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COMPLETE MAXIMAL SPACE-LIKE HYPERSURFACES IN AN ANTI-DE SITTER SPACE OF DIMENSION 4

By

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Introduction.

Let $\boldsymbol{R}_1^{n+1}$ be an $(n+1)$-dimensional Minkowski space and $S_1^{n+1}(c)$ (resp. $H_1^{n+1}(c)$) an $(n+1)$-dimensional de Sitter space (resp. an anti-de Sitter space) of constant curvature c. The class of these indefinite Riemannian manifolds of constant curvature c and with index 1 is called a *Lorentz space form*, which is denoted by $M_1^{n+1}(c)$. A submanifold M of a Lorentz space form $M_1^{n+1}(c)$ is said to be *space-like* if an induced metric on M from that of the ambient space is positive definite. After the study of Calabi [3] and Cheng and Yau [6] about the Bernstein type property for maximal space-like hypersurfaces in a Minkowski space $\boldsymbol{R}_1^{n+1}$, complete space-like hypersurfaces with constant mean curvature in a Lorentz space form have been studying by many geometers. As standard models of not totally umbilic space-like hypersurfaces with constant mean curvature in a Lorentz space form $M_1^{n+1}(c)$ there exists a class of hypersurfaces $H^k(c_1)\times M^{n-k}(c_2)$, where $k=1, \cdots, n-1$, where $H^m(c)$ (resp. $M^m(c)$) is an m-dimensional hyperbolic space (resp. a space form) of constant curvature c. In the case of $k=1$, it is called a *hyperbolic cylinder*. In particular, when it is maximal, c_1 and c_2 satisfy $c_1=nc/k$ and $c_2=nc/(n-k)$.

Now, for a complete minimal hypersurface in $S^{n+1}(1)$ with constant scalar curvature, Chern pointed out that it seems to be interesting to study the distribution of the value of the squared norm of the second fundamental form, and Peng and Terng [10] and Cheng [4] partially realized the aim in $S^4(1)$. Relative to the problem the similar case for space-like hypersurfaces with constant mean curvature H of $M_1^4(c)$ is recently classified by Aiyama and Cheng [2]. Numbers S_0 and $S_\pm$ are defined by

$$S_0=\frac{h^2}{3} \quad \text{and} \quad S_\pm=-3c+\frac{1}{4}\left(3h^2\pm\sqrt{h^4-8ch^2}\right),$$

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where $h=3H$ and $S_0<S_-\leqq S_+$. They prove that a 3-dimensional hyperbolic cylinder is the only complete space-like hypersurface with non-zero constant mean curvature, constant scalar curvature and $S>S_-$. However, there are no informations about the case $S\leqq S_-$. The purpose of this paper is to investigate the case $S<S_-$ in the maximal hypersurface and to prove the following theorem which is the Lorentz version in $H_1^4(c)$ about Chern's problem.

THEOREM. *Let M be a 3-dimensional complete maximal space-like hypersurface with constant scalar curvature in an anti-de Sitter space $H_1^4(c)$. If $-kc<S\leqq -3c$, $k\doteqdot 2.64$, then M is congruent to the hyperbolic cylinder $H^1(c_1)\times H^2(c_2)$.*

1. Preliminaries.

Let (M, g) be a space-like hypersurface in an $(n+1)$-dimensional Lorentz space form $M_1^{n+1}(c)$. We choose a local field of orthonormal frames $e_1, \cdots, e_n$ adapted to the Riemannian metric induced from the indefinite Riemannian metric on the ambient space and let $\omega_1, \cdots, \omega_n$ denote the dual coframes on M. The connection forms $\{\omega_{ij}\}$ on M are characterized by the structure equations

$$(1.1)\qquad \begin{cases} d\omega_i+\sum\omega_{ij}\wedge\omega_j=0\,, \quad \omega_{ij}+\omega_{ji}=0\,, \\ d\omega_{ij}+\sum\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}\,, \\ \Omega_{ij}=-\dfrac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l\,, \end{cases}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. components of the Riemannian curvature tensor R) of M. The second fundamental form α with values in the normal bundleu is given by $\alpha=-\sum h_{ij}\omega_i\omega_j e_0$, where e_0 is a unit time-like normal vector and the mean curvature H of M is given by $H=h/n=\sum h_{jj}/n$.

The Gauss equation, the Codazzi equation and the Ricci formula for the second fundamental form are given by

$$(1.2)\qquad R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})-h_{il}h_{jk}+h_{ik}h_{jl}\,,$$

$$(1.3)\qquad h_{ijk}-h_{ikj}=0\,,$$

$$(1.4)\qquad h_{ijkl}-h_{ijlk}=-\sum h_{rj}R_{rikl}-\sum h_{ir}R_{rjkl}\,,$$

$$(1.5)\qquad h_{ijklm}-h_{ijkml}=-\sum h_{rjk}R_{rilm}-\sum h_{irk}R_{rjlm}-\sum h_{ijr}R_{rklm}\,,$$

where h_{ijk}, h_{ijkl} and h_{ijklm} denote components of the covariant differentials $\nabla\alpha$, $\nabla^2\alpha$ and $\nabla^3\alpha$ of α, respectively.

We denote by R_{ij} components of the Ricci curvature tensor *Ric*. The Ricci tensor R_{ij} and the scalar curvature r are given by

$$R_{ij}=(n-1)c\delta_{ij}-hh_{ij}+\sum h_{ik}h_{kj}, \tag{1.6}$$

$$r=n(n-1)c-h^2+\sum h_{ij}^2, \tag{1.7}$$

Now, we compute some local formulas under the assumption that the mean curvature of M is constant. First of all, by making use of (1.4) and by taking account of the Codazzi equation (1.3), the Gauss equation (1.2) and the Bianchi equation, it is well known that the Laplacian of the second fundamental form is given by

$$\Delta h_{ij}=c(nh_{ij}-h\delta_{ij})-h\sum h_{ik}h_{kj}+f_2h_{ij}, \tag{1.8}$$

where $f_2=\sum h_{ij}^2$. For simplicity we put $f_m=\sum h_{ik_1}h_{k_1k_2}\cdots h_{k_{m-1}i}$ for any positive integer m. In particular, we denote by S the square of the length of the second fundamental form α, i.e., $S=f_2$. By utilizing (1.8), the Laplacian of the non-negative function S can be determined as follows:

$$\frac{1}{2}\Delta S=\sum h_{ijk}^2+\sum h_{ij}\Delta h_{ij}=\sum h_{ijk}^2-hf_3+S(S+nc)-ch^2. \tag{1.9}$$

On the other hand we easily see that

$$\frac{1}{2}\Delta h_{ijk}^2=\sum h_{ijkl}^2+\sum h_{ijk}\Delta h_{ijk}.$$

By a similar and direct computation to the argument above we have

$$\begin{aligned}\Delta h_{ijk}=&\sum h_{ir}R_{rjk}+\sum h_{ikr}R_{rj}+\sum h_{ijr}R_{rk}\\ &-\sum h_{rs}R_{rijsk}-\sum h_{jr}R_{riks s}-\sum h_{ir}R_{rjkss}\\ &-\sum h_{krs}R_{rijs}-2\sum h_{jrs}R_{riks}-2\sum h_{irs}R_{rjks},\end{aligned}$$

where R_{ijk} denote components of the covariant differential ∇Ric of the Ricci tensor Ric. Thus one finds

PROPOSITION 1.1. *Let M be an n-dimensional space-like hypersurface with constant mean curvature in a Lorentz space form $M_1^{n+1}(c)$. Then we have*

$$\frac{1}{2}\Delta S=|\nabla\alpha|^2-hf_3+S(S+nc)-ch^2, \tag{1.10}$$

$$\frac{1}{2}\Delta|\nabla\alpha|^2=|\nabla^2\alpha|^2+\{S+(2n+3)c\}|\nabla\alpha|^2+3A-6B-3hC+\frac{3}{2}|\nabla S|^2, \tag{1.11}$$

$$\frac{1}{3}\Delta f_3=-hf_4+(S+nc)f_3-chS+2C, \tag{1.12}$$

$$\frac{1}{4}\Delta f_4=-hf_5+(S+nc)f_4-chf_3+2A+B, \tag{1.13}$$

where we have put

$$A=\sum h_{ij}{}^2 h_{ikl}h_{jkl},\quad B=\sum h_{ijk}h_{irs}h_{jr}h_{ks}\quad \text{and}\quad C=\sum h_{ijk}h_{ijl}h_{kl}.$$

REMARK. The equation (1.11) is obtained by Treibergs [11] in the case where the ambient space is a Minkowski space. These equations are recently obtained also by Aiyama and Cheng [2].

The generalized maximum principle due to Omori [9] and Yau [12] and a Lorentz version due to Nomizu [8] of Cartan's formula for isoparametric hypersurfaces are next introduced. A space-like hypersurface in a Lorentz space form is said to be *isoparametric,* if all principal curvatures are constant.

THEOREM 1.2. *Let M be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from above on M. For any positive number ε there exists a point p in M such that*

$$F(p)>\sup F-\varepsilon,\quad |\nabla F(p)|<\varepsilon,\quad \Delta F(p)<\varepsilon,$$

where ∇F denotes a gradient of the function F.

THEOREM 1.3. *Let M be an isoparametric space-like hypersurface in a Lorentz space form $M_1^{n+1}(c)$. Let $\lambda_1, \cdots, \lambda_p$ are all constant distinct principal curvatures of M with multiplicities $m_1, \cdots, m_p$, respectively. Then we have*

$$\sum_{j\neq i} m_j \frac{c-\lambda_j\lambda_i}{\lambda_j-\lambda_i}=0.$$

2. Isoparametric hypersurfaces.

This section is concerned with isoparametric space-like hypersurfaces in $H_1^4(c)$. Let M be a 3-dimensional space-like hypersurface with constant mean curvature in a 4-dimensional anti-de Sitter space $H_1^4(c)$. For any point x in M we can choose a local field $\{e_1, \cdots, e_4\}$ of orthonormal frames in such a way that $h_{ij}=\lambda_i\delta_{ij}$, where λ_i denotes a principal curvature. Without loss of generality we may assume that

$$\lambda_1\leqq\lambda_2\leqq\lambda_3.$$

PROPOSITION 2.1. *There does not exist an isoparametric space-like hypersurface in an anti-de Sitter space $H_1^4(c)$ with distinct principal curvatures with each other.*

PROOF. Let M be an isoparametric space-like hypersurface in $H_1^4(c)$ and

λ_1, λ_2 and λ_3 distinct principal curvatures. By Theorem 1.2 we have

$$\frac{c-\lambda_1\lambda_2}{\lambda_2-\lambda_1}+\frac{c-\lambda_1\lambda_3}{\lambda_3-\lambda_1}=0,$$

from which combining with $\lambda_1+\lambda_2+\lambda_3=h$ it follows that we have

$$\lambda_1^3-h\lambda_1^2+(2\lambda_2\lambda_3+3c)\lambda_1-ch=0.$$

Similarly the following equations are given by the Cartan-Nomizu formula:

$$\tag{2.1}\begin{cases}\lambda_1^3-h\lambda_1^2+(2\lambda_2\lambda_3+3c)\lambda_1-ch=0,\\ \lambda_2^3-h\lambda_2^2+(2\lambda_3\lambda_1+3c)\lambda_2-ch=0,\\ \lambda_3^3-h\lambda_3^2+(2\lambda_1\lambda_2+3c)\lambda_3-ch=0.\end{cases}$$

The first and the second equations of (2.1) and the assumption $\lambda_1\neq\lambda_2$ give us

$$(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)-h(\lambda_1+\lambda_2)+3c=0.$$

So, by the equation (2.1) we have

$$\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1=3c.$$

Accordingly we obtain $S=h^2-6c$. It is seen by Cheng and Nakagawa [5] that the estimate of the above bound of the squared norm S is given as $S\leqq S_+$, from which combining with $S=h^2-6c$ it follows that we have $c(h^2-9c)\geqq 0$, a contradiction. □

REMARK. In the case of the sphere $S^4(c)$, there exists an isoparametric hypersurface in $S^4(c)$ with distinct principal curvatures with each other.

3. Proof of Theorem.

In this section we shall prove the main theorem in the introduction. Let M be a 3-dimensional complete maximal space-like hypersurface in an anti-de Sitter space $H_1^4(c)$ and let λ_1, λ_2 and λ_3 be principal curvatures. Without loss of generality we may assume that

$$\lambda_1\leqq\lambda_2\leqq\lambda_3.$$

As is easily seen, the fact that the scalar curvature r is constant is equivalent to the property that the squared norm S of α is constant. By the assumption of the theorem and by the definition of the function f_3 we have

$$\tag{3.1}\begin{cases}\lambda_1+\lambda_2+\lambda_3=0,\\ \lambda_1^2+\lambda_2^2+\lambda_3^2=S=\mathit{constant},\\ \lambda_1^3+\lambda_2^3+\lambda_3^3=f_3.\end{cases}$$

PROOF OF THEOREM. First of all we notice that the assumption $0<S$ means that M is not totally geodesic. Suppose that f_3 is constant. From the assumption of the theorem, M is isoparametric. So, by Proposition 2.1 two of principal curvatures λ_1, λ_2 and λ_3 are equal. By a theorem due to Abe, Koike and Yamaguchi [1], M is congruent to the hyperbolic cylinder $H^1(c_1)\times H^2(c_2)$ and $S=-3c$.

Next, we show that if $0<-2.64c<S\leqq-3c$, then f_3 is constant. We suppose that f_3 is not constant. Then $S<-3c$. First we suppose that there does not exist a point q at which $f_3(q)=0$. By the continuity of the function f_3 we may suppose that f_3 is negative without loss of generality. By the Gauss equation the Ricci curvature is bounded from below and the function f_3 is bounded from above by 0, and so we can apply Theorem 1.2 to f_3. For any positive number ε there exists a point p in M such that

$$(3.2) \qquad |\nabla f_3(p)|<\varepsilon, \quad \Delta f_3(p)<\varepsilon, \quad f_3(p)>\sup f_3-\varepsilon.$$

By the first and the second equations of (3.1), solving the problem for the conditional extremum we lead to

$$|f_3|\leqq\sqrt{\frac{S^3}{6}},$$

where the equality holds if and only if two of principal curvatures are equal. Since f_3 is not constant, we get

$$(3.3) \qquad -\sqrt{\frac{S^3}{6}}<\sup_M f_3\leqq 0\,.$$

We observe from (3.1) and (3.3) that λ_1, λ_2 and λ_3 are mutually distinct on $M'=\{x\in M: f_3(x)\neq-\sqrt{S^3/6}\}$. By taking account of the assumption that $\sum h_{ii}=0$ and $\sum h_{ij}{}^2=S=constant$, the exterior differentiation implies

$$(3.4) \qquad \sum h_{iik}=0, \quad \sum \lambda_i h_{iik}=0$$

for any index k at any point x in M. Also we define numbers $\delta_k(x)$ ($k=1, 2, 3$) by

$$(3.5) \qquad \sum \lambda_i{}^2 h_{iik}(x)=\delta_k(x)$$

for any index k. Then equations (3.4) and (3.5) can be regarded simultaneous equations with 3 unknown $h_{11k}(x)$, $h_{22k}(x)$ and $h_{33k}(x)$, and the unique solution is given by

$$h_{iik}(x)=a_i(x)\delta_k(x)$$

for index i, $k=1, 2, 3$ at any point x in M'. For any positive number $\varepsilon(<\sup f_3+\sqrt{S^3/6})$ in (3.2), let M_0 be the connected component containing the point p

in (3.2) of $\{x\in M: f_3(x)>\sup f_3-\varepsilon\}$. Then M_0 is contained in M'. Since all principal curvatures λ_1, λ_2 and λ_3 satisfy

$$|\lambda_i|\leqq\sqrt{S} \tag{3.6}$$

by (3.1), it follows from (3.4) that there exists a positive number $c_1=c_1(p, \varepsilon)$ such that $|a_k(x)|<c_1$ for any point x in M_0 and $i=1, 2, 3$. Furthermore we have $|\delta_k(p)|<\varepsilon/3$ at the point p in M satisfying (3.2) and we also have

$$|h_{iik}(p)|<\frac{1}{3}c_1\varepsilon \tag{3.7}$$

for any indices i and k. By (1.12) we have

$$\begin{aligned}\Delta f_3&=3\{(S+3c)f_3+2\sum\lambda_i h_{ijk}{}^2\}=3(S+3c)f_3+2\sum(\lambda_i+\lambda_j+\lambda_k)h_{ijk}{}^2\\&=3(S+3c)f_3+12(\lambda_1+\lambda_2+\lambda_3)h_{123}{}^2+6\sum_{i\neq k}(2\lambda_i+\lambda_k)h_{iik}{}^2+6\sum_i\lambda_i h_{iii}{}^2,\end{aligned}$$

from which combining with (3.2), (3.6) and (3.7) it follows that we have

$$\varepsilon>\Delta f_3(p)>3(S+3c)f_3(p)-14\sqrt{S}c_1{}^2\varepsilon^2 .$$

Thus there exists a positive constant $c_2=1/3+14/3\sqrt{S}c_1{}^2\varepsilon$ such that

$$(S+3c)f_3(p)<c_2\varepsilon ,$$

where c_2 converges to 1/3 if ε tends to zero.

For any convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m\to 0$ $(m\to\infty)$ and $\varepsilon>0$, there exists a point sequence $\{p_m\}$ such that the sequence $\{f_3(p_m)\}$ converges to $\sup f_3$ by (3.2), from which together with the last inequality we have a positive constant $c_2'(m)=c_2'(S, \varepsilon_m)$ such that

$$(S+3c)f_3(p_m)<c_2'(m)\varepsilon_m .$$

It implies that $\sup_M f_3\geqq 0$ because $S+3c$ is negative and hence, by means of the supposition that the function f_3 is negative we have

$$\sup_M f_3=0 . \tag{3.8}$$

Now, under this condition we observe the estimation of the function $|\nabla^2\alpha|$. Since $|\nabla\alpha|$ is constant by (1.10) and the assumption, (1.11) means that the value of $|\nabla^2\alpha|$ is determined by the function $A-2B$. So, in order to obtain the estimate of the lower bound of the function we have

$$\begin{aligned}3(A-2B)&=\sum(\lambda_i^2+\lambda_j^2+\lambda_k^2-2\lambda_i\lambda_j-2\lambda_j\lambda_k-2\lambda_k\lambda_i)h_{ijk}^2\\&=\sum_{i\neq j\neq k\neq i}\{2(\lambda_i^2+\lambda_j^2+\lambda_k^2)-(\lambda_i+\lambda_j+\lambda_k)^2\}h_{ijk}^2\\&\quad+3\sum_{i\neq k}(\lambda_k^2-4\lambda_i\lambda_k)h_{iik}^2-3\sum\lambda_i^2h_{iii}^2\\&=2S\sum h_{ijk}^2+3\sum_{i\neq k}(\lambda_k^2-4\lambda_i\lambda_k-2S)h_{iik}^2-\sum(3\lambda_i^2+2S)h_{iii}^2.\end{aligned}$$

Accordingly it follows from this equation, (1.10), (3.6) and (3.7) that there exists a positive constant $c_3=(41/3)Sc_1^2$ such that

$$3(A-2B)(p)>-2S^2(S+3c)-c_3\varepsilon^2. \tag{3.9}$$

On the other hand, in order to estimate the value of $|\nabla^2\alpha|$ from itself, we shall analyze the detail of its term. First we put

$$t_{ij}=h_{ijij}-h_{jiji}. \tag{3.10}$$

Then the Ricci formula (1.4) and the Gauss equation (1.2) imply

$$t_{ij}=(\lambda_i-\lambda_j)(c-\lambda_i\lambda_j). \tag{3.11}$$

Hence the direct calculation implies

$$\sum_{i\neq j}t_{ij}^2=S^3+4cS^2+6c^2S-2f_3^2 \tag{3.12}$$

because of $n=3$, $H=0$ and $f_4=S^2/2$. Moreover we obtain

$$\sum_{i\neq j}h_{ijij}^2=\sum_{i<j}h_{ijij}^2+\sum_{i<j}(h_{ijij}-t_{ij})^2=\sum_{i\neq j}\left(h_{ijij}-\frac{1}{2}t_{ij}\right)^2+\frac{1}{4}\sum_{i\neq j}t_{ij}^2.$$

Thus, because of $\sum h_{ijkl}^2\geqq 3\sum_{i\neq j}h_{ijij}^2$ it is reduced to

$$\sum h_{ijkl}^2\geqq 3\sum_{i\neq j}\left(h_{ijij}-\frac{1}{2}t_{ij}\right)^2+\frac{3}{4}\sum_{i\neq j}t_{ij}^2. \tag{3.13}$$

This is the estimation of the lower bound of $|\nabla^2\alpha|$.

Taking account of (3.2) and combining (1.11) with (3.8), (3.12) and (3.13) we get

$$\begin{aligned}&S(S+3c)(S+9c)+2S^2(S+3c)+c_3\varepsilon^2\\&\geqq 3\sum_{i\neq j}\left(h_{ijij}-\frac{1}{2}t_{ij}\right)^2(p)+\frac{3}{4}S(S^2+4cS+6c^2)-\frac{3}{2}\varepsilon^2.\end{aligned} \tag{3.14}$$

Therefore, in order to investigate the range of S, it suffices to estimate the lower bound of the first term of the right hand side in the above inequality. However, by the direct computation there exists a positive constant $c_4=c_4(S, c)$ such that

$$\sum_{i\neq j}\left(h_{ijij}-\frac{1}{2}t_{ij}\right)^2>(h_{1212}-h_{2323})^2-c_4\varepsilon\,. \tag{3.15}$$

In fact, we have

$$\sum_{i\neq j}\left(h_{ijij}-\frac{1}{2}t_{ij}\right)^2=\left(h_{1212}-\frac{1}{2}t_{12}\right)^2+\left(h_{2323}-\frac{1}{2}t_{23}\right)^2+\cdots$$

$$=(h_{1212}-h_{2323})^2+2h_{1212}h_{2323}-(t_{12}h_{1212}+t_{23}h_{2323})+\frac{1}{4}(t_{12}{}^2+t_{23}{}^2)+\cdots$$

$$=(h_{1212}-h_{2323})^2+2\left(h_{1212}-\frac{1}{2}t_{23}\right)\left(h_{2323}-\frac{1}{2}t_{12}\right)+\frac{1}{4}(t_{12}-t_{23})^2+\cdots\,.$$

By (3.11) we see $t_{12}-t_{23}=-(S+3c)\lambda_2$. Because of (3.8) there exist sufficiently small numbers ε_1, ε_2 and ε_3 such that ε_1, ε_2 and ε_3 converge to zero as ε tends to zero and

$$\lambda_1(p)=-\sqrt{\frac{S}{2}}+\varepsilon_1,\quad \lambda_2(p)=\varepsilon_2,\quad \lambda_3(p)=\sqrt{\frac{S}{2}}+\varepsilon_3.$$

Accordingly, there exists a positive constant $c_5=c_5(S,c)$ such that

$$|t_{12}-t_{23}|<c_5\varepsilon\,.$$

Since the function $|\nabla^2\alpha|$ is bounded by (1.11), its upper bound depends only on S and c and we have

$$\sum_{i\neq j}\left(h_{ijij}-\frac{1}{2}t_{ij}\right)^2(p)$$

$$>(h_{1212}-h_{2323})^2(p)+2\left(h_{1212}-\frac{1}{2}t_{12}\right)\left(h_{2323}-\frac{1}{2}t_{23}\right)(p)-c_4\varepsilon+\cdots$$

$$=(h_{1212}-h_{2323})^2(p)+2\left(h_{2121}-\frac{1}{2}t_{21}\right)\left(h_{3232}-\frac{1}{2}t_{32}\right)(p)-c_4\varepsilon+\cdots$$

$$=(h_{1212}-h_{2323})^2(p)+\left\{\left(h_{2121}-\frac{1}{2}t_{21}\right)+\left(h_{3232}-\frac{1}{2}t_{32}\right)\right\}^2(p)-c_4\varepsilon+\cdots$$

$$\geqq(h_{1212}-h_{2323})^2(p)-c_4\varepsilon$$

for some positive integer $c_4=c_4(S,c)$.

Accordingly we need next the estimate of the lower bound of the first term of the above relation. We notice that $\sum h_{ijk}{}^2\geqq 6h_{123}{}^2$. Moreover we get by (1.10) and (3.7)

$$\sum_{i,j}h_{ij2}{}^2(p)<2h_{123}{}^2(p)+\frac{7}{9}c_1{}^2\varepsilon^2\leqq-\frac{1}{3}S(S+3c)+\frac{7}{9}c_1{}^2\varepsilon^2. \tag{3.16}$$

Differentiating $S=\sum h_{ij}{}^2$ twice, we have $\sum_i\lambda_i h_{iikk}+\sum_{i,j}h_{ijk}{}^2=0$ for $k=1,2,3$. So there exists a positive constant $c_6=c_6(S,c)$ such that

$$\sum_{i,j} h_{ij2}{}^2(p) > \sqrt{\frac{S}{2}}(h_{1122}-h_{3322})(p) - c_6\varepsilon\,,$$

because $|\nabla^2\alpha|$ is bounded, from which together with (3.16) it follows that there is a positive constant $c_7=c_7(S, c, \varepsilon)$ such that

$$(h_{1122}-h_{3232})(p) < -\frac{\sqrt{2S}}{3}(S+3c)+c_7\varepsilon\,.$$

Furthermore, by (3.11) we have a constant $c_8=c_8(\varepsilon_1, \varepsilon_2, \varepsilon_3, S, c)=c_8(S, c, \varepsilon)$ such that

$$t_{23}(p) > -c\sqrt{\frac{S}{2}}+c_8\varepsilon\,.$$

Thus, combining above two inequalities we have

$$(h_{1212}-h_{2323})(p)=(h_{1212}-h_{3232})(p)-t_{23}(p) < -\frac{\sqrt{2S}}{6}(2S+3c)+c_9\varepsilon \tag{3.17}$$

for a certain constant $c_9=c_9(c_7, c_8)=c_9(S, c, \varepsilon)$. Because of $2S+3c>0$, we can suppose that the right hand side of the above inequality is negative for a sufficiently small positive number ε. Thus we can get the lower bound of the first term of the right hand side of (3.15). Combining some results obtained above, we can show the existence of the zero point of the function f_3. In fact, from (3.14), (3.15) and the above equation (3.17) we have

$$S(S+6c)(19S+42c) > -c_{10}\varepsilon$$

for a certain constant $c_{10}=c_{10}(c_3, c_4, c_9)$ and any positive number ε, that is,

$$S(S+6c)(19S+42c)\geqq 0\,,$$

which shows $S\leqq -42c/19$, a contradiction. Thus there is a point q such that $f_3(q)=0$.

Before proving the theorem we give some formulas at the point q. First, the values of principal curvatures at that point are given by

$$\lambda_1=-\sqrt{\frac{S}{2}}\,,\quad \lambda_2=0\,,\quad \lambda_3=\sqrt{\frac{S}{2}}\,.$$

Similar to (3.4) we have $\sum_i h_{iik}=0$ and $\sum_i \lambda_i h_{iik}=0$ at the point q. The following relations can be verified from these equations and the value of principal curvatures at that point:

$$h_{11k}=h_{33k}\,,\quad h_{22k}=-2h_{11k} \tag{3.18}$$

for every k. By substituting (3.18) into $\sum h_{ijk}{}^2$, it gives

$$\sum h_{ijk}{}^2 = 6h_{123}{}^2+16h_{111}{}^2+\frac{5}{2}h_{222}{}^2+16h_{333}{}^2=-S(S+3c)\,. \tag{3.19}$$

Furthermore we see

$$\sum_{i,j} h_{ij2}{}^2 \geqq -\frac{1}{3} S(S+3c). \tag{3.20}$$

In order to prove (3.20), it suffices to notice that

$$\sum_{i,j} h_{ij2}{}^2 = 2h_{123}{}^2 + 8h_{111}{}^2 + \frac{3}{2} h_{222}{}^2 + 8h_{333}{}^2 .$$

From (3.19) and the last equation we have

$$\sum_{i,j} h_{ij2}{}^2 \leqq -\frac{3}{5} S(S+3c). \tag{3.21}$$

We are now in position to prove the theorem. Since $f_3(q)=0$, we have

$$\sum h_{ijkl}{}^2 \geqq 3 \sum_{i \neq j} \left(h_{ijij} - \frac{1}{2} t_{ij}\right)^2 + \frac{3}{4} S(S^2+4cS+6c^2) \tag{3.22}$$

by (3.12) and (3.13). By (1.13) and $f_4=(1/2)S^2$ we obtain $2A+B=-(1/2)S^2(S+3c)$, from which combining with (1.11) it follows that

$$\begin{aligned}\sum h_{ijkl}{}^2 &= S(S+3c)(S+9c) - 3(A-2B) \\ &= S(S+3c)(S+9c) - 4(2A+B) + 5(A+2B) \\ &= 3S(S+3c)^2 + 5(A+2B).\end{aligned}$$

Thus we have

$$3S(S+3c)^2 + 5(A+2B) \geqq 3 \sum_{i \neq j} \left(h_{ijij} - \frac{1}{2} t_{ij}\right)^2 + \frac{3}{4} S(S^2+4cS+6c^2). \tag{3.23}$$

Since we have

$$t_{12} = t_{23} = -\sqrt{\frac{S}{2}}\, c \tag{3.24}$$

at that point q, we can estimate the first term in the right hand side of (3.23) by the similar method to that by which the estimation of (3.15) is given and we have

$$\sum_{i \neq j} \left(h_{ijij} - \frac{1}{2} t_{ij}\right)^2 \geqq (h_{1212} - h_{2323})^2, \tag{3.25}$$

where we used (3.24). Differentiating $S=\sum h_{ij}{}^2$, we have $\sum_i \lambda_i h_{iikk} + \sum_{i,j} h_{ijk}{}^2 = 0$ for $k=1, 2, 3$. Substituting the value of principal curvatures into the above equation we obtain

$$\sqrt{\frac{S}{2}}(h_{11kk} - h_{33kk}) = \sum_{i,j} h_{ijk}{}^2 \tag{3.26}$$

for $k=1, 2, 3$. In particular, by (3.21) we have

$$\sqrt{\frac{S}{2}}(h_{1122}-h_{3322})=\sum_{i,j} h_{ij2}{}^2 \leqq -\frac{3}{5}S(S+3c).$$

Thus we have

$$h_{1212}-h_{2323}=h_{1212}-h_{3232}-t_{23}\leqq -\frac{\sqrt{2S}}{10}(6S+13c).$$

By assumption, $6S+13c>0$. Since the right hand side is negative, it follows from (3.23), (3.24) and (3.26) that

$$(3.27)\qquad 3S(S+3c)^2+5(A+2B)\geqq 3\left\{\frac{\sqrt{2S}}{10}(6S+13c)\right\}^2+\frac{3}{4}S(S^2+4cS+6c^2).$$

We shall next estimate the second term of the left hand side in (3.27). Since we get

$$\begin{aligned}A+2B&=\sum \lambda_i{}^2 h_{ijk}{}^2+2\sum \lambda_i\lambda_j h_{ijk}{}^2=\frac{1}{3}\sum(\lambda_i+\lambda_j+\lambda_k)^2 h_{ijk}{}^2\\ &=\frac{1}{3}\left\{\sum_{i\neq j\neq k\neq i}(\lambda_i+\lambda_j+\lambda_k)^2 h_{ijk}{}^2+3\sum_{i\neq k}(2\lambda_i+\lambda_k)^2 h_{iik}{}^2+9\sum \lambda_i{}^2 h_{iii}{}^2\right\}\\ &=\sum_{i\neq k}(2\lambda_i+\lambda_k)^2 h_{iik}{}^2+3\sum \lambda_i{}^2 h_{iii}{}^2,\end{aligned}$$

we have by (3.18) $A+2B=S(4h_{111}{}^2+h_{222}{}^2+4h_{333}{}^2)$, and hence we obtain

$$A+2B\leqq \frac{2}{5}S\left(16h_{111}{}^2+\frac{5}{2}h_{222}{}^2+16h_{333}{}^2\right).$$

It follows from the last inequality together with (3.19) that we have

$$A+2B\leqq -\frac{2}{5}S^2(S+3c).$$

By (3.27) and this inequality we have

$$S(191S^2+36cS-1236c^2)\leqq 0,$$

which shows that the range of S is contained in $[0, -k_0c]$, where $-k_0c$ is a positive root of the equation of order 3 and $k_0 \doteqdot 2.64$. This is a contradiction.

Therefore the function f_3 is constant and hence, by the first discussion, the theorem is completely proved. □

REMARK. In their paper [7] Ki, Kim and Nakagawa proved recently the following theorem: Let M be an n-dimensional complete maximal space-like hypersurface in an anti-de Sitter space $H_1^{n+1}(c)$. If the scalar curvature of M is constant, then there exists a positive number k which is depending on only the dimension such that if $-kc<S\leqq -3c$, then M is congruent to the hyperbolic cylinder $H^1(c_1)\times H^{n-1}(c_2)$. In the case of $n=3$ in the above theorem we see $k\doteqdot 2.98$.

References

[1] M. Abe, N. Koike and S. Yamaguchi, Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form, Yokohama Math. J. **35** (1987), 123–136.

[2] R. Aiyama and Q.M. Cheng, Complete space-like hypersurfaces with constant mean curvature in a Lorentz space form of dimension 4, Kodai Math. J. **15** (1992), 375–386.

[3] E. Calabi, Examples of Bernstein problems for some nonlinear equations, Proc. Pure Appl. Math. **15** (1970), 223–230

[4] Q.M. Cheng, Complete minimal hypersurfaces in $S^4(1)$, Osaka Math. J. **27** (1990), 885–892.

[5] Q.M. Cheng and H. Nakagawa, Totally umbilic hypersurfaces, Hiroshima Math. J. **20** (1990), 1–10.

[6] S.Y. Cheng and S.T. Yau, Maximal space-like hypersurfaces in the Lorentz- Minkowski spaces, Ann. of Math. **104** (1976), 407–419.

[7] U-Hang Ki, H.S. Kim and H. Nakagawa, Complete maximal space-like hypersurfaces of an anti-de Sitter space, Kyungpook Math. J. **31** (1991), 131–141.

[8] K. Nomizu, On isoparametric hypersurfaces in the Lorentzian space form, Japan J. Math. **7** (1981), 217–226.

[9] H. Omori, Isometric immersions of Riemannian manifolds, J. Math. Soc. Japan **19** (1967), 205–214.

[10] C.K. Peng and C.L. Terng, Minimal hypersurfaces of spheres with constant scalar curvature, Seminar on minimal submanifolds, Princeton Univ. Press, Prenceton, 1983, pp. 177–198.

[11] A.E. Treibergs, Entire hypersurfaces of constant mean curvature in Minkowski 3-space, Invent. Math. **66** (1982), 39–56.

[12] S.T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure and Appl. Math. **28** (1975), 201–208.

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