L^2) as $T_s := \Gamma_1(\mathrm{e}^{-s})$, $s \geqq 0$. There exists its inverse operator T_s^{-1} being self-adjoint, which is given qy $T_s^{-1} = \Gamma_1(\mathrm{e}^s)$, $s \geqq 0$. Moreover, its natural extension to $\Lambda_2^p(K)$ is written as $\tilde{T}_s^{-1} := \Gamma_1(\mathrm{e}^s) \otimes I_f$, which appeared in the above (7.7). □

As a direct corollary of Lemma 7.3 we readily obtain

LEMMA 7.4. *Under the assumption* (7.6), *for all* $p \in \boldsymbol{N}_+$ *and* $k \in \boldsymbol{N}$ *there exists a positive constant* $C_1(\varepsilon, k)$ *such that the inequality*

$$\|(I+\Delta_p(\Theta))^k \omega\|_{\Lambda_2^p(K)} \leqq C_1(\varepsilon, k) \cdot \|(I+\Gamma_p(\Theta))^k \omega\|_{\Lambda_2^p(K)}$$

holds for every $\omega \in \mathrm{Dom}(\Gamma_p(\Theta)^k)$. □

Therefore, by repeating the reduction to the subspace $\mathcal{K}_n^p := K_n \otimes \Lambda^p K^c$ and employing the limiting proceeding for the acquired relative to $\mathcal{P}(\Lambda^p K^c)$, we can easily see that

LEMMA 7.5. *Under the assumption* (7.6) *we have*

$$\mathrm{Dom}(\Gamma_p(\Theta)^k) \subset \mathrm{Dom}(\Delta_p(\Theta)^k)$$

for all $k \in \boldsymbol{N}$ *and* $p \in \boldsymbol{N}_+$. □

The next proposition is a comparison theorem for the spaces $H^{2,\infty}(\Lambda_2^p(K))$ and $(S)(\Lambda^p K)$, whereby our second decomposition can be derived according to the Arai-Mitoma theory. One may find some of familiar techniques and methods useful and effective in this argument as well, and those have been used well in the Malliavin calculus [33].

PROPOSITION 7.6. *Suppose* (7.6). *Then the inclusion*

$$(S)(\Lambda^p K)\subset H^{2,\infty}(\Lambda_2{}^p(K)) \tag{7.8}$$

holds for all $p\in \boldsymbol{N}_+$.

As to the proof it is sufficient to show that $\|\gamma\|_k\leqq\tilde{C}|||\gamma|||_N$, $(\gamma\in\mathscr{P}(\Lambda^p K^c))$, for any $N>k$ $(N, k\in\boldsymbol{N})$, each $p\in\boldsymbol{N}_+$, and some positive constant $\tilde{C}$. In fact, an application of Khinchin's inequalities yields the assertion by virtue of hyper-contractivity of T_s. The next assertion indicates that our Laplacians have such a nice property as stated in § 1.

THEOREM 7.7. *Under the assumption* (7.6) *we have*

$$\Delta_p(\Theta)[(S)(\Lambda^p K)]\subset(S)(\Lambda^p K) \tag{7.9}$$

for all $p\in\boldsymbol{N}_+$.

It is sufficient to prove

$$\Delta_p{}^n(\Theta)\omega\in\mathrm{Dom}\,(\Gamma_p{}^n(\Theta)^k)\,,$$

for $\omega\in(S)_k(\Lambda^p K)\cap\mathcal{K}_n{}^p$, all $k\in\boldsymbol{N}$, and each $p\in\boldsymbol{N}_+$. It is easy, hence omitted. Ultimately, we are now in a position to state our R-H-K type decomposition theorem for $(S)(\Lambda^p K)$.

THEOREM 7.8 ([8], 1992). *Assume the condition* (7.6). *Then the space* $(S)(\Lambda^p K)$ *admits the decomposition*

$$(S)(\Lambda^p K)=\mathrm{Im}\,[\Delta_p(\Theta)\restriction(S)(\Lambda^p K)]\oplus\mathrm{Ker}\,\Delta_p(\Theta) \tag{7.10}$$

for all $p\in\boldsymbol{N}_+$.

PROOF. According to Theorem 7.1 and Proposition 7.6 the element ω of $(S)(\Lambda^p K)$ is decomposed into

$$\omega=\omega_1+\omega_2=\Delta_p(\Theta)\eta+\omega_2\,,$$

with $\omega_1\in\mathrm{Im}\,[\Delta_p(\Theta)\restriction H^{2,\infty}(\Lambda_2{}^p(K))]$, $\omega_2\in\mathrm{Ker}\,\Delta_p(\Theta)$, and

$$\eta=Q_p(\Theta)\omega=\int_0^\infty(K_s(p\,;\,\Theta)\omega-\omega_2)ds\in H^{2,\infty}(\Lambda_2{}^p(K))\,,$$

where $K_s(p\,;\,\Theta):=\int_0^\infty e^{-s\lambda}dE_p(\Theta\,;\,\lambda)$, $(s\geqq0)$ and $\{E_p(\Theta\,;\,\lambda)\,;\,\lambda\in\boldsymbol{R}\}$ is a family of spectral measures associated with the operator $\Delta_p(\Theta)$. Because of (7.9), it results from the following lemma:

LEMMA 7.9. *Under the condition* (7.6) *we have*

$$Q_p(\Theta)\omega\in(S)_k(\Lambda^p K)\,, \qquad (\omega\in\mathrm{Dom}\,(\Gamma_p(\Theta)^k))$$

for all $k\geqq 1$, *each* $p\in \boldsymbol{N}_+$. *q.e.d.* □

§ 8. Concluding remarks.

After having finished writing this paper, the author learned that H.-H. Kuo, J. Potthoff, and J.-A. Jan had obtained very useful and important results in "Continuity of affine transformations of white noise test functionals and applications", Stochastic Processes and their Applications **43** (1992), 85–98. They succeeded in obtaining a direct simple proof of the fact that the space of white noise test functionals is infinitely differentiable in Fréchet sense, which is closely related to our results in § 3. We found it very interesting and suggestive, and stimulating as well.

In addition, we were informed of the publication of H.-H. Kuo's paper entitled "Lectures on white noise analysis", which appeared as Special Invited Paper in Soochow J. Math. **18** (1992), 229–300. There can be found at pp. 251–266 very interesting and remarkable descriptions about a variety of differential operators in white noise analysis, which are deeply connected with the contents of § 3 and § 5 in our paper (cf. [12–15]). Especially so excellent are his works on the characteristics of various sorts of Laplacians (pp. 279–249) via an infinite dimensional version of the Fourier transform which is compatible with Hida calculus (see [7], [26]).

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