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STABILITY OF MINIMAL SUBMANIFOLDS IN SYMMETRIC SPACES

By

Makiko Sumi TANAKA

1. Introduction.

We determine the stability of totally geodesic submanifolds in a compact symmetric space, which are called polars and meridians (see 2.1). These subspaces were introduced by Chen and Nagano ([CN-1]) and we have many interesting results after that ([CN-2], [N-1], [N-2], [NS-1], [NS-2] and [NS-3]). Recently, several results have been obtained about the stability of totally geodesic submanifolds in compact symmetric spaces. Ohnita gave the formula for the index, the nullity and the Killing nullity of a totally geodesic submanifold in a compact symmetric space in [O], in which he also proved that the Helgason sphere in a compact symmetric space is stable. Tasaki proved that the Helgason sphere in a compact Lie group is homologically volume-minimizing in its real homology class in [Ts-1]. He used the calibration theory. And there are studies about the stability of certain closed subgroups in a compact Lie group by Mashimo and Tasaki ([MT-1] and [MT-2]). Mashimo determined all the unstable Cartan embeddings of compact symmetric spaces in [M]. And there is a result about the stability of symmetric R-spaces in Hermitian symmetric spaces and totally complex submanifolds in quaternionic Kähler symmetric spaces of classical type by Takeuchi ([Tk-2]). Recently Nagano and the author have obtained a result on a relationship between the stability of minimal submanifolds and that of p-harmonic maps ([NS-3]). In the present paper we study the stability of all the polars and meridians in every compact symmetric space by using Ohnita's method in Section 3. We will also study the stability of totally complex totally geodesic submanifolds in quaternionic Kähler symmetric spaces of exceptional type in Section 4.

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2. Preliminaries.

DEFINITION 2.1 ([CN-1]). Let M be a compact symmetric space and o a point in M. Then we call a connected component of the fixed points set of s_o, the symmetry at o, in M *a polar* of o. We denote it by M^+ or by $M^+(p)$ if M^+ contains a point p. We also call a connected component of the fixed point set of $s_p \cdot s_0$ in M through p *a meridian* of $M^+(p)$ in M and denote it by $M^-(p)$ or simply by M^-. When a polar consists of a single point, we call it *a pole*.

REMARK 2.2. The congruence class of $M^-(p)$ is independent of p.

Polars and meridians are totally geodesic submanifolds in M; they are thus symmetric spaces. And they were determined for every compact connected irreducible symmetric space ([CN-1], [N-1] and [N-2]). One of the most important properties of these subspaces is that M can be determined by any pair of $(M^+(p), M^-(p))$ completely (1.15 in [N-2]). M^+ relates to M in its topology and on the other hand M^- does in its local structure. For example, M is orientable if and only if each M^+ has an even dimension. And M^- has the same rank as M (see [N-1]).

DEFINITION 2.3 ([CN-2]). Let M be a compact symmetric space and o a point in M. And suppose there is a pole p of o in M. Then we call a set consisting of the midpoints of the geodesics from o to p *a centrosome* and denote it by $C(o, p)$ or simply by C.

A centrosome is also a totally geodesic submanifold of M.

REMARK 2.4. When there is a pole p of o, there exists a double covering map $\pi: M \to M''$ from M to another space M'' such that $\pi(o)=\pi(p)$. And the image of each connected component of $C(o, p)$ is a polar of $\pi(o)$ in M''.

DEFINITION 2.5. Let (M, g) and (N, h) be compact Riemannian manifolds and $\phi: M \to N$ be a minimal immersion. Then we say ϕ (or M) is *stable* if the second derivative of the volume function $V(M, \phi_t{}^*h)$ at $t=0$ is non-negative for every smooth variation $\{\phi_t\}$ of ϕ with $\phi_0=\phi$.

The second variation formula of $V(M, \phi_t{}^*h)$ reads as follows:

$$\frac{d^2}{dt^2} V(M, \phi_t{}^*h)_{|t=0} = \int_M \langle J(v), v \rangle \, d\nu$$

where v is an element of $\Gamma(T^\perp(M))$, the space of all smooth sections of the normal bundle $T^{\perp}(M)$ of M in N, and $d\nu$ is the Riemannian measure of (M, g). And J is *the Jacobi operator* defined by

$$J=-\Delta^{\perp}+A_{\phi}+R_{\phi}\,,$$

where $\Delta^{\perp}$ is the Laplacian of the normal connection $\nabla^{\perp}$ of $T^{\perp}(M)$, and A_{ϕ}, R_{ϕ} are smooth sections of $\mathrm{End}\,(T^{\perp}(M))$ (refer to [O]). J is a self-adjoint strongly elliptic linear differential operator and hence J has discrete eigenvalues $\mu_1<\mu_2<\cdots<\infty$. The eigenspaces of J have finite dimensions.

DEFINITION 2.6. *The index of ϕ (or of M) is the sum of the multiplicities of the negative eigenvalues of J.*

Obviously ϕ is stable if and only if the index of ϕ vanishes.

From now on we assume that $\phi: M=G/K\to N=U/L$ is a totally geodesic isometric immersion between compact symmetric spaces. We choose U so that G is a subgroup of U. We denote the Lie algebra of G and U by $\mathfrak{g}$ and $\mathfrak{u}$ respectively. And let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ and $\mathfrak{u}=\mathfrak{l}\oplus\mathfrak{p}$ be the canonical decompositions. We have the decomposition $\mathfrak{u}=\mathfrak{g}\oplus\mathfrak{g}^{\perp}$ as a G-module as well as the K-module decompositions $\mathfrak{l}=\mathfrak{k}\oplus\mathfrak{k}^{\perp}$ and $\mathfrak{p}=\mathfrak{m}\oplus\mathfrak{m}^{\perp}$, where $\mathfrak{m}$ (resp. $\mathfrak{m}^{\perp}$) is isomorphic to T_0M (resp. $T_0{}^{\perp}M$) as a K-module. Decompose $\mathfrak{g}^{\perp}$ into the sum of simple G-modules $\mathfrak{g}_i{}^{\perp}$ and denote by μ and μ_i the corresponding representations of G $(1\leqq i\leqq k)$. We have the decompositions $\mathfrak{g}_i{}^{\perp}=\mathfrak{k}_i{}^{\perp}\oplus\mathfrak{m}_i{}^{\perp}$ as K-modules where $\mathfrak{k}_i{}^{\perp}=\mathfrak{k}^{\perp}\cap\mathfrak{g}_i{}^{\perp}$ and $\mathfrak{m}_i{}^{\perp}=\mathfrak{m}^{\perp}\cap\mathfrak{g}_i{}^{\perp}$.

THEOREM 2.7 ([O]). *Let $\phi: M=G/K\to N=U/L$ be a totally geodesic isometric immersion from a compact symmetric space M into another compact symmetric spave N. Then we haue*

$$\text{(2.7)}\qquad \mathrm{index}\,(M)=\sum_{i=1}^{k}\sum_{\lambda\in D(G),\,a_{\lambda}>a_i}\dim \mathrm{Hom}_K(V_{\lambda},\,(\mathfrak{m}_i{}^{\perp})^C)\cdot\dim V_{\lambda}$$

where $D(G)$ denotes all the equivalence classes of complex irreducible representations of G and V_{λ} denotes its representation space for an element λ in $D(G)$ and a_{λ} denotes the eigenvalue of the Casimir operator of λ. While a_i denotes the eigenvalue of the Casimir operator of μ_i. $\mathrm{Hom}_K(V_{\lambda},\,(\mathfrak{m}_i{}^{\perp})^C)$ denotes the K-module homomorphisms from V_{λ} into the complexification $(\mathfrak{m}_i{}^{\perp})^C$ of $\mathfrak{m}_i{}^{\perp}$.

Now we apply (2.7) to the inclusion maps $\iota^{+}: M^{+}\to M$ and $\iota^{-}: M^{-}\to M$ of a polar $M^{+}=G^{+}/K^{+}$ and the meridian $M^{-}=G^{-}/K^{-}$ in $M=G/K$. Here we may take $K^{+}=K^{-}$ ([N-1], [N-2]). We fix a point o with $K(o)=o$. We note that we may consider M^+ containing o. Let $\mathfrak{g}^+$, $\mathfrak{g}^-$ and $\mathfrak{g}$ be the Lie algebras of G^+, G^- respectively. And let $\mathfrak{g}^+=\mathfrak{k}^+\oplus\mathfrak{m}^+$, $\mathfrak{g}^-=\mathfrak{k}^-\oplus\mathfrak{m}^-$ and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ be the canonical decompositions, where $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are isomorphic to T_oM^+ and T_oM^- as K^+-modules respectively. Since $\mathfrak{m}=\mathfrak{m}^+\oplus\mathfrak{m}^-$, we have $(\mathfrak{m}^+)^\perp=\mathfrak{m}^-$ and $(\mathfrak{m}^-)^\perp=\mathfrak{m}^+$. $\mathfrak{m}^+=\mathfrak{m}_1{}^+\oplus\cdots\oplus\mathfrak{m}_k{}^+$ and $\mathfrak{m}^-=\mathfrak{m}_1{}^-\oplus\cdots\oplus\mathfrak{m}_h{}^-$ as K^+-modules.

The next lemma is an immediate consequence of the theorem 2.7.

LEMMA 2.8. *With the above notation, the indices of M^+ and M^- are given as follows*:

$$\text{index}(M^+)=\sum_{i=1}^{k}\sum_{\lambda\in D(G^+),a_\lambda>a_i}\dim\mathrm{Hom}_{K^+}(V_\lambda,(m_i{}^-)^C)\cdot\dim V_\lambda \tag{2.8.1}$$

$$\text{index}(M^-)=\sum_{j=1}^{h}\sum_{\nu\in D(G^-),a_\nu>a_j}\dim\mathrm{Hom}_{K^-}(V_\nu,(m_j{}^+)^C)\cdot\dim V_\nu \tag{2.8.2}$$

where we follow the notation of Theorem 2.7.

3. Stability of polars and meridians in symmetric spaces.

In this section we determine the stability of all polars and meridians in every compact connected irreducible symmetric space by using Lemma 2.8.

We denote by $M=G/K$ a compact connected irreducible symmetric space, by $M^+=M^+(p)=G^+/K^+$ one of the polars of a point o in M and by $M^-=M^-(p)=G^-/K^-$ its corresponding meridian, where $K^-=K^+$ as mentioned in Section 2. We also denote by $\mathfrak{g}$, $\mathfrak{g}^+$ and $\mathfrak{g}^-$ the Lie algebras of G, G^+ and G^- respectively.

When K or K^+ is not connected, we denote its identity component by K_o or $K_o{}^+$ respectively. Since $M^+(p)$ is a K_o-orbit (see 1.5a(ii) in [N-2]), we may assume $G^+=K_o$.

In order to apply (2.8.1) (resp. (2.8.2)) to study the stability of M^+ (resp. M^-) in M, what we should do is the following (3.0.1) through (3.0.3):

(3.0.1) To determine every representation of G^+ on $\mathfrak{g}/\mathfrak{g}^+$ (resp. G^- on $\mathfrak{g}/\mathfrak{g}^-$) which is denoted by μ (resp. ρ) and to decompose μ (resp. ρ) into the irreducible representations.

Here $\mathfrak{g}/\mathfrak{g}^+$ is isomorphic to T_oM as a G^+-module, that is, μ is equivalent to the isotropy representation of K. Hence μ is irreducible. On the other hand, G/G^- is another symmetric space and ρ is the isotropy representation of G^-. ρ is irreducible or the sum of two irreducible representations which are equivalent to each other. (One can check it case by case.) So even if ρ is not irreducible, we denote its irreducible component by the same ρ.

(3.0.2) To find every complex irreducible representation λ of G^+ (resp. ν of G^-) which satisfies the condition $a_\lambda > a_\mu$ (resp. $a_\nu > a_\rho$) where a_λ denotes the eigenvalue of the Casimir operator of λ and similarly for a_μ, a_ν and a_ρ. (We can get the eigenvalue of the Casimir operator by Freudenthal's formula.)

(3.0.3) To examine whether each representation λ (resp. ν) in (3.0.2) satisfies the following condition or not: when we restrict λ (resp. ν) to K^+, it includes at least one of the simple K^+-submodules of $(\mathfrak{m}^-)^c$ (resp. $(\mathfrak{m}^+)^c$).

If a representation satisfies both conditions in (3.0.2) and in (3.0.3), we say that this representation is *admissible*. If there is no admissible representation of G^+ (resp. G^-), we conclude that M^+ (resp. M^-) is stable.

NOTATION. We follow the notation of [B] concerning the numbering of the fundamental weights and that of [N-1] concerning the symmetric spaces.

The isotropy representation of K^+ on the tangent space of M^+ is denoted by μ^+ and on that of M^- by μ^-. (Refer to Appendix for the isotropy representations.)

For two representations λ' and λ'' of some groups G' and G'', the represtation $\lambda'+\lambda''$ denotes the representation of a group $G'\times G''$ whose representation space is $V_{\lambda'}\otimes V_{\lambda''}$. Note that the eigenvalue of the Casimir operator satisfies $a_{\lambda'+\lambda''}=a_{\lambda'}+a_{\lambda''}$. We denote by $\lambda'\oplus\lambda''$ the representation of a group $G'\times G''$ whose representation space is $V_{\lambda'}\oplus V_{\lambda''}$ such that G'' acts trivially on $V_{\lambda'}$ and G' does on $V_{\lambda''}$.

Though T denotes $U(1)$, we also denote its representation by T.

LEMMA 3.1. *If M is a Hermitian symmetric space, all the polars and the meridians of M are stable.*

PROOF. When M is a Hermitian symmetric space, all the polars and the meridians are hermitian (See 2.30 in [N-2]). As a Hermitian symmetric space is Kählerian, we get their stability from the well-known fact; the complex submanifolds of a Kählerian manifold are homologically volume-minimizing in their real homology class. q.e.d.

A 4n-dimensional symmetric space M is called *a quaternionic Kähler symmetric space* if M has the following property: there is a point x in M such that, through an identification of T_xM with $\boldsymbol{H}^n$, the linear holonomy group of M at x is contained in $Sp(n)\cdot Sp(1)$.

LEMMA 3.2. *Suppose M is a quaternionic Kähler symmetric space and one of its polars is also quaternionic Kähler, then this polar and its corresponding meridian are stable.*

PROOF. We can check that when M is a quaternionic Kähler symmetric space and one of its polars is also quaternionic Kähler, its corresponding meridian is also qaternionic Kähler. Since the quaternionic Kähler submanifolds in a quaternionic Kähler manifold are homological volumeminimizing in their homology class (see [Ts-2]), they are stable. q.e.d.

LEMMA 3.3. *Let $M^+=K_0/K^+$ be a polar in a compact connected symmetric space $M=G/K$ and M^- its meridian. If K^+ is connected and M^- is a local direct product of the circle S^1 and a semisimple symmetric space, then M^+ is unstable.*

PROOF. When M^- is a local direct product of S^1 and a semisimple symmetric space, $(m^-)^C$ includes one dimensional subspace as a K^+-module. Because of the connectedness of K^+, a trivial representation is admissible. So M^+ is unstable. q.e.d.

LEMMA 3.4. *Let $M^\wedge$ be a finite covering space of M and $\pi: M^\wedge \to M$ the projection such that $\pi(o^\wedge)=o$ for a point $o^\wedge$ in $M^\wedge$ and o in M. If a polar $M^{\wedge +}$ of $o^\wedge$ in $M^\wedge$ is stable, then the image $\pi(M^{\wedge +})$ is a stable polar of o in M. And similarly if a meridian $M^{\wedge -}$ of $o^\wedge$ in $M^\wedge$ is stable, then so is the image $\pi(M^{\wedge -})$, a meridian in M.*

PROOF. Suppose $M^+ := \pi(M^{\wedge +})$ is unstable, then there exist a normal vector field v on M^+ which contributes to the index of M^+. When we consider the lift of v, it is a vector field on $M^\wedge$ which also contributes to the index of $M^{\wedge +}$ because π is locally isometric. It contradicts the assumption. As for a meridian, we can prove it in the same manner. q.e.d.

LEMMA 3.5. *Let $M^\wedge$ be a double covering space of M and $\pi: M^\wedge \to M$ the projection such that $\pi(o^\wedge)=o$ for a point $o^\wedge$ in $M^\wedge$ and o in M and $\pi(o^\wedge)=\pi(p)$ for a point p in $M^\wedge$. If a connected component $C := C(o^\wedge, p)_0$ of the centrosome $C(o^\wedge, p)$ in $M^\wedge$ is stable then the image $\pi(C)$ is a stable polar of o in M. And similarly if an orthogonal space $C^\perp$ of C in $M^\wedge$ is stable, then the image $\pi(C^\perp)$ is a stable meridian in M.*

PROOF. We can prove it similarly to Lemma 3.4.

DEFINITION 3.6. We say a connected component of centrosome $C(o, p)$ is of *s-size* when it meets one of the shortest geodesic segments joining o to p in M.

LEMMA 3.7. *Let M be a compact connected irreducible symmetric space. And suppose that there exists at least one pole p of a point o in M. If $C(o, p)_o$ is of s-size, then its orthogonal complement is the space of local direct product of S^1 and a compact simply connected symmetric space.*

PROOF. One can check this by using the classification in [CN-1] and [N-1]. For another proof, the shortest geodesic through $x \in C(o, p)_o$ is unique and hence its tangent vector at x is fixed by the isotropy subgroup. q.e.d.

PROPOSITION 3.8. *Let M be a compact connected irreducible symmetric space. And suppose that there exists at least one pole p of a point o in M. If $C(o, p)_o$ is of s-size, then it has a trivial line bundle as a subbundle of the normal bundle in M. Hence, $C(o, p)_o$ is unstable.*

PROOF. For all M except for $AI(2n)$ and EV, the isotropy subgroup of the automorphism group of $C := C(o, p)_o$ is connected. So we get their instability by Lemma 3.3 and 3.7. As for $M = AI(2n)$ and EV, we also get the instability of C in M by examining the action of each component of the isotropy subgroup of the automorphism group of C. q.e.d.

LEMMA 3.9. *Let $M\hat{}$ be a double covering space of a compact connected symmetric space M and $\pi : M\hat{} \to M$ the projection such that $\pi(o\hat{}) = o$ for a point $o\hat{}$ in $M\hat{}$ and o in M. Where we assume that $M\hat{}$ is neither $SO(2n)$ with $n=odd$ nor $G_n(\boldsymbol{R}^{2n})$ with $n=odd$. And suppose that there exists a pole p of $o\hat{}$ in $M\hat{}$ and $\pi(o\hat{}) = \pi(p)$. If $C(o\hat{}, p)_o$ is of s-size, the image $\pi(C(o\hat{}, p)_o)$ never has a trivial line bundle as a subbundle of a normal bundle in M.*

PROOF. We can construct a totally geodesic sphere S^m which contains S^1 in Lemma 3.7 except for the case in which $M\hat{}$ is $SO(2n)$ with n=odd or $G_n(\boldsymbol{R}^{2n})$ with n=odd, and which has the same dimension as the Helgason sphere of $M\hat{}$ (refer to [NS-2]). Since this S^1 is a shortest geodesic through $o\hat{}$ and p, S^m meets the s-sized $C(o\hat{}, p)_o$ in S^{m-1}. The image is $\pi(S^m) = \boldsymbol{R}P^m$ with its polar $\boldsymbol{R}P^{m-1}$ which is contained in $M^+ := \pi(C(o\hat{}, p)_o)$. Its meridian is $\boldsymbol{R}P^1 = S^1$. By the relationship between the orientations of $\boldsymbol{R}P^m$ and of its hypersurface, we can conclude that the normal bundle of $\boldsymbol{R}P^{m-1}$ is not orientable, that is, M^+

does not have a trivial line bundle as a subbundle of a normal bundle in M.

q.e.d.

Now we examine the stability of M^{+} and M^{-} of each M case by case. Here we arrange the spaces according to the root system of M.

Type A

[1] $SU(n)/\boldsymbol{Z}_k$ ($\boldsymbol{Z}_k$ is a subgroup of the center of $SU(n)$)

CASE 1. k is odd.

$$(M^{+}, M^{-})=(G_r(\boldsymbol{C}^n), \{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k), \quad 0<r=\text{even}\leqq n.$$

Every polar is unstable by Lemma 3.3 since $K^{+}=\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k$ is connected and M^{-} is a local direct product of S^1 and the other space. On the other hand, every meridian is stable. In fact, we have $G^{-}=\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k\times\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k$ with $\rho=\{(T'+\omega_1(A'_{r-1})+\omega_1(A'_{n-r-1}))+0\}\oplus\{0+(T''+\omega_1(A''_{r-1})+\omega_1(A''_{n-r-1}))\}$ and $\mu^{+}=T+\omega_1(A_{r-1})+\omega_1(A_{n-r-1})$, where T' (resp. T''), A'_{r-1} (resp. A''_{r-1}) and A'_{n-r-1} (resp. A''_{n-r-1}) denote T, $SU(r)$ and $SU(n-r)$ which are subgroups of G^{-} acting on M^{-} to the left (resp. right) side respectively and T, A_{r-1} and A_{n-r-1} denote T, $SU(r)$ and $SU(n-r)$ which are subgroups of K^{-}, the diagonal subgroup of G^{-}, respectively. If an element $\nu\in D(\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k\times\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k)$ satisfies $a_\nu>a_\rho$, then $\nu=0+\sigma$ or $\sigma+0$, where $\sigma\in D(\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k)$ which satisfies $a_\sigma>a_\rho$, or $\nu=\omega_1(A'_{r-1})+\omega_1(A''_{r-1})$ if $r<n-r$ and $\nu=\omega_1(A'_{n-r-1})+\omega_1(A''_{n-r-1})$ if $r>n-r$. As for $\nu=0+\sigma$ or $\sigma+0$, we have $V_{0+\sigma}=W_\sigma$ and $V_{\sigma+0}=W_\sigma$ as K^{+}-modules where W_σ is the representation space of the representation σ of K^{+}. As for $\nu=\omega_1(A'_{r-1})+\omega_1(A''_{r-1})$ if $r<n-r$ and $\nu=\omega_1(A'_{n-r-1})+\omega_1(A''_{n-r-1})$ if $r<n-r$, A_{n-r-1} in K^{+} is trivial in the former and non-trivial in μ^{+}. A_{r-1} in K^{+} is trivial in the latter and non-trivial in μ^{+}. So we conclude that any of the above ν is not admissible.

CASE 2. k is even and n/k is odd.

$$(M^{+}, M^{-})=(G_r(\boldsymbol{C}^n)^*, \{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k), \quad 0<r\leqq n/2.$$

When we consider the projection π from $SU(n)/\boldsymbol{Z}_{k/2}$ to $SU(n)/\boldsymbol{Z}_k$, the restriction of the projection to each polar is diffeomorphism if $r\neq n/2$. So the stability of $M^{+}=G_r(\boldsymbol{C}^n)^*=G_r(\boldsymbol{C}^n)$ with $r\neq n/2$ is reduced to the case 1, that is, all polars $G_r(\boldsymbol{C}^n)^*$ with $r\neq n/2$ are unstable. But $G_{n/2}(\boldsymbol{C}^n)^*$ is stable. Since it is the image of $G_{n/2}(\boldsymbol{C}^n)$, one of the connected components of the centrosome of s-size, by the above projection π, the trivial representation is not admissible by Lemma 3.9. And we can see that the other representations which satisfy the condition in (3.0.2) are not admissible either by restricting them to K^+. On the other hand, every meridian is stable from the result of the case 1 and by Lemma 3.4.

CASE 3. k and n/k are even.

$$(M^+, M^-)=(G_r(\boldsymbol{C}^n)^*, \{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k), \quad 0<r=\text{even}\leqq n.$$

Every polar except for $G_{n/2}(\boldsymbol{C}^n)^*$ is unstable and every meridian is stable as in case 2.

[2] $U(n)/\boldsymbol{Z}_k$

We can get the following results in a similar way to [1].

CASE 1. k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{C}^n), (U(r)\times U(n-r))/\boldsymbol{Z}_k), \quad 0<r\leqq n.$$

Every polar is unstable but every meridian is stable.

CASE 2. k is even and n/k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{C}^n)^*, (U(r)\times U(n-r))\boldsymbol{Z}_k), \quad 0<r\leqq n/2.$$

Every polar except for $G_{n/2}(\boldsymbol{C}^n)^*$ is unstable and every meridian is stable.

CASE 3. k and n/k are even.

$$(M^+, M^-)=(G_r(\boldsymbol{C}^n)^*, (U(r)\times U(n-r))/\boldsymbol{Z}_k), \quad 0<r\leqq n.$$

Every polar except for $G_{n/2}(\boldsymbol{C}^n)^*$ is unstable and every meridian is stable.

[3] $AI(n)/\boldsymbol{Z}_k$

CASE 1. k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{R}^n), \{T\cdot(AI(r)\times AI(n-r))\}/\boldsymbol{Z}_k), \quad 0<r=\text{even}\leqq n.$$

Every polar is unstable since the trivial representation of K is admissible. In fact, each connected component of K^+ acts on T in M^- trivially though K^+ is not connected. On the other hand, every meridian is stable. In fact, we have $G^-=\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k$ with $\rho=T+\omega_1(A_{r-1})+\omega_1(A_{n-r-1})$ and $\mu^+=\omega_1(SO(r))+\omega_1(SO(n-r))$. If an element $\nu\in D(\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k)$ satisfies $a_\nu>a_\rho$, then ν is a representation in which at least $SU(r)$ or $SU(n-r)$ acts trivially. So when we restrict these representations to K^+, at least $SO(r)$ or $SO(n-r)$ acts trivially. But in μ^+ both $SO(r)$ and $SO(n-r)$ act non-trivially. So we conclude that none of such ν is admissible.

REMARK 3.10. By using homomorphisms between symmetric spaces, we can also prove the instability of $M^+=G_r(\boldsymbol{R}^n)$ in $M=AI(n)/\boldsymbol{Z}_k$ as follows. There exists an inclusion $\iota: M\to SU(n)/\boldsymbol{Z}_k$ which carries a polar $G_r(\boldsymbol{R}^n)$ of $AI(n)/\boldsymbol{Z}_k$ into a polar $G_r(\boldsymbol{C}^n)$ of $SU(n)/\boldsymbol{Z}_k$ for each r respectively. And ι also carries a meridian $\{T\cdot(AI(r)\times AI(n-r))\}/\boldsymbol{Z}_k$ into a meridian $\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k$ for each r respectively, where we note that the image of T of $\{T\cdot(AI(r)\times AI(n-r))\}/\boldsymbol{Z}_k$ coincides with T of $\{T\cdot(SU(r)\times SU(n-r))\}/\boldsymbol{Z}_k$, in particular it is tangent to the image of $AI(n)/\boldsymbol{Z}_k$. Since $G_r(\boldsymbol{C}^n)$ has a trivial line bundle as a subbundle of the normal bundle in $SU(n)$ by the result in [1], we can conclude that $G_r(\boldsymbol{R}^n)$ also has a trivial line bundle by restricting the trivial line bundle to the image of a polar $G_r(\boldsymbol{R}^n)$.

CASE 2. k is even and n/k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{R}^n)^*, \{T\cdot(AI(r)\times AI(n-r))\}/\boldsymbol{Z}_k), \quad 0<r\leqq n/2.$$

Every polar $G_r(\boldsymbol{R}^n)^*=G_r(\boldsymbol{R}^n)$ with $r\neq n/2$ is unstable but every meridian is stable since the similar arguments work as the case 2 in [1]. And also we get that $M^+=G_{n/2}(\boldsymbol{R}^n)^*$ is stable. In fact, we have $G=SU(n)/\boldsymbol{Z}_k$, $K_0=SO(n)^*$ and $K^+{}_0=S(O(n/2)\cdot O(n/2))$ with $\mu=2\omega_1(SO(n))$ and $\mu^-=0\oplus 2\omega_1(SO(n/2))\oplus 2\omega_1(SO(n/2))$. If $\lambda\in D(SO(n)^*)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$ or ω_2. And we have the next decomposition of $\lambda=\omega_2$ as a K^+-module.

$$V_{\omega_2(SO(n))}=W_{\omega_2(SO(n/2))}\oplus W_{\omega_2(SO(n/2))}\oplus W_{\omega_1(SO(n/2))}\otimes W_{\omega_1(SO(n/2))}.$$

So $\lambda=\omega_2$ is not admissible. As for $\lambda=0$, since K^+ acts on T as ± 1 times the identity, we conclude that $\lambda=0$ is not admissible either.

CASE 3. k and n/k are even.

$$(M^+, M^-)=(G_r(\boldsymbol{R}^n)^*, \{T\cdot(AI(r)\times AI(n-r))\}/\boldsymbol{Z}_k), \quad 0<r=\text{even}\leqq n.$$

Every polar except for $G_{n/2}(\boldsymbol{R}^n)^*$ is unstable and every meridian is stable similarly to the case 2.

[4] $UI(n)/\boldsymbol{Z}_k$

We can get the following results similarly to [3].

CASE 1. k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{R}^n), (UI(r)\times UI(n-r))/\boldsymbol{Z}_k), \quad 0<r\leqq n.$$

Every polar is unstable but every meridian is stable.

CASE 2. k is even and n/k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{R}^n)^*, (UI(r)\times UI(n-r))/\boldsymbol{Z}_k), \quad 0<r\leqq n/2.$$

Every polar except for $G_{n/2}(\boldsymbol{R}^n)^*$ is unstable and every meridian is stable

CASE 3. k and n/k are even.

$$(M^+, M^-)=(G_r(\boldsymbol{R}^n)^*, (UI(r)\times UI(n-r))/\boldsymbol{Z}_k), \quad 0<r\leqq n.$$

Every polar except for $G_{n/2}(\boldsymbol{R}^n)^*$ is unstable and every meridian is stable.

[5] $AII(n)/\boldsymbol{Z}_k$

CASE 1. k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{H}^n), \{T\cdot(AII(r)\times AII(n-r))\}/\boldsymbol{Z}_k), \quad 0<r=\text{even}\leqq n.$$

Every polar is unstable since we can conclude that it has a trivial line bundle as in Remark 3.10. In this case the inclusion map: $M\to SU(2n)/\boldsymbol{Z}_k$ carries a polar $G_r(\boldsymbol{H}^n)$ of $AII(n)/\boldsymbol{Z}_k$ into a polar $G_{2r}(\boldsymbol{C}^{2n})$ of $SU(2n)/\boldsymbol{Z}_k$ for each r. On the other hand, every meridian is stable as in [1].

CASE 2. k is even and n/k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{H}^n)^*, \{T\cdot(AII(r)\times AII(n-r))\}/\boldsymbol{Z}_k), \quad 0<r\leqq n/2.$$

Every polar except for $G_{n/2}(\boldsymbol{H}^n)^*$ is unstable and every meridian is stable as in case 2 of [1].

CASE 3. k and n/k are even.

$$(M^+, M^-)=(G_r(\boldsymbol{H}^n)^*, \{T\cdot(AII(r)\times AII(n-r))\}/\boldsymbol{Z}_k), \quad 0<r=\text{even}\leqq n.$$

Every polar except for $G_{n/2}(\boldsymbol{H}^n)^*$ is unstable and every meridian is stable as in case 2.

[6] $UII(n)/\boldsymbol{Z}_k$

We can get the following results as in [5].

CASE 1. k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{H}^n), (UII(r)\times UII(n-r))/\boldsymbol{Z}_k). \quad 0<r\leqq n.$$

Every polar is unstable but every meridian is stable.

CASE 2. k is even and n/k is odd.

$$(M^+, M^-)=(G_r(\boldsymbol{H}^n)^*, (UII(r)\times UII(n-r))/\boldsymbol{Z}_k), \quad 0<r\leqq n/2.$$

Every polar except for $G_{n/2}(\boldsymbol{H}^n)^*$ is unstable and every meridian is stable.

CASE 3. k and n/k are even.

$$(M^+, M^-)=(G_r(\boldsymbol{H}^n)^*, (UII(r)\times UII(n-r))/\boldsymbol{Z}_k), \quad 0<r\leqq n.$$

Every polar except for $G_{n/2}(\boldsymbol{H}^n)^*$ is unstable and every meridian is stable.

[7] EIV

$$(M^+, M^-)=(FII, T\cdot S^9).$$

$M^+=FII$ is unstable by Lemma 3.3 since $K^+=SO(9)^\sim$ is connected and M^- is the local direct product of T and S^9. On the other hand, $M^-=T\cdot S^9$ is stable. In fact, we have $G^-=T\cdot SO(10)^\sim$ with $\rho=T+\omega_5(D_5)$ and $\mu^+=\omega_4(B_4)$. And if $\nu\in D(T\cdot SO(10)^\sim)$ satisfies $a_\nu<a_\rho$, then $\nu=0$ or $T+\omega_1$. Then we can conclude that neither one is admissible.

Type B or D

[8.1] $SO(n)^\sim$

$$(M^+, M^-)=(G_{2r}{}^o(\boldsymbol{R}^n),\ SO(2r)^\sim\cdot SO(n-2r)^\sim),\quad 0<r=\text{even}\leqq n/2.$$

The polar $G_{2r}{}^o(\boldsymbol{R}^n)$ is stable unless $n=4m+3$ with $r=2m$ or $n=2r+2$. When $n=4m+3$ with $r=2m$, the polar is unstable since $\lambda=\omega_1(B_{2m+1})$ is admissible. (Note that ω_1 has the smallest eigenvalue of the Casimir operator among the fundamental weights of B_{2m+1} if $m\geqq 2$). When $n=2r+2$, we have the instability of the polar by Lemma 3.3 since K^+ is connected and its corresponding meridian is $SO(2)\cdot SO(2r)^\sim$. In general, we have $G=SO(n)^\sim\times SO(n)^\sim$, $K=SO(n)^\sim$, $K^+=SO(2r)^\sim\cdot SO(n-2r)^\sim$, $\mu=\omega_2(SO(n)^\sim)$ and $\mu^-=\omega_2(SO(2r)^\sim)\oplus\omega_2(SO(n-2r)^\sim)$. If $\lambda\in D(SO(n)^\sim)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$ or ω_1 if $n\geqq 16$. Since we have the next decomposition as a K^+-module

$$V_{\omega_1(SO(n)^\sim)}=W_{\omega_1(SO(2r)^\sim)}\oplus W_{\omega_1(SO(n-2r)^\sim)}\ ,$$

we conclude that $\lambda=\omega_1$ is not admissible when $2r$ and $n-2r$ are sufficiently large. If $n<16$, the spin (or the half spin) representation of $SO(n)^\sim$ satisfies the condition $a_\lambda<a_\mu$, but we can see that it is not admissible. Obviously $\lambda=0$ is not admissible either. On the other hand, each meridian $M^-=SO(2r)^\sim\cdot SO(n-2r)^\sim$ is stable. In fact, we have $G^-=SO(2r)^\sim\cdot SO(n-2r)^\sim\times SO(2r)^\sim\cdot SO(n-2r)^\sim$ with $\rho=\{(\omega_1(SO'(2r)^\sim)+\omega_1(SO'(n-2r)^\sim))+0\}\oplus\{0+(\omega_1(SO''(2r)^\sim)+\omega_1(SO''(n-2r)^\sim))\}$ and $\mu^+=\omega_1(SO(2r)^\sim)+\omega_1(SO(n-2r)^\sim)$, where we refer to [1] for the notations SO' and SO''. If $\nu\in D(SO'(2r)^\sim\cdot SO'(n-2r)^\sim\times SO''(2r)^\sim\cdot SO''(n-2r)^\sim)$ satisfies $a_\nu<a_\rho$, then $\nu=0+\sigma$ or $\sigma+0$, where $\sigma\in D(SO(2r)^\sim\cdot SO(n-2r)^\sim)$ which satisfies $a_\sigma>a_\rho$ or $\nu=\omega_1(SO'(2r)^\sim)+\omega_1(SO''(2r)^\sim)$ if $2r<n-2r$ and $\nu=\omega_1(SO'(n-2r)^\sim)+\omega_1(SO''(n-2r)^\sim)$ if $2r>n-2r$. We can conclude that none of these representations is admissible as in [1].

[8.2] $SO(4n)^\#$

[8.2-1] $(M^+, M^-)=(G_{2r}{}^o(\boldsymbol{R}^{4n}),\ (SO(2r)^\sim\cdot SO(4n-2r)^\sim)/\{1, \delta\}),\ 0<r=\text{even}\leqq n.$

[8.2-2] $(M^+, M^-)=(G_{2n}(\boldsymbol{R}^{4n})^\#,\ SO(2n)^\sim\cdot SO(2n)^\sim)/\{1, \delta\}).$

These polars and meridians are stable from the result in [8.1] and by Lemma 3.4.

[8.2-3] $(M^+, M^-)=(DIII(2n)^*, U^\frown(2n)/\boldsymbol{Z}_2)$ where $U^\frown(2n)$ is the connected subgroup of $SO(4n)^\sim$ which doubly covers $U(2n)$ in $SO(4n)$.

The polar is stable by Lemma 3.9. And so is the meridian. In fact, we have $G=SO(4n)^\# \times SO(4n)^\#$, $G^-=U^\frown(2n)/\boldsymbol{Z}_2\times U^\frown(2n)/\boldsymbol{Z}_2$, $K^+=U^\frown(2n)/\boldsymbol{Z}_2$, $\rho=(T'+\omega_2(A'_{2n-1})+0)\oplus(0+T''+\omega_2(A''_{2n-1}))$ and $\mu^+=T+\omega_2(A_{2n-1})$. If $\nu\in D(U^\frown(2n)/\boldsymbol{Z}_2\times U^\frown(2n)/\boldsymbol{Z}_2$ satisfies $a_\nu<a_\rho$, then ν is a trivial representation since $T+\omega_1$ cannot be a representation of $U^\frown(2n)/\boldsymbol{Z}_2$. Clearly the trivial representation is not admissible.

[8.3] $SO(n)$

$$(M^+, M^-)=(G_r(\boldsymbol{R}^n),\ SO(r)\times SO(n-r)),\quad 0<r=\text{even}\leqq n.$$

All polars and all meridians are stable. In fact, when $r\equiv 0 \pmod 4$, we can conclude that all polars and all meridians except for some cases are stable from the results in [8.1] and by Lemma 3.4. So we may consider the following three cases only: (1) $r\equiv 2 \pmod 4$, (2) $n=4m+3$ with $r=4m$, and (3) $n\equiv 2 \pmod 4$ with $r=n-2$. As for the cases (2) and (3), the representations which are admissible in [8.1] can not be admissible in this case because of the disconnectedness of K^+. And as for the case (1), the similar arguments work as in [8.1].

[8.4] $SO(2n)^*$

[8.4-1] $(M^+, M^-)=(G_r(\boldsymbol{R}^{2n})^*,\ SO(r)\cdot SO(2n-r)),\quad 0<r\leqq n.$

We can conclude that these polars and meridians are stable by the results in [8.3] and Lemma 3.4.

[8.4-2] $(M^+, M^-)=(DIII(n)^*,\ U(n)/\boldsymbol{Z}_2)$.

When n is even, the polar is stable by Lemma 3.9. And when n is odd, the polar is unstable by Lemma 3.3 since the isotropy subgroup of $DIII(n)^*=DIII(n)$ is connected. On the other hand, the meridian is stable by the results in [8.2-3] and Lemma 3.4 if n is even and the similar argument works if n is odd.

[9.1] $G^o{}_r(\boldsymbol{R}^n)$

$$(M^+, M^-)=(G_a{}^o(\boldsymbol{R}^r)\cdot G_b{}^o(\boldsymbol{R}^{n-r}),\ G_a{}^o(\boldsymbol{R}^{n-2b})\cdot G_b{}^o(\boldsymbol{R}^{2b})),\quad a+b=r,\ b=\text{even}.$$

The polar and the meridian are stable unless $r=2m+1$ with $a=1$. In fact, we have $G=SO(n)^\sim$, $K=SO(r)^\sim\times SO(n-r)^\sim$, $G^-=SO(n-2b)^\sim\times SO(2b)^\sim$, $K^+=SO(a)^\sim\times SO(b)^\sim\times SO(b)^\sim\times SO(n-r-b)^\sim$, $\mu=\omega_1(SO(r)^\sim)+\omega_1(SO(n-r)^\sim)$, $\rho=\omega_1(SO(n-2b)^\sim)+\omega_1(SO(2b)^\sim)$, $\mu^+=\{\omega_1(SO(a)^\sim)+\omega_1(SO(b)^\sim)\}\oplus\{\omega_1(SO(b)^\sim)+\omega_1(SO(n-r-b)^\sim)\}$ and $\mu^-=\{\omega_1(SO(a)^\sim)+\omega_1(SO(n-r-b)^\sim)\}\oplus\{\omega_1(SO(b)^\sim)+\omega_1(SO(b)^\sim)\}$.

When $r=2m+1$ with $a=1$, the polar is $S^{2m}\cdot G_{2m}{}^{o}(\boldsymbol{R}^{n-2m-1})$ and we can see that $\omega_1(SO(r)^{\sim})$ is admissible. And in that case the meridian is $S^{n-4m-1}\cdot G_{2m}{}^{o}(\boldsymbol{R}^{4m})$ and $\omega_1(SO(n-2b)^{\sim})$ is also admissible.

[9.2] $G_r(\boldsymbol{R}^n)^{\#}$

[9.2-1] $(M^+, M^-)=(G_a{}^{o}(\boldsymbol{R}^n)\cdot G_b(\boldsymbol{R}^n),\ (G_a{}^{o}(\boldsymbol{R}^{2a})\cdot G_b{}^{o}(\boldsymbol{R}^{2b}))/\{1, \delta\})$,
$a+b=n,\ 0<b=\text{even}<n/2$.

[9.2-2] $(M^+, M^-)=(G_b(\boldsymbol{R}^n)^{\#}\cdot G_b(\boldsymbol{R}^n)^{\#},\ G_b(\boldsymbol{R}^n)^{\#}\cdot G_b(\boldsymbol{R}^n)^{\#})$.

[9.2-3] $(M^+, M^-)=(SO(n)^*,\ UI^{\wedge}(n)/\{1, \delta\})$ where $UI^{\wedge}(n)=U^{\wedge}(n)/SO(n)$ and see [8.2-3] for $U^{\wedge}(n)$.

We can conclude that these polars and meridians are stable similarly to the case [8.2].

[9.3] $G_r(\boldsymbol{R}^n)$

$$(M^+, M^-)=(G_a(\boldsymbol{R}^r)\times G_b(\boldsymbol{R}^{n-r}),\ G_a(\boldsymbol{R}^{n-2b})\times G_b(\boldsymbol{R}^{2b})),\quad a+b=r.$$

All polars and all meridians are stable. In fact, when b is even, we can conclude that all polars and all meridians except for some cases are stable from the results in [9.1] and by Lemma 3.4. So we may consider the following two cases only: (1) b is odd and (2) $r=2m+1$ with $a=1$. As for the case (1), the similar arguments work as in [9.1]. As for the case (2), the representations which are admissible in [9.1] cannot be admissible in this case because of the disconnectedness of K^+.

[9.4] $G_r(\boldsymbol{R}^n)^*$

[9.4-1] $(M^+, M^-)=(G_a(\boldsymbol{R}^n)\times G_b(\boldsymbol{R}^n),\ G_a(\boldsymbol{R}^{2a})\cdot G_b(\boldsymbol{R}^{2b})),\quad a+b=n,\ 0<a<n/2$.

[9.4-2] $(M^+, M^-)=(G_a(\boldsymbol{R}^n)\cdot G_a(\boldsymbol{R}^n),\ G_a(\boldsymbol{R}^n)\cdot G_a(\boldsymbol{R}^n))$.

We can conclude that these polars and meridians are stable from the results in [9.3] and by Lemma 3.4.

[9.4-3] $(M^+, M^-)=(SO(n)^*,\ UI(n)/\boldsymbol{Z}_2)$.

The polar is stable if n is even by Lemma 3.9. And when n is odd, the polar is unstable by Lemma 3.3 since the isotropy subgroup of $SO(n)^*=SO(n)$ is connected. On the other hand, the meridian is stable from the results in [9.2-3] and by Lemma 3.4 if n is even and the similar argument works if n is odd.

Type C or BC

[10.1] $Sp(n)$

$$(M^{+}, M^{-})=(G_{r}(\boldsymbol{H}^{n}), Sp(r)\times Sp(n-r)), \quad 0<r\leqq n.$$

Every polar is stable. In fact, we have $G=Sp(n)\times Sp(n)$, $K=Sp(n)$, $K^{+}=Sp(r)\times Sp(n-r)$, $\mu=2\omega_{1}(C_{n})$ and $\mu^{-}=2\omega_{1}(C_{r})\oplus 2\omega_{1}(C_{n-r})$. If $\lambda\in D(Sp(n))$ satisfies $a_{\lambda}<a_{\mu}$, then $\lambda=0$, ω_{1} or ω_{2} and if $n=3$, ω_{3}. We have the next decompositions as K^{+}-modules.

$$V_{\omega_{1}(C_{n})}=W_{\omega_{1}(C_{r})}\oplus W_{\omega_{1}(C_{n-r})}.$$

$$V_{\omega_{2}(C_{n})}=W_{\omega_{2}(C_{r})}\oplus W_{\omega_{2}(C_{n-r})}\oplus W_{\omega_{1}(C_{r})}\otimes W_{\omega_{1}(C_{n-r})}\oplus C$$

$$V_{\omega_{3}(C_{3})}=W_{\omega_{1}(A_{1})}\otimes W_{\omega_{2}(C_{2})}\oplus W_{\omega_{1}(C_{2})}.$$

So we can conclude that each λ is not admissible. Every meridian is also stable. In fact, we have $G^{-}=Sp(r)\times Sp(n-r)\times Sp(r)\times Sp(n-r)$ with $\rho=\{(\omega_{1}(C'_{r})+\omega_{1}(C'_{n-r}))+0\}\oplus\{0+(\omega_{1}(C''_{r})+\omega_{1}(C''_{n-r}))\}$ and $\mu^{+}=\omega_{1}(C_{r})+\omega_{1}(C_{n-r})$. If $\nu\in D(Sp(r)\times Sp(n-r)\times Sp(r)\times Sp(n-r))$ satisfies $a_{\nu}<a_{\rho}$, then $\nu=0+\sigma$ or $\sigma+0$ where $\sigma\in D(Sp(r)\times Sp(n-r))$ which satisfies $a_{\sigma}<a_{\rho}$, $\nu=\omega_{1}(C'_{r})+\omega_{1}(C''_{r})$ if $r<n-r$, or $\nu=\omega_{1}(C'_{n-r})+\omega_{1}(C''_{n-r})$ if $r>n-r$. Then we can conclude that these representations are not admissible as in [1].

[10.2] $Sp(n)^{*}$

[10.2-1] $(M^{+}, M^{-})=(G_{r}(\boldsymbol{H}^{n})^{*}, Sp(r)\cdot Sp(n-r))$, $0<r\leqq n/2$.

Both the polar and the meridian are stable from the results of [10.1] and by Lemma 3.5.

[10.2-2] $(M^{+}, M^{-})=(CI(n)^{*}, U(n)/\boldsymbol{Z}_{2})$.

The polar is stable. In fact, we have $G=Sp(n)^{*}\times Sp(n)^{*}$, $K_{0}=Sp(n)^{*}$, $K^{+}{}_{o}=U(n)/\boldsymbol{Z}_{2}$, $\mu=2\omega_{1}(C_{n})$ and $\mu^{-}=0\oplus(\omega_{1}+\omega_{n-1})(A_{n-1})$. If $\lambda\in D(Sp(n)^{*})$ satisfies $a_{\lambda}<a_{\mu}$, then $\lambda=0$ or ω_{2}. We have the next decomposition as a $K^{+}{}_{o}$-module.

$$V_{\omega_{2}(C_{n})}=W_{\omega_{2}(A_{n-1})}\oplus W_{\omega_{n-2}(A_{n-1})}\oplus W_{(\omega_{1}+\omega_{n-1})(A_{n-1})}.$$

Because $CI(n)$ is the centrosome of s-size in $Sp(n)$, we can conclude that $\lambda=0$ is not admissible by Lemma 3.10. And $\lambda=\omega_{2}$ is not admissible either by the disconnectedness of K^{+} despite of the above decomposition. The meridian is also stable. In fact, we have $G^{-}=U(n)/\boldsymbol{Z}_{2}\times U(n)/\boldsymbol{Z}_{2}$ with $\rho=\{(T'+2\omega_{1}(A'_{n-1}))+0\}\oplus\{0+(T''+2\omega_{1}(A''_{n-1}))\}$ and $\mu^{+}=T+2\omega_{1}(A_{n-1})$. If $\nu\in D(U(n)/\boldsymbol{Z}_{2}\times U(n)/\boldsymbol{Z}_{2})$ satisfies $a_{\nu}<a_{\rho}$, then $\nu=0+\sigma$ or $\sigma+0$ where $\sigma\in D(U(n)/\boldsymbol{Z}_{2})$ which satisfies $a_{\sigma}<a_{\rho}$. These representations are not admissible as in [1].

[11.1] $G_{r}(\boldsymbol{H}^{n})$

$$(M^{+}, M^{-})=(G_{a}(\boldsymbol{H}^{r})\times G_{b}(\boldsymbol{H}^{n-r}), G_{a}(\boldsymbol{H}^{n-2b})\times G_{b}(\boldsymbol{H}^{2b})), \quad a+b=r.$$

Every polar is stable. In fact, we have $G=Sp(n)$, $K=Sp(r)\times Sp(n-r)$, $K^{+}=Sp(a)\times Sp(b)\times Sp(b)\times Sp(n-r-b)$, $\mu=\omega_1(C_r)+\omega_1(C_{n-r})$ and $\mu^{-}=(\omega_1(C_a)+\omega_1(C_{n-2b-a}))\oplus(\omega_1(C_b)+\omega_1(C_b))$. If $\lambda\in D(Sp(r)\times Sp(n-r))$ satisfies $a_\lambda<a_\mu$, then $\lambda=0+0$, $\sigma+0$ or $0+\sigma'$, where $\sigma\in D(Sp(r))$ and $\sigma'\in D(Sp(n-r))$ which satisfy $a_\sigma<a_\mu$ and $a_{\sigma'}<a_\mu$. Because at least $Sp(r)$ or $Sp(n-r)$ acts trivially in these representations and non-trivially in μ^{-}, we can conclude that these representations are not admissible. Every meridian is also stable. In fact, we have $G'=Sp(2b)\times Sp(n-2b)$ with $\rho=\omega_1(C_{2b})+\omega_1(C_{n-2b})$ and $\mu^{+}=(\omega_1(C_a)+\omega_1((C_b))\oplus(\omega_1(C_b)+\omega_1(C_{n-r-b}))$. If $\nu\in D(Sp(2b)\times Sp(n-2b))$ satisfies $a_\nu<a_\rho$, then $\nu=0+0$, $\sigma+0$ or $0+\sigma'$, where $\sigma\in D(Sp(2b))$ and $\sigma'\in D(Sp(n-2b))$ which satisfy $a_\sigma<a_\mu$ and $a_{\sigma'}<a_\mu$. Because at least $Sp(2b)$ or $Sp(n-2b)$ acts trivially in these representations and non-trivially in μ^{+}, we can conclude that these representations are not admissible.

[11.2] $G_n(\boldsymbol{H}^{2n})^*$

[11.2-1] $(M^{+}, M^{-})=(G_a(\boldsymbol{H}^n)\times G_b(\boldsymbol{H}^n), G_a(\boldsymbol{H}^{2a})\cdot G_b(\boldsymbol{H}^{2b}))$, $a+b=n, 0<a<n/2$.

[11.2-2] $(M^{+}, M^{-})=(G_a(\boldsymbol{H}^n)\cdot G_a(\boldsymbol{H}^n), G_a(\boldsymbol{H}^n)\cdot G_a(\boldsymbol{H}^n))$.

Both the polars and the meridians are stable from the results of [11.1] and by Lemma 3.5.

[11.2-3] $(M^{+}, M^{-})=(Sp(n)^*, UII(n)/\boldsymbol{Z}_2)$.

The meridian is stable. In fact, we have $G=Sp(2n)^*$, $K_o=Sp(n)^*\times Sp(n)^*$, $K^{+}{}_o=Sp(n)^*$, $\mu=\omega_1(C'_n)+\omega_1(C''_n)$ and $\mu^{-}=0\oplus\omega_2(C_n)$. If $\lambda\in D(Sp(n)^*\times Sp(n)^*)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0+0$, ω_2+0 or $0+\omega_2$. And we can see that these representations are not admissible by the disconnectedness of K^{+}. And the meridian is also stable. In fact, we have $G^{-}=U(n)/\boldsymbol{Z}_2$ with $\rho=T+2\omega_1(A_{n-1})$ and $\mu^{+}=2\omega_1(C_n)$. If $\nu\in D(U(n)/\boldsymbol{Z}_2)$ satisfies $a_\nu<a_\rho$, then $\nu=0$ or $T+\omega_2(A_{n/2-1})$. We can see that neither one is admissible.

[12.1] $G_r(\boldsymbol{C}^n)$

$$(M^{+}, M^{-})=(G_a(\boldsymbol{C}^r)\times G_b(\boldsymbol{C}^{n-r}), G_a(\boldsymbol{C}^{n-2b})\times G_b(\boldsymbol{C}^{2b})),\quad a+b=r.$$

[12.2] $G_r(\boldsymbol{C}^n)^*$

[12.2-1] $(M^{+}, M^{-})=(G_a(\boldsymbol{C}^n)\times G_b(\boldsymbol{C}^n), G_a(\boldsymbol{C}^{2a})\cdot G_b(\boldsymbol{C}^{2b}))$, $a+b=n$, $0<a<n/2$.

[12.2-2] $(M^{+}, M^{-})=(G_a(\boldsymbol{C}^n)\cdot G_a(\boldsymbol{C}^n), G_a(\boldsymbol{C}^n)\cdot G_a(\boldsymbol{C}^n))$.

[12.2-3] $(M^{+}, M^{-})=(U(n)/\boldsymbol{Z}_2, U(n)/\boldsymbol{Z}_2)$.

[13.1] $CI(n)$

$$(M^+, M^-)=(G_r(\boldsymbol{C}^n),\ CI(r)\times CI(n-r)),\quad 0<r\leqq n.$$

[13.2] $CI(n)^*$

[13.2-1] $(M^+, M^-)=(G_r(\boldsymbol{C}^n)^*,\ CI(r)\cdot CI(n-r)),\quad 0<r\leqq n/2.$

[13.2-2] $(M^+, M^-)=(UI(n)/\boldsymbol{Z}_2,\ UI(n)\boldsymbol{Z}_2).$

[14.1] $DIII(n)$

$$(M^+, M^-)=(G_r(\boldsymbol{C}^n),\ DIII(r)\times DIII(n-r)),\quad 0<r=\text{even}\leqq n.$$

[14.2] $DIII(2n)^*$

[14.2-1] $(M^+, M^-)=(G_r(\boldsymbol{C}^{2n})^*,\ DIII(r)\cdot DIII(2n-r)),\quad 0<r=\text{even}\leqq n.$

[14.2-2] $(M^+, M^-)=(UII(n)/\boldsymbol{Z}_2,\ UII(n)/\boldsymbol{Z}_2).$

[15.1] $EVII$

$$(M^+, M^-)=(EIII,\ S^2\times G_2{}^o(\boldsymbol{R}^{12})).$$

[15.2] $EVII^*$

[15.2-1] $(M^+, M^-)=(EIII,\ S^2\cdot G_2{}^o(\boldsymbol{R}^{12})).$

[15.2-2] $(M^+, M^-)=((T\cdot EIV)/\boldsymbol{Z}_2,\ (T\cdot EIV)/\boldsymbol{Z}_2).$

[16] $EIII$

[16-1] $(M^+, M^-)=(G_2{}^o(\boldsymbol{R}^{10}),\ G_2{}^o(\boldsymbol{R}^{10})).$

[16-2] $(M^+, M^-)=(DIII(5),\ S^2\times G_1(\boldsymbol{C}^6)).$

Because [12.1], [13.1], [14.1], [15.1] and [16] are Hermitian symmetric spaces, we can conclude that both their every polar and meridian are stable by Lemma 3.1. As for [12.2], [13.2], [14.2] and [15.2], their polars and meridians of [12.2-1], [12.2-2], [13.2-1], [14.2-1] and [15.2-1] are stable by Lemma 3.5. And their polars of [12.2-3], [13.2-2], [14.2-2] and [15.2-2] which are congruent to their meridians corresponds to the connected centrosomes of the spaces [12.1], [13.1], [14.1] and [15.1] respectively and we can see that the indices of these centrosomes are equal to one. So we conclude that both polars and meridians of these spaces are stable by Lemma 3.10.

[17] FII

$$(M^+, M^-)=(S^8, S^8).$$

The polar which is congruent to the meridian is stable. In fact, we have $G=F_4$, $K=SO(9)^\sim$, $K^+=SO(8)^\sim$, $\mu=\omega_4(B_4)$ and $\mu^-=\omega_3(D_4)$ or $\omega_4(D_4)$. If $\lambda\in D(SO(9)^\sim)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$ or ω_1. As for $\lambda=\omega_1$, we have the next

decomposition as a K^+-module.

$$V_{\omega_1(B_4)}=W_{\omega_1(D_4)}\oplus C.$$

So we can conclude that ω_1 is not admissible. Clearly the trivial representation is not admissible either.

REMARK 3.12. These S^8 are congruent with the Helgason sphere of FII. Ohnita studied the stability of Helgason spheres of compact symmetric spaces in [O] and got the result that they are all stable minimal submanifolds.

Type E_6

[18] E_6

[18-1] $(M^+, M^-)=(EII, Sp(1)\cdot SU(6))$.

The polar is stable. Since we have $G=E_6\times E_6$, $K=E_6$, $K^+=Sp(1)\cdot SU(6)$, $\mu=\omega_2(E_6)$ and $\mu^-=2\omega_1(A_1)\oplus(\omega_1+\omega_5)(A_5)$. If $\lambda\in D(E_6)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$, ω_1 or ω_6. And we have the next decompositions as K^+-modules.

$$V_{\omega_1(E_6)}=W_{\omega_1(A_1)}\otimes W_{\omega_1(A_5)}\oplus W_{\omega_4(A_5)}$$

$$V_{\omega_6(E_6)}=W_{\omega_1(A_1)}\otimes W_{\omega_5(A_5)}\oplus W_{\omega_2(A_5)}.$$

So we can conclude that each λ is not admissible. The meridian is also stable. Since we have $G^-=Sp(1)\cdot SU(6)\times Sp(1)\cdot SU(6)$ with $\rho=\{(\omega_1(A'_1)+\omega_3(A'_5))+0\}\oplus\{0+(\omega_1(A''_1)+\omega_3(A''_5))\}$ and $\mu^+=\omega_1(A_1)+\omega_3(A_5)$. If $\nu\in D(Sp(1)\cdot SU(6)\times Sp(1)\cdot SU(6))$ satisfies $a_\nu<a_\rho$, then $\nu=\sigma+0$ or $0+\sigma$ where $\sigma\in D(Sp(1)\cdot SU(6))$ such that $a_\sigma<a_\rho$, $\nu=2\omega_1(A'_1)+2\omega_1(A''_1)$, $\nu=2\omega_1(A'_1)+(\omega_1(A''_1)+\omega_1(A''_5))$ or $\nu=2\omega_1(A'_1)+(\omega_1(A''_1)+\omega_5(A''_5))$. We can see $\nu=\sigma+0$ or $0+\sigma$ is not admissible and $\nu=2\omega_1(A'_1)+2\omega_1(A''_1)$ is not admissible either because $SU(6)$ acts trivially in this representation and non-trivially in μ^+. By the following decompositions:

$$V_{2\omega_1(A_1)}\otimes(V_{\omega_1(A'_1)}\otimes V_{\omega_1(A'_5)})=W_{3\omega_1(A_1)}\otimes W_{\omega_1(A_5)}\oplus W$$

$$V_{2\omega_1(A_1)}\otimes(V_{\omega_1(A'_1)}\otimes V_{\omega_5(A'_5)})=W_{3\omega_1(A_1)}\otimes W_{\omega_5(A_5)}\oplus W'$$

where $\dim W=\dim W'=12$, we can conclude that $\nu=2\omega_1(A'_1)+(\omega_1(A''_1)+\omega_1(A''_5))$ or $\nu=2\omega_1(A'_1)+(\omega_1(A''_1)+\omega_5(A''_5))$ is not admissible because of $\dim M^+=40$.

[18-2] $(M^+, M^-)=(EIII, T\cdot SO(10)^\sim)$.

The polar is unstable by Lemma 3.4 since $K^+=T\cdot SO(10)^\sim$ is connected. On the other hand, the meridian is stable. Since we have $G^-=T'\cdot SO'(10)^\sim\times T''\cdot SO''(10)^\sim$ with $\rho=\{(T'+\omega_5(D'_5))+0\}\oplus\{0+(T''+\omega_5(D''_5))\}$ and $\mu^+=T+\omega_5(D_5)$. If

$\nu\in D(T'\cdot SO'(10)^{\sim}\times T''\cdot SO''(10)^{\sim})$ satisfies $a_\nu<a_\rho$, then $\nu=0+0$, $(T'+\omega_1(D'_5))+0$, or $0+(T''+\omega_1(D''_5))$. But we can see easily that these representations are not admissible.

[19] EI

[19-1] $(M^+, M^-)=(CI(4)^*, S^2\cdot AI(6))$.

The polar is stable. Since we have $G=E_6$, $K=Sp(4)^*$, $K^+{}_o=U(4)/\boldsymbol{Z}_2$, $\mu=\omega_4(C_4)$ and $\mu^-=T\oplus 2\omega_2(A_3)$. If $\lambda\in D(Sp(4)^*)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$, ω_2 or $2\omega_1$. And we have the next decompositions as $K^+{}_o$-modules.

$$V_{\omega_4(C_4)}=W_{(\omega_1+\omega_3)(A_3)}\oplus W_{\omega_2(A_3)}\oplus W_{\omega_2(A_3)}.$$

$$V_{2\omega_1(C_4)}=W_{2\omega_1(A_3)}\oplus W_{2\omega_3(A_3)}\oplus W_{(\omega_1+\omega_3)(A_3)}\oplus C.$$

So we can conclude that there is no admissible representation. The meridian is also stable. Since we have $G^-=Sp(1)\cdot SU(6)$ with $\rho=\omega_1(A_1)+\omega_3(A_5)$ and $\mu^+=T+2\omega_1(A_3)$. If $\nu\in D(Sp(1)\cdot SU(6))$ satisfies $a_\nu<a_\rho$, then $\nu=\sigma+0$ or $0+\sigma'$, where $\sigma\in D(Sp(1))$ and $\sigma'\in D(SU(6))$ which satisfies $a_\sigma<a_\rho$ and $a_{\sigma'}<a_\rho$, $\nu=\omega_1+\omega_1$, $\nu=\omega_1+\omega_5$, $\nu=3\omega_1+\omega_1$, or $\nu=3\omega_1+\omega_5$. Because both T and A_3 act non-trivially in μ^+, we may consider only $\nu=\omega_1+\omega_1$ and $\nu=3\omega_1+\omega_1$ since $\omega_1(A_3)$ and $\omega_3(A_3)$ are equivalent under the outer automorphism of A_3. We have the next decompositions as $K^+{}_o$-modules.

$$V_{\omega_1(A_1)}\otimes V_{\omega_1(A_5)}=(V(\omega_1(A_1))\oplus V(-\omega_1(A_1)))\otimes(W_{\omega_1(A_3)}\oplus C\oplus C),$$

$$V_{3\omega_1(A_1)}\otimes V_{\omega_1(A_5)}=(V(3\omega_1(A_1))\oplus V(\omega_1(A_1))$$
$$\oplus V(-\omega_1(A_1))\oplus V(-3\omega_1(A_1)))\otimes(W_{\omega_1(A_3)}\oplus C\oplus C)$$

where $V(a)$ denotes a weight space of a weight a. So we conclude that there is no admissible representation.

[19-2] $(M^+, M^-)=(G_2(\boldsymbol{H}^4)^*, T\cdot G_5{}^o(\boldsymbol{R}^{10}))$.

The polar is unstable. Since we have the inclusion $\iota: EI\to E_6$ which carries the polar $G_2(\boldsymbol{H}^4)^*$ of EI into the polar $EIII$ of E_6 and the corresponding meridian $T\cdot G_5{}^o(\boldsymbol{R}^{10})$ of EI into the corresponding meridian $T\cdot SO(10)^{\sim}$ of E_6. We can conclude that $M^+=G_2(\boldsymbol{H}^4)^*$ has a trivial line bundle as in Remark 3.11. On the other hand, the meridian is stable. Since we have $G=E_6$, $G^-=T\cdot SO(10)^{\sim}$, $K^+=SO(5)^{\sim}\cdot SO(5)^{\sim}$, $\rho=T+\omega_5(D_5)$ and $\mu^+=\omega_1(C_2)+\omega_1(C''_2)$. If $\nu\in D(T\cdot SO(10)^{\sim})$ satisfies $a_\nu<a_\rho$, then $\nu=0$ or $T+\omega_1$. And we have the next decomposition as a K^+-module.

$$V_{\omega_1(D_5)}=W_{\omega_1(B_2)}\oplus W_{\omega_1(B'_2)}.$$

Because $\omega_1(B_2)=\omega_2(C_2)$ and $\omega_1(B'_2)=\omega_2(C'_2)$, $\nu=T+\omega_1$ is not admissible.

Type E_7

[20.1] E_7

$$(M^+, M^-)=(EVI, Sp(1)\cdot SO(12)^\sim).$$

The polar is stable. Since we have $G=E_7\times E_7$, $K=E_7$, $K^+=Sp(1)\cdot SO(12)^\sim$, $\mu=\omega_1(E_7)$ and $\mu^-=2\omega_1(A_1)\oplus\omega_2(D_6)$. If $\lambda\in D(E_7)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$ or ω_7. And we have the next decomposition as a K^+-module.

$$V_{\omega_7(E_7)}=W_{\omega_1(A_1)}\otimes W_{\omega_1(D_6)}\oplus W_{\omega_6(D_6)}.$$

So we can conclude that $\lambda=\omega_7$ is not admissible. The meridian is also stable. Since we have $G^-=Sp(1)\cdot SO(12)^\sim\times Sp(1)\cdot SO(12)^\sim$ with $\rho=\{(\omega_1(A'_1)+\omega_6(D'_6))+0\}\oplus\{0+(\omega_1(A''_1)+\omega_6(D''_6))\}$ and $\mu^+=\omega_1(A_1)+\omega_6(D_6)$. If $\nu\in D(Sp(1)\cdot SO(12)^\sim\times Sp(1)\cdot SO(12)^\sim)$ satisfies $a_\nu<a_\rho$, then $\nu=\sigma+0$ or $0+\sigma$ where $\sigma\in D(Sp(1)\cdot SO(12)^\sim)$ such that $a_\sigma<a_\rho$, or $\nu=2\omega_1(A_1)+2\omega_1(A'_1)$. As for $\nu=\sigma+0$ or $0+\sigma$, we can see that these representations are not admissible. And $\nu=2\omega_1(A_1)+2\omega_1(A'_1)$ is not admissible either because $SO(12)^\sim$ acts trivially in the representation and non-trivially in μ^+.

[20.2] $E_7{}^*$

[20.2-1] $(M^+, M^-)=(EVI, Sp(1)\cdot SO(12)^*)$.

Both M^+ and M^- are stable from the result of [20.1] and by Lemma 3.5.

[20.2-2] $(M^+, M^-)=(EV^*, SU(8)/\boldsymbol{Z}_4)$.

The polar is stable. Since we have $G=E_7{}^*\times E_7{}^*$, $K=E_7{}^*$, $K^+=SU(8)/\boldsymbol{Z}_4$, $\mu=\omega_1(E_7{}^*)$ and $\mu^-=(\omega_1+\omega_7)(A_7)$. If $\lambda\in D(E_7{}^*)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$ because ω_7 is not a representation of $E_7{}^*$. As μ^- does not include a trivial representation, $\lambda=0$ is not admissible. The meridian is also stable. Since we have $G^-=SU(8)/\boldsymbol{Z}_4\times SU(8)/\boldsymbol{Z}_4$ with $\rho=(\omega_4(A'_7)+0)\oplus(0+\omega_4(A''_7))$ and $\mu^+=\omega_4(A_7)$. If $\nu\in D(SU(8)/\boldsymbol{Z}_4\times SU(8)/\boldsymbol{Z}_4)$ satisfies $a_\nu<a_\rho$, then $\nu=0+0$ because $\omega_j(1\leqq j\leqq n-1,\ j\neq 4)$ is not a representation of $SU(8)/\boldsymbol{Z}_4$. As μ^+ does not include a trivial representation, $\nu=0+0$ is not admissible.

[20.2-3] $(M^+, M^-)=(EVII^*, T\cdot E_6)$.

Because $EVII$ is the centrosome of s-size in E_7 whose index$=1$, the polar $EVII^*$ is stable by Lemma 3.10. The meridian is also stable. Since we have $G^-=T\cdot E_6\times T\cdot E_6$ with $\rho=\{(T'+\omega_1(E'_6))+0\}\oplus\{0+(T''+\omega_1(E''_6))$ and $\mu^+=T+\omega_1(E_6)$. If $\nu\in D(T\cdot E_6\times T\cdot E_6)$ satisfies $a_\nu<a_\rho$, then $\nu=0+0$. Because μ^+

does not include a trivial representation, $\nu=0+0$ is not admissible.

[21.1] *EV*

$$(M^{+}, M^{-})=(G_4(\boldsymbol{C}^8)^*, S^2\cdot G_6{}^o(\boldsymbol{R}^{12})).$$

The polar is stable. Since we have $G=E_7{}^*$, $K=SU(8)/\boldsymbol{Z}_4$, $K^{+}{}_o=S(U(4)/\boldsymbol{Z}_2\cdot U(4)/\boldsymbol{Z}_2)$, $\mu=\omega_4(A_7)$ and $\mu^{-}=T\oplus(\omega_2(A_3)+\omega_2(A'_3))$. If $\lambda\in D(SU(8)/\boldsymbol{Z}_4)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$ or $\omega_1+\omega_7$. And we have the next decomposition of $\lambda=\omega_1+\omega_7$ as a $K^{+}{}_o$-module.

$$V_{(\omega_1+\omega_7)(A_7)}=W_{(\omega_1+\omega_3)(A_3)}\oplus W_{(\omega_1+\omega_3)(A'_3)}\oplus W_{\omega_1(A_3)}$$
$$\otimes W_{\omega_1(A'_3)}\oplus W_{\omega_3(A_3)}\otimes W_{\omega_3(A'_3)}\oplus C.$$

So $\lambda=\omega_1+\omega_7$ is not admissible. The meridian is also stable. Since we have $G^{-}=Sp(1)\cdot SO(12)^{\sim}$ with $\rho=\omega_1(A_1)+\omega_6(D_6)$ and $\mu^{+}=T+(\omega_2(A_3)+\omega_2(A'_3))$. If $\nu\in D(Sp(1)\cdot SO(12)^{\sim})$ satisfies $a_\nu<a_\rho$, then $\nu=0+0$, $\omega_1+\omega_1$, $2\omega_1+0$, $3\omega_1+\omega_1$ or $4\omega_1+0$. Because $SU(4)/\boldsymbol{Z}_2\cdot SU(4)/\boldsymbol{Z}_2$ acts trivially if $\nu=0+0$, $2\omega_1+0$ or $4\omega_1+0$ and non-trivially in μ^{+}, these representations are not admissible. And we have the next decompositions of $\nu=\omega_1+\omega_1$ and $\nu=3\omega_1+\omega_1$ as $K^{+}{}_o$-modules.

$$V_{\omega_1(A_1)}\otimes V_{\omega_1(D_6)}=(V(\omega_1)\oplus V(-\omega_1))\otimes(W_{\omega_1(D_3)}\oplus W_{\omega_1(D'_3)}).$$

$$V_{3\omega_1(A_1)}\otimes V_{\omega_1(D_6)}=(V(3\omega_1)\oplus V(\omega_1)\oplus V(-\omega_1)\oplus V(-3\omega_1))\otimes(W_{\omega_1(D_3)}\oplus W_{\omega_1(D'_3)})$$

where $V(\alpha)$ denotes a weight space of a weight α. So both $\nu=\omega_1+\omega_1$ and $\nu=3\omega_1+\omega_1$ are not admissible.

[21.2] *EV**

[21.2-1] $(M^{+}, M^{-})=(G_4(\boldsymbol{C}^8)^*, S^2\cdot G_6(\boldsymbol{R}^{12})^{\#})$.

We can conclude that both M^{+} and M^{-} are stable by the result of [21.1] and Lemma 3.5.

[21.2-2] $(M^{+}, M^{-})=(AI(8)/\boldsymbol{Z}_4, AI(8)/\boldsymbol{Z}_4)$.

The polar which is congruent to the meridian is stable. Since we have $G=E_7{}^*$, $K=SU(8)/\boldsymbol{Z}_4$, $K^{+}{}_o=SO(8)^*$, $\mu=\omega_4(A_7)$ and $\mu^{-}=2\omega_3(D_4)$ or $2\omega_4(D_4)$. If $\lambda\in D(SU(8)/\boldsymbol{Z}_4)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$ or $\omega_1+\omega_7$. And we have the next decomposition of $\lambda=\omega_1+\omega_7$ as a $K^{+}{}_o$-module.

$$V_{(\omega_1+\omega_7)(A_7)}=W_{\omega_2(D_4)}\oplus W_{2\omega_1(D_4)}.$$

So $\lambda=\omega_1+\omega_7$ is not admissible.

[21.2-3] $(M^{+}, M^{-})=(AII(4)^*, T\cdot EI)$.

The polar is stable by Lemma 3.10, since $AII(4)$ is one of the connected components of the centrosome of EV which is of s-size and whose index=1. The meridian is also stable. Since we have $G=E_7{}^*$, $G^-=T\cdot E_6$, $K^+{}_0=Sp(4)^*$, $\rho=T+\omega_1(E_6)$ and $\mu^+=\omega_2(C_4)$. If $\nu\in D(T\cdot E_6)$ satisfies $a_\nu<a_\rho$, then ν is a trivial representation or a representation in which only T acts non-trivially. We can see these representations are not admissible because $K^+{}_0$ acts trivially in these representations and non-trivially in μ^+.

Type E_8

[22] E_8

[22-1] $(M^+, M^-)=(EVIII, SO(16)^\#)$.

The polar is stable. Since we have $G=E_8\times E_8$, $K=E_8$, $K^+=SO(16)^\#$, $\mu=\omega_8(E_8)$ and $\mu^-=\omega_2(D_8)$. If $\lambda\in D(E_8)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$. But we can easily see that the trivial representation is not admissible. The meridian is also stable. Since we have $G^-=SO(16)^\#\times SO(16)^\#$ with $\rho=\omega_7(D'_8)\oplus\omega_8(D''_8)$ and $\mu^+=\omega_7(D_8)$. If $\nu\in D(SO(16)^\#\times SO(16)^\#)$ satisfies $a_\nu<a_\rho$, then $\nu=\sigma+0$ or $0+\sigma$ where $\sigma\in D(SO(16)^\#)$ is ω_1 or ω_2. We can see each ν is not admissible, since at least one simple factor of G^- acts trivially in ν and both simple factors act non-trivially in μ^+.

[22-1] $(M^+, M^-)=(EIX, Sp(1)\cdot E_7)$.

The polar is stable. Since we have $G=E_8\times E_8$, $K=E_8$, $K^+=Sp(1)\cdot E_7$, $\mu=\omega_8(E_8)$ and $\mu^-=2\omega_1(A_1)\oplus\omega_1(E_7)$. If $\lambda\in D(E_8)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$. Because μ^- does not include a trivial representation, $\lambda=0$ is not admissible. The meridian is also stable. Since we have $G^-=Sp(1)\cdot E_7\times Sp(1)\cdot E_7$ with $\rho=\{(\omega_1(A'_1)+\omega_7(E'_7))+0\}\oplus\{0+(\omega_1(A''_1)+\omega_7(E''_7))\}$ and $\mu^+=\omega_1(A_1)+\omega_7(E_7)$. If $\nu\in D(Sp(1)\cdot E_7\times Sp(1)\cdot E_7)$ satisfies $a_\nu<a_\rho$, then $\nu=\sigma+0$ or $0+\sigma$ where $\sigma\in D(Sp(1)\cdot E_7)$ is $0+0$, $0+\omega_1$ or $0+\omega_6$. We can see these representations are not admissible, since $Sp(1)$ acts trivially in each ν and non-trivially in μ^+.

[23] $EVIII$

[23-1] $(M^+, M^-)=(G_8(\boldsymbol{R}^{16})^\#, G_8(\boldsymbol{R}^{16})^\#)$.

The polar which is congruent to the meridian is stable. Since we have $G=E_8$, $K=SO(16)^\#$, $K^+=SO^\sim(8)\cdot SO^\sim(8)$, $\mu=\omega_8(D_8)$ and $\mu^-=\omega_3(D_4)+\omega_3(D'_4)$ or $\omega_4(D_4)+\omega_4(D'_4)$. If $\lambda\in D(SO(16)^\#)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$, ω_1 or ω_2. And we have the next decompositions of $\lambda=\omega_1$ and $\lambda=\omega_2$ as K^+-modules.

$$V_{\omega_1(D_8)}=W_{\omega_1(D_4)}\oplus W_{\omega_1(D'_4)}.$$

$$V_{\omega_2(D_8)}=W_{\omega_2(D_4)}\oplus W_{\omega_2(D'_4)}\oplus(W_{\omega_1(D_4)}\otimes W_{\omega_1(D'_4)}).$$

Since $\omega_1(D_4)$, $\omega_3(D_4)$ and $\omega_4(D_4)$ are not equivalent to each other (they are equivalent under the outer automorphism of D_4), we can conclude that there is no admissible representation.

[23-2] $(M^+, M^-)=(DIII(8)^*, S^2\cdot EV)$.

The polar is stable. Since we have $G=E_8$, $K=SO(16)^\#$, $K^+{}_o=U(8)/\boldsymbol{Z}_2$, $\mu=\omega_8(D_8)$ and $\mu^-=T\oplus\omega_4(A_7)$, $\lambda\in D(SO(16)^\#)$ which satisfies the condition $a_\lambda<a_\mu$ is same as in [23-1]. And we have the next decompositions of $\lambda=\omega_1$ and $\lambda=\omega_2$ as $K^+{}_o$-modules.

$$V_{\omega_1(D_8)}=W_{\omega_1(A_7)}\oplus W_{\omega_7(A_7)}.$$

$$V_{\omega_2(D_8)}=W_{\omega_2(A_7)}\oplus W_{\omega_6(A_7)}\oplus W_{(\omega_1+\omega_7)(A_7)}\oplus C.$$

So we conclude that each λ is not admissible. The meridian is also stable. Since we have $G^-=Sp(1)\cdot E_7$ with $\rho=\omega_1(A_1)+\omega_7(E_7)$ and $\mu^+=T+\omega_2(A_7)$. If $\nu\in D(Sp(1)\cdot E_7)$ satisfies $a_\nu<a_\rho$, then $\nu=\sigma+0$ or $0+\sigma'$ where $\sigma\in D(Sp(1))$ and $\sigma'\in D(E_7)$ such that $a_\sigma<a_\rho$ and $a_\sigma'<a_\rho$. Since at least one of simple factor of G^- acts trivially in each ν and both simple factors act non-trivially in μ^+, neither one is admissible.

Type F_4

[24] F_4

[24-1] $(M^+, M^-)=(FI, Sp(1)\cdot Sp(3))$.

The polar is stable. Since we have $G=F_4\times F_4$, $K=F_4$, $K^+=Sp(1)\cdot Sp(3)$, $\mu=\omega_1(F_4)$ and $\mu^-=2\omega_1(A_1)\oplus 2\omega_1(C_3)$. If $\lambda\in D(F_4)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$ or ω_4. And we have the next decomposition as a K^+-module.

$$V_{\omega_4(F_4)}=W_{\omega_1(A_1)}\otimes W_{\omega_1(C_3)}\oplus W_{\omega_2(C_3)}.$$

So we conclude that there is no admissible representation. The meridian is also stable. Since we have $G^-=Sp(1)\cdot Sp(3)\times Sp(1)\cdot Sp(3)$ with $\rho=\{(\omega_1(A'_1)+\omega_3(C'_3))+0\}\oplus\{0+(\omega_1(A''_1)+\omega_3(C''_3))\}$ and $\mu^+=\omega_1(A_1)+\omega_3(C_3)$. If $\nu\in D(Sp(1)\cdot Sp(3)\times Sp(1)\cdot Sp(3))$ satisfies $a_\nu<a_\rho$, then $\nu=\sigma+0$ or $0+\sigma$ where $\sigma\in D(Sp(1)\cdot Sp(3))$ such that $a_\sigma<a_\rho$, or $\nu=2\omega_1(A'_1)+2\omega_1(A''_1)$. As for $\nu=\sigma+0$ or $0+\sigma$, we can see that it is not admissible as in [1]. $\nu=2\omega_1(A'_1)+2\omega_1(A''_1)$, is not admissible either because $Sp(3)$ acts trivially in the representation and non-trivially in μ^+.

[24-2] $(M^+, M^-)=(FII, SO(9)^\sim)$.

The polar is stable. Since we have $G=F_4\times F_4$, $K=F_4$, $K^+=SO(9)^\sim$, $\mu=\omega_1(F_4)$ and $\mu^-=\omega_2(B_4)$. If $\lambda\in D(F_4)$ which satisfies $a_\lambda<a_\mu$ is same as in [24-1]. We can see that $\lambda=\omega_4$ is not admissible because $\dim V_\lambda=26$ and $\dim M^-=36$. The meridian is also stable. Since we have $G^-=SO(9)^\sim\times SO(9)^\sim$ with $\rho=(\omega_4(B'_4)+0)\oplus(0+\omega_4(B''_4))$ and $\mu^+=\omega_4(B_4)$. If $\nu\in D(SO(9)^\sim\times SO(9)^\sim)$ satisfies $a_\nu<a_\rho$, then $\nu=0+0$, $\omega_1(B'_4)+0$ or $0+\omega_1(B''_4)$. We can see these representations are not admissible as in [1].

[25] FI

[25-1] $(M^+, M^-)=(G_1(\boldsymbol{H}^3), G_4{}^o(\boldsymbol{R}^9))$.

[25-2] $(M^+, M^-)=(S^2\cdot CI(3), S^2\cdot CI(3))$.

[26] EII

[26-1] $(M^+, M^-)=(G_2(\boldsymbol{C}^6), G_4{}^o(\boldsymbol{R}^{10}))$.

[26-2] $(M^+, M^-)=(S^2\cdot G_3(\boldsymbol{C}^6), S^2\cdot G_3(\boldsymbol{C}^6))$.

[27] EVI

[27-1] $(M^+, M^-)=(G_4{}^o(\boldsymbol{R}^{12}), G_4{}^o(\boldsymbol{R}^{12}))$.

[27-2] $(M^+, M^-)=(S^2\cdot DIII(6), S^2\cdot DIII(6))$.

[28] EIX

[28-1] $(M^+, M^-)=(EVI, G_4{}^o(\boldsymbol{R}^{16}))$.

[28-2] $(M^+, M^-)=(S^2\cdot EVII, S^2\cdot EVII)$.

Type G_2

[29] GI

$$(M^+, M^-)=(S^2\cdot S^2, S^2\cdot S^2).$$

The spaces [25], [26], [27], [28] and [29] are quaternionic Kähler symmetric spaces. Since both the polars and meridians of [25-1], [26-1], [27-1] and [28-1] are quaternionic Kähler, we conclude that these spaces are stable by Lemma 3.2. And we can conclude that among the polars in [25-2], [26-2], [27-2], [28-2] and [29] which are totally complex subspaces (see Section 4) only the polar and the meridian in [29] are unstable. Since in the case of [25-2], [26-2], [27-2] or [28-2], $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are not isomorphic to each other as K^+-modules, we can see that there is no admissible representation. And in [29], $2\omega_1(A_1)$ is admissible.

[30] G_2

$$(M^+, M^-)=(GI, SO(4)).$$

The polar is unstable. Since we have $G=G_2\times G_2$, $K=G_2$, $K^+=SO(4)$, $\mu=\omega_2(G_2)$ and $\mu^-=2\omega_1(A'_1)\oplus 2\omega_1(A''_1)$. If $\lambda\in D(G_2)$ satisfies $a_\lambda<a_\mu$, then $\lambda=0$ or ω_1. And we have the next decomposition as a K^+-module.

$$V_{\omega_1(G_2)}=W_{\omega_1(A_1)}\otimes W_{\omega_1(A'_1)}\oplus W_{2\omega_1(A'_1)}.$$

So we can see $\lambda=\omega_1$ is admissible. The meridian is also unstable. Since we have $G^-=SO(4)\times SO(4)$ with $\rho=\{(\omega_1(A^L{}_1)+3\omega_1(A'^L{}_1))+0\}\oplus\{0+(\omega_1(A^R{}_1)+3\omega_1(A'^R{}_1))$ and $\mu^+=\omega_1(A_1)+3\omega_1(A'_1)$. We can choose $\nu=(\omega_1(A^L{}_1)+\omega_1(A'^L{}_1))+(2\omega_1(A^R{}_1)+0)$ as $\nu\in D(SO(4)\times SO(4))$ which satisfies $a_\nu<a_\rho$. And we have the next decomposition as a K^+-module.

$$V_{\omega_1(A^L{}_1)}\otimes V_{\omega_1(A'^L{}_1)}\otimes V_{2\omega_1(A^R{}_1)}=W_{\omega_1(A_1)}\otimes W_{3\omega_1(A'_1)}\oplus W,$$

where W is certain K^+-module. So we conclude that ν is admissible.

Now we have the next theorem:

THEOREM 3.12. *Let M be a compact connected irreducible symmetric space and N a polar or a meridian of M. If N is unstable, then it falls into one of the following five cases:*

Case 1. $M=G_2$ *and* $N=GI$ *or* $SO(4)$.

Case 2. $M=SO^\sim(4n+3)$ *and* $N=G_{4n}{}^o(\boldsymbol{R}^{4n+3})$.

Case 3. $M=G_{2m+1}{}^o(\boldsymbol{R}^n)$ *and* $N=S^{2m}\cdot G_{2m}{}^o(\boldsymbol{R}^{n-2m-1})$ *or* $S^{n-4m-1}\cdot G_{2m}{}^o(\boldsymbol{R}^{4m})$.

Case 4. $M=GI$ *and* $N=S^2\cdot S^2$.

Case 5. N has a trivial line bundle as a subbundle of the normal bundle in M.

4. Some other results.

We have next proposition.

PROPOSITION 4.1. *Let $M=G/K$ be a compact irreducible quaternionic Kähler symmetric space. Assume that a meridian $M^-(p)$ is not quaternionic Kähler. Then $M^-(p)$ is congruent with $M^+(p)$.*

PROOF. Let $Q(p)$ denote $s_p\cdot s_0$. Then $adQ(p)$ does not trivially act on the normal subgroup, $Sp(1)$ of K. In fact, if $adQ(p)$ does, $Sp(1)$ can act on $M^-(p)$ because $Q(p)(kx)=Q(p)kQ(p)^{-1}Q(p)x=adQ(p)(k)(Q(p)x)=kx$ for any k in $Sp(1)$ and x in $\mathfrak{m}^-$. This is contrary to the assumption. Thus $ad(s_p)$ does not act on $Sp(1)$ as identity because $ad(s_0)$ does. Then there exists an one-parameter subgroup S_1 of $Sp(1)$ such that the restriction of $ad(s_p)$ to S_1 is s_1, the point symmetry at the unit element of S_1. Take J in S_1 with $J^2=-1$, then we have $s_p(Jx)=s_pJs_ps_px=ad(s_p)(J)(s_px)=J^{-1}(-x)=(-J)(-x)=Jx$ for any x in $\mathfrak{m}^-$. That is $Jx\in\mathfrak{m}^+$ and so $J\mathfrak{m}^-$ is contained in $\mathfrak{m}^+$. On the other hand, for any y in $\mathfrak{m}^+$, we have $s_p(Jy)=(-J)y=-Jy$, that is, $Jy\in\mathfrak{m}^-$. Here $J\mathfrak{m}^+$ is contained in $\mathfrak{m}^-$ and $\mathfrak{m}^+$ in $J\mathfrak{m}^-$. Now we have $\mathfrak{m}^+=J\mathfrak{m}^-$ and we conclude that $M^-(p)$ is congruent with $M^+(p)$ by J because of their connectedness. q.e.d.

REMARK 4.1.1. We have a similar fact in case of a symmetric R-space of a Hermitian symmetric space (see 2.23 in [N-2]).

In the above proposition, $M^+(p)$ or $M^-(p)$ is a totally complex submanifold in M. Let M be a compact irreducible quaternionic symmetric space with $\dim M=m$ and N its totally complex totally geodesic submanifolds with $\dim N=m/2$ (see [Tk-2] for the definition of "totally complex"). We call such N totally complex subspace for brevity. In [Tk-2] Takeuchi classified the totally complex subspaces in each compact irreducible quaternionic symmetric space and studied their stability when M is of classical type. Here we have the result for the case when M is of exceptional type.

THEOREM 4.2. *Let M be a compact irreducible quaternionic symmetric space of exceptional type and N its totally complex subspace. Then the stability of N in M is the following:*

(1.1) *$N=S^2\cdot G_3(C^6)$ in $M=EII$ is stable.*
(1.2) *$N=DIII(5)$ in $M=EII$ is stable.*
(1.3) *$N=CI(4)^*$ in $M=EII$ is stable.*

(2.1) *$N=S^2\cdot DIII(6)$ in $M=EVI$ is stable.*
(2.2) *$N=EIII$ in $M=EVI$ is stable.*
(2.3) *$N=G_4(C^8)^*$ in $M=EVI$ is stable.*

(3.1) *$N=S^2\cdot EVII$ in $M=EIX$ is stable.*
(3.2) *$N=DIII(8)^*$ in $M=EIX$ is stable.*

(4) *$N=S^2\cdot CI(3)$ in $M=FI$ is stable.*
(5) *$N=S^2\cdot S^2$ in $M=GI$ is unstable.*

PROOF. We can find that these spaces are totally complex subspaces in each irreducible quaternionic symmetric space of exceptional type by [Tk-2].

Refer to the results in the previous section for the cases (1.1), (2.1), (3.1), (4) and (5) since N is a polar of M in these cases.

CASE (1.2). The automorphism group of N, G_N is Spin (10) and that of M, G_M is E_6. Then we can see the representation of G_N on $\mathfrak{g}^{\perp}$ is equivalent to the restriction of the isotropy representation, that is, $\omega_4 \oplus \omega_5$ where $\mathfrak{g}^{\perp}$ is the orthogonal complement of $\mathfrak{g}_N$ in $\mathfrak{g}_M$ which are the Lie algebras of G_N and G_M respectively. If $\lambda \in D(\mathrm{Spin}(10))$ satisfies that the eigenvalue of its Casimir operator is less than that of ω_4 or ω_5, then $\lambda=0$ or ω_1. On the other hand, $\mathfrak{m}^{\perp}$ is isomorphic to the tangent space of N at a point o T_oN as an $U^{\wedge}(5)$-module, where $\mathfrak{m}^{\perp}$ is the orthogonal complement of T_oN in T_oM. And we have

$$(\mathfrak{m}^{\perp})^C = W_{\omega_2(A_4)} \oplus W_{\omega_3(A_4)}$$

as an $U^{\wedge}(5)$-module. Since we get the next decomposition as an $U^{\wedge}(5)$-module

$$V_{\omega_1(D_5)} = W_{\omega_1(A_4)} \oplus W_{\omega_4(A_4)},$$

we can conclude that $DIII(5)$ is stable.

CASE (1.3). The automorphism group of N, G_N is $Sp(4)^*$ and the automorphism group of M, G_M is E_6. Then we can see the representation of G_N on $\mathfrak{g}^{\perp}$ is equivalent to the isotropy representation, that is, ω_4 where $\mathfrak{g}^{\perp}$ is the orthogonal complement of $\mathfrak{g}_N$ in $\mathfrak{g}_M$ which are the Lie algebras of G_N and G_M respectively. If $\lambda \in D(Sp(4)^*)$ satisfies that the eigenvalue of its Casimir operator is less than that of ω_4, then $\lambda=0$, ω_2 or $2\omega_1$. On the other hand, $\mathfrak{m}^{\perp}$ is isomorphic to the tangent space of N at a point o T_oN only as a vector space not as an $U(4)/\boldsymbol{Z}_2$-module, where $\mathfrak{m}^{\perp}$ is the othogonal complement of T_oN in T_oM. In fact we have

$$(\mathfrak{m}^{\perp})^C = (V(\lambda) \otimes W_{2\omega_3(A_3)}) \oplus (V(-\lambda) \otimes W_{2\omega_1(A_3)})$$

$$\mathfrak{m}^C = (V(\lambda) \otimes W_{2\omega_1(A_3)}) \oplus (V(-\lambda) \otimes W_{2\omega_3(A_3)})$$

as an $U(4)/\boldsymbol{Z}_2$-module where $V(\lambda)$ and $V(-\lambda)$ are the weight spaces of $\omega_1(A_1)$. So $2\omega_1$ is not admissible. Since we get the next decomposition as an $U(4)/\boldsymbol{Z}_2$-module

$$V_{\omega_2(C_4)} = W_{\omega_2(A_3)} \oplus W_{\omega_2(A_3)} \oplus W_{(\omega_1+\omega_3)(A_3)},$$

ω_2 is not also admissible. Now we conclude that $CI(4)^*$ is stable.

For the case (2.2) the similar argument works as the case (1.2). And for the case (2.3) and (3.2) the similar arguments work as the case (1.3). q.e.d.

We discussed the stability of p-harmonic maps in [NS-3] and got the following result.

THEOREM 4.3 (2.12 in [NS-3]). *Let $f: M \to N$ be a non-constant smooth map of a compact connected Riemannian manifold M into another. Assume that f is isometric and totally geodesic immersion. Then f is stable as a p-harmonic map for a sufficiently large p if and only if f is stable as a minimal immersion.*

We get the immediate corollary of this theorem.

COROLLARY 4.4. *Let M be a compact connected irreducible symmetric space and N a polar or a meridian of M which does not belong to the five cases in Theorem 3.12 in the previous section. Then the inclusion map $f: N \to M$ is a stable p-harmonic map for a sufficiently large p.*

APPENDIX. The isotropy representations (cf. [Be] for example)

$M=G/K$	*isotropy representation*
$SU(n)=SU(n)\times SU(n)/SU(n)$	$(\omega_1+\omega_{n-1})(A_{n-1})$
$Spin(n)=Spin(n)\times Spin(n)/SO(n)$	$\omega_2(SO(n))$
$Sp(n)=Sp(n)\times Sp(n)/Sp(n)$	$2\omega_1(C_n)$
$E_6=E_6\times E_6/E_6$	$\omega_2(E_6)$
$E_7=E_7\times E_7/E_7$	$\omega_1(E_7)$
$E_8=E_8\times E_8/E_8$	$\omega_8(E_8)$
$F_4=F_4\times F_4/F_4$	$\omega_1(F_4)$
$G_2=G_2\times G_2/G_2$	$\omega_2(G_2)$
$AI(n)=SU(n)/SO(n)$	$2\omega_1(SO(n))$
$AII(n)=SU(2n)/Sp(n)$	$\omega_2(C_r)$
$G_r^o(\boldsymbol{R}^n)=SO(n)/SO(r)\times SO(n-r)$	$\omega_1(SO(r))+\omega_1(SO(n-r))$
$G_r(\boldsymbol{C}^n)=SU(n)/T\cdot(SU(r)\times SU(n-r))$	$T+\omega_1(A_{r-1})+\omega_1(A_{n-r-1})$
$G_r(\boldsymbol{H}^n)=Sp(n)/Sp(r)\times Sp(n-r)$	$\omega_1(C_r)+\omega_1(C_{n-r})$
$CI(n)=Sp(n)/U(n)$	$T+2\omega_1(A_{n-1})$
$DIII(n)=SO(2n)/U(n)$	$T+\omega_2(A_{n-1})$
$EI=E_6/Sp(4)^*$	$\omega_4(C_4)$

$EII=E_6/Sp(1)\cdot SU(6)$	$\omega_1(A_1)+\omega_3(A_5)$
$EIII=E_6/T\cdot Spin(10)$	$T+\omega_5(D_5)$
$EIV=E_6/F_4$	$\omega_4(F_4)$
$EV=E_7/SU(8)/\boldsymbol{Z}_2$	$\omega_4(A_7)$
$EVI=E_7/Sp(1)\cdot Spin(12)$	$\omega_1(A_1)+\omega_6(D_6)$
$EVII=E_7/T\cdot E_6$	$T+\omega_1(E_6)$
$EVIII=E_8/SO(16)^{\#}$	$\omega_7(D_8)$
$EIX=E_8/Sp(1)\cdot E_7$	$\omega_1(A_1)+\omega_7(E_7)$
$FI=F_4/Sp(1)\cdot Sp(3)$	$\omega_1(A_1)+\omega_3(C_3)$
$FII=F_4/Spin(9)$	$\omega_4(B_4)$
$GI=G_2/SO(4)$	$\omega_1(A_1)+3\omega_1(A'_1)$

References

[B] N. Bourbaki, Groups et Algebres de Lie, Chap. 4-8.

[Be] M. Berger, Les espaces symetriques non compacts, Ann. Sci. Ecole Norm. Sup. **74** (1957), 85-177.

[CN-1] B. Y. Chen and T. Nagano, Totally geodesic submanifolds of symmetric spaces, II, Duke Math. J. **45** (1978), 405-425.

[CN-2] B. Y. Chen and T. Nagano, A Riemannian geometric invariant and its applications to a problem of Borel and Serre, Trans. Amer. Math. Soc. **308** (1988), 273-297.

[M] K. Mashimo, On the stability of Cartan embeddings of compact symmetric spaces, Arch. Math., **58** (1992), 500-508.

[MT-1] K. Mashimo and H. Tasaki, Stability of closed Lie subgroups in compact Lie groups, Kodai Math. J. **13** (1990), 181-203.

[MT-2] K. Mashimo and H. Tasaki, Stability of maximal tori in compact Lie groups, Algebras, Groups and Geometries, **7** (1990), 114-126.

[N-1] T. Nagano, The involutions of compact symmetric spaces, Tokyo J. Math. **11** (1988), 57-79.

[N-2] T. Nagano, The involutions of compact symmetric spaces, II, Tokyo J. Math. **15** (1992), 39-82.

[NS-1] T. Nagano and M. Sumi, The structure of the symmetric space with applications, Ceometry of Manifolds (K. Shiohama, ed.), Academic Press (1989), 111-128.

[NS-2] T. Nagano and M. Sumi, The spheres in symmetric spaces, Hokkaido Math. J., **20** (1991), 331-352.

[NS-3] T. Nagano and M. Sumi, Stability of p-harmonic maps, Tokyo J. Math., **15** (1992), 475-482.

[O] Y. Ohnita, On stability of minimal submanifolds in compact symmetric spaces, Composito Mathematica **64** (1987), 157-189.

[Tk-1] M. Takeuchi, Stability of certain minimal submanifolds of compact Hermitian symmetric spaces, Tohoku Math. J. **36** (1984), 293-314.

[Tk-2] M. Takeuchi, Totally complex submanifolds of quaternionic symmetric spaces, Japan J. Math. **12** (1986), 161-189.

[Ts-1] H. Tasaki, Certain minimal or homologically volume minimizing submanifolds in compact symmetric spaces, Tsukuba J. Math. **9** (1985), 117-131..

[Ts-2] H. Tasaki, Quaternionic submanifolds in quaternionic symmetric spaces, Tohoku Math. J. **38** (1986), 513-538.

Present Address:
Division of Natural Sciences,
College of Liberal Arts,
International Christian University,
Mitaka, Tokyo, 181 JAPAN