

## FUNCTION SPACES WHICH ARE STRATIFIABLE<sup>(\*)</sup>

By

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**Abstract.** Let  $X$  be a compact metric space and  $Y$  a stratifiable space. By  $C(X, Y)$ , we denote the space of continuous maps from  $X$  to  $Y$  with the compact-open topology. In general,  $C(X, Y)$  is not stratifiable. In this paper, we show that  $C(X, Y)$  is stratifiable if  $Y$  satisfies the condition given by Mizokami [Mi]. And we construct a stratifiable space  $Y$  such that  $C(X, Y)$  is not stratifiable even if  $X$  is countable and compact.

### 1. Introduction.

Let  $X$  and  $Y$  be topological spaces. By  $\mathfrak{F}(X)$ ,  $\mathfrak{R}(X)$  and  $\mathfrak{O}(X)$ , we denote the families of all nonempty finite subsets, all compact subsets and all open subsets of  $X$ , respectively. By  $C(X, Y)$ , we denote the space of all continuous maps of  $X$  to  $Y$  admitting the compact-open topology, whose open base is

$$\{M(K_1, \dots, K_n; U_1, \dots, U_n) \mid n \in \mathbf{N}, \\ K_i \in \mathfrak{R}(X), U_i \in \mathfrak{O}(Y) \quad \text{for } i=1, \dots, n\}$$

where

$$M(K_1, \dots, K_n; U_1, \dots, U_n) \\ = \{f \in C(X, Y) \mid f(K_i) \subset U_i \quad \text{for } i=1, \dots, n\}$$

A regular space  $Y$  is *stratifiable* if it has a  $\sigma$ -closure preserving (abbrev.  $\sigma$ -CP) quasi-base  $\mathcal{B}$  [Ce] (cf. [Bo<sub>1</sub>]), where  $\mathcal{B}$  is a *quasi-base* for  $Y$  if for any  $y \in Y$  and each neighborhood  $U$  of  $y$ , there exists  $B \in \mathcal{B}$  such that  $y \in \text{Int } B \subset B \subset U$ . In general,  $C(X, Y)$  is not stratifiable even if  $X$  is compact metric and  $Y$  is stratifiable. In fact, Borges [Bo<sub>2</sub>] constructed a stratifiable space  $Y$  such that  $C(\mathbf{I}, Y)$  is not normal, where  $\mathbf{I}=[0, 1]$  is the unit interval.

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Similarly to  $C(X, Y)$ , the hyperspace  $\mathfrak{R}(Y)$  with the Vietoris topology is not stratifiable even if  $Y$  is stratifiable (cf. [MK] and [Mi]). In [Mi], Mizokami gave a condition for  $Y$  such that  $\mathfrak{R}(Y)$  is stratifiable. In this paper, we show that if  $Y$  satisfies this Mizokami's condition then  $C(X, Y)$  is stratifiable for any compact metric space  $X$ . Cauty [Ca<sub>3</sub>] proved that if  $Y$  is a  $CW$ -complex then  $C(X, Y)$  is stratifiable for any compact space  $X$ . But any non-metrizable  $CW$ -complex  $Y$  does not satisfy the Mizokami's condition by [Mi, Theorem 4.3] (or cf. [GS, Example 3.2]). Therefore our result is independent from Cauty's result.

By  $C_p(X, Y)$ , we denote the space of all continuous maps from  $X$  to  $Y$  admitting the pointwise convergence topology, that is,  $C_p(X, Y)$  is a subspace of the product space  $Y^X$ . Note that if  $Y$  is stratifiable then  $C_p(X, Y)$  is stratifiable for a countable space  $X$ , since it can be embedded in the countable product space  $Y^\omega$  of  $Y$  (cf. [Ce]). Thus it is natural to ask whether  $C(X, Y)$  is stratifiable for a compact countable space  $X$  and a stratifiable space  $Y$ . However it can be seen in Section 3 that  $C(X, Y)$  is not stratifiable for a compact countable space  $X$  and a stratifiable space  $Y$  which is constructed by Mizokami in [Mi, Example 2.1].

## 2. Main Result.

For a family  $\mathcal{B}$  of subsets of  $Y$  and  $A \subset Y$ , let  $\mathcal{B}|A = \{B \cap A \mid B \in \mathcal{B}\}$ . We say that  $\mathcal{B}$  is *finite on compact sets* (abbrev. *CF*) in  $Y$  if  $\mathcal{B}|K$  is finite for each  $K \in \mathfrak{R}(X)$ . And  $\mathcal{B}$  is  $\sigma$ -*CP-CF* if it can be written as  $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$  such that each  $\mathcal{B}_n$  is CP (closure-preserving) and CF in  $Y$ . In this section, we show the following theorem.

**THEOREM 2.1.** *Let  $X$  be a compact metric space and  $Y$  a stratifiable space which has a  $\sigma$ -CP-CF quasi-base consisting of closed sets. Then  $C(X, Y)$  has a  $\sigma$ -CP quasi-base, hence it is stratifiable.*

To prove this theorem, we need some lemmas.

**LEMMA 2.2.** *Let  $\mathcal{U}$  be an open set in  $C(X, Y)$ ,  $f \in \mathcal{U}$  and  $\mathcal{B}$  a quasi-base for  $Y$ . Then there exist  $K_1, \dots, K_n \in \mathfrak{R}(X)$  and  $B_1, \dots, B_n \in \mathcal{B}$  such that*

$$\begin{aligned} f &\in M(K_1, \dots, K_n; \text{Int } B_1, \dots, \text{Int } B_n) \\ &\subset M(K_1, \dots, K_n; B_1, \dots, B_n) \subset \mathcal{U}, \end{aligned}$$

that is, the family

$$\{M(K_1, \dots, K_n; B_1, \dots, B_n) \mid n \in \mathbf{N}, \\ K_i \in \mathfrak{R}(X), B_i \in \mathfrak{B} \ (i=1, \dots, n)\}$$

is a quasi-base for  $C(X, Y)$ .

PROOF. Since  $\mathcal{U}$  is open in  $C(X, Y)$ , we have  $K_1, \dots, K_m \in \mathfrak{R}(X)$  and  $U_1, \dots, U_m \in \mathfrak{D}(Y)$  such that

$$f \in M(K_1, \dots, K_m; U_1, \dots, U_m) \subset \mathcal{U}.$$

For any  $i=1, \dots, m$  and  $x \in K_i$ , since  $f(x) \in U_i$ , there is  $B_i^x \in \mathfrak{B}$  such that  $f(x) \in \text{Int } B_i^x \subset B_i^x \subset U_i$ , whence

$$x \in f^{-1}(\text{Int } B_i^x) \subset f^{-1}(B_i^x) \subset f^{-1}(U_i).$$

By compactness of  $K_i$ , there are  $x_1^i, \dots, x_{n(i)}^i \in K_i$  such that

$$K_i \subset \bigcup_{j=1}^{n(i)} f^{-1}(\text{Int } B_{i,j}) = f^{-1}\left(\bigcup_{j=1}^{n(i)} \text{Int } B_{i,j}\right) \\ \subset f^{-1}\left(\bigcup_{j=1}^{n(i)} B_{i,j}\right) \subset f^{-1}(U_i).$$

where  $B_{i,j} = B_{x_j^i}^i$ . Then  $K_i$  has a closed cover  $\{K_{i,j}\}_{j=1}^{n(i)}$  such that  $K_{i,j} \subset f^{-1}(\text{Int } B_{i,j})$ . Note that  $K_{i,j} \in \mathfrak{R}(X)$ . It is clear that

$$f \in M(K_{i,1}, \dots, K_{i,n(i)}; \text{Int } B_{i,1}, \dots, \text{Int } B_{i,n(i)}) \\ \subset M(K_{i,1}, \dots, K_{i,n(i)}; B_{i,1}, \dots, B_{i,n(i)}) \subset M(K_i, U_i)$$

Therefore we have

$$f \in \bigcap_{i=1}^m M(K_{i,1}, \dots, K_{i,n(i)}; \text{Int } B_{i,1}, \dots, \text{Int } B_{i,n(i)}) \\ \subset \bigcap_{i=1}^m M(K_{i,1}, \dots, K_{i,n(i)}; B_{i,1}, \dots, B_{i,n(i)}) \\ \subset M(K_1, \dots, K_m; U_1, \dots, U_m) \subset \mathcal{U}. \quad \square$$

LEMMA 2.3. Let  $\mathfrak{B}$  be a CP (resp. CF) family of closed sets in  $X$ . Then  $\mathfrak{B}^* = \{\bigcap \mathcal{A} \mid \mathcal{A} \in \mathfrak{F}(\mathfrak{B})\}$  is also CP (resp. CF).

PROOF. The CF case is obvious. To see the CP case, let  $\mathfrak{C} \subset \mathfrak{F}(\mathfrak{B})$ . We prove that

$$\bigcup_{\mathcal{A} \in \mathfrak{C}} (\bigcap \mathcal{A}) = \overline{\bigcup_{\mathcal{A} \in \mathfrak{C}} (\bigcap \mathcal{A})}.$$

To this end, let  $x \notin \bigcup_{\mathcal{A} \in \mathfrak{C}} (\bigcap \mathcal{A})$ . For each  $\mathcal{A} \in \mathfrak{C}$ , since  $x \notin \bigcap \mathcal{A}$ , we can choose  $B_{\mathcal{A}} \in \mathcal{A}$  such that  $x \notin B_{\mathcal{A}}$ . Since  $\{B_{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{C}\} \subset \mathfrak{B}$  and  $\mathfrak{B}$  is CP, we have

$$\overline{\bigcup_{\mathcal{A} \in \mathfrak{G}} (\bigcap \mathcal{A})} \subset \overline{\bigcup_{\mathcal{A} \in \mathfrak{G}} B_{\mathcal{A}}} = \bigcup_{\mathcal{A} \in \mathfrak{G}} B_{\mathcal{A}}.$$

Since  $x \notin \bigcup_{\mathcal{A} \in \mathfrak{G}} B_{\mathcal{A}}$ ,  $x \notin \overline{\bigcup_{\mathcal{A} \in \mathfrak{G}} (\bigcap \mathcal{A})}$ . Therefore it follows that

$$\bigcup_{\mathcal{A} \in \mathfrak{G}} (\bigcap \mathcal{A}) = \overline{\bigcup_{\mathcal{A} \in \mathfrak{G}} (\bigcap \mathcal{A})}. \quad \square$$

REMARK. In the CP case of the above lemma, it is necessary to assume that members of  $\mathfrak{B}$  are closed in  $X$ . In fact, let

$$X = \{0\} \cup \left\{ \left\{ \frac{1}{n} \right\} \mid n \in \mathbf{N} \right\} \quad \text{and} \quad \mathfrak{B} = \{B \subset X \mid |B| = \aleph_0\}.$$

Note that  $X \setminus \{0\} \in \mathfrak{B}$ , but it is not closed in  $X$ . For any  $\emptyset \neq \mathfrak{B}_0 \subset \mathfrak{B}$ , we have

$$\bigcup \overline{\mathfrak{B}_0} = \{0\} \cup \bigcup \mathfrak{B}_0 = \overline{\bigcup \mathfrak{B}_0},$$

that is,  $\mathfrak{B}$  is CP. On the other hand,  $\left\{ \left\{ \frac{1}{n} \right\} \mid n \in \mathbf{N} \right\} \subset \mathfrak{B}^*$  and

$$\bigcup \overline{\left\{ \left\{ \frac{1}{n} \right\} \mid n \in \mathbf{N} \right\}} \not\subset \bigcup \left\{ \left\{ \frac{1}{n} \right\} \mid n \in \mathbf{N} \right\},$$

whence  $\mathfrak{B}^*$  is not CP.

The following lemma is easy.

LEMMA 2.4. *If  $\mathcal{A}$  and  $\mathfrak{B}$  are CP (resp. CF) families, then  $\mathcal{A} \cup \mathfrak{B}$  is also CP (resp. CF).*  $\square$

PROOF OF THEOREM 2.1. Let  $\mathfrak{B}$  be a  $\sigma$ -CP-CF quasi-base for  $Y$  consisting of closed sets. By Lemma 2.4, we can write  $\mathfrak{B} = \bigcup_{n \in \mathbf{N}} \mathfrak{B}_n$ , where  $\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \dots$  are CP and CF. Using the compactness,  $X$  has a sequence  $\{C_n\}_{n=1}^\infty$  of finite closed covers of  $X$  such that  $\text{mesh } C_n \rightarrow 0$  if  $n \rightarrow \infty$ . For each  $m, n \in \mathbf{N}$  and  $(C_1, \dots, C_m) \in (C_n)^m$ , we define

$$\mathcal{A}_{(C_1, \dots, C_m)}^n = \{M(C_1, \dots, C_m; B_1, \dots, B_m) \mid B_i \in \mathfrak{B}_n \ i=1, \dots, m\}.$$

We shall show that

$$\mathcal{A} = \bigcup_{n \in \mathbf{N}} \bigcup_{m \in \mathbf{N}} \bigcup \{ \mathcal{A}_{(C_1, \dots, C_m)}^n \mid (C_1, \dots, C_m) \in (C_n)^m \}$$

is a quasi-base for  $C(X, Y)$  and that each  $\mathcal{A}_{(C_1, \dots, C_m)}^n$  is CP. Then  $\mathcal{A}$  is  $\sigma$ -CP quasi-base since each  $(C_n)^m$  is finite.

First to prove that  $\mathcal{A}$  is a quasi-base for  $C(X, Y)$ , let  $\mathcal{U}$  be open in  $C(X, Y)$  and  $f \in \mathcal{U}$ . By Lemma 2.2, there are  $K_1, \dots, K_l \in \mathfrak{K}(X)$  and  $B_1, \dots, B_l \in \mathfrak{B}$  such that

$$f \in M(K_1, \dots, K_l; \text{Int } B_1, \dots, \text{Int } B_l) \\ \subset M(K_1, \dots, K_l; B_1, \dots, B_l) \subset \mathcal{U}.$$

Let

$$\eta = \min \{ \text{dist}(K_i, X \setminus f^{-1}(\text{Int } B_i)) \mid i=1, \dots, l \} > 0,$$

where  $\text{dist}(A, \emptyset) = \text{diam } X$ . Since  $\text{mesh } \mathcal{C}_n \rightarrow 0$  ( $n \rightarrow \infty$ ) and  $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots$  we can choose  $n \in \mathbf{N}$  such that  $\text{mesh } \mathcal{C}_n < \eta$  and  $B_1, \dots, B_l \in \mathcal{B}_n$ . For each  $i=1, \dots, l$ , write

$$\{C \in \mathcal{C}_n \mid C \cap K_i \neq \emptyset\} = \{C_{i,1}, \dots, C_{i,m_i}\}.$$

Then  $K_i \subset \bigcup_{j=1}^{m_i} C_{i,j} \subset f^{-1}(\text{Int } B_i)$ , whence

$$f \in M(C_{i,1}, \dots, C_{i,m_i}; \overbrace{\text{Int } B_i, \dots, \text{Int } B_i}^{m_i}) \\ \subset M(C_{i,1}, \dots, C_{i,m_i}; \overbrace{B_i, \dots, B_i}^{m_i}) \subset M(K_i, B_i).$$

Hence we have

$$f \in \bigcap_{i=1}^l M(C_{i,1}, \dots, C_{i,m_i}; \overbrace{\text{Int } B_i, \dots, \text{Int } B_i}^{m_i}) \\ \subset \text{Int} \bigcap_{i=1}^l M(C_{i,1}, \dots, C_{i,m_i}; \overbrace{B_i, \dots, B_i}^{m_i}) \\ \subset \bigcap_{i=1}^l M(C_{i,1}, \dots, C_{i,m_i}; \overbrace{B_i, \dots, B_i}^{m_i}) \\ \subset M(K_1, \dots, K_l; B_1, \dots, B_l) \subset \mathcal{U}.$$

Let  $m = \sum_{i=1}^l m_i$  and

$$(C_1, \dots, C_m) = (C_{1,1}, \dots, C_{1,m_1}, \dots, C_{l,1}, \dots, C_{l,m_l}) \in (\mathcal{C}_n)^m.$$

Then

$$\bigcap_{i=1}^l M(C_{i,1}, \dots, C_{i,m_i}; \overbrace{B_i, \dots, B_i}^{m_i}) \\ = M(C_1, \dots, C_m; \overbrace{B_1, \dots, B_1}^{m_1}, \dots, \overbrace{B_l, \dots, B_l}^{m_l}) \\ \in \mathcal{A}_{(C_1, \dots, C_m)}^n \subset \mathcal{A}.$$

Next to show that each  $\mathcal{A}_{(C_1, \dots, C_m)}^n$  is CP, let  $\mathcal{B}' \subset (\mathcal{B}_n)^m$  and

$$\mathcal{A}' = \{M(C_1, \dots, C_m; B_1, \dots, B_m) \mid (B_1, \dots, B_m) \in \mathcal{B}'\} \\ \subset \mathcal{A}_{(C_1, \dots, C_m)}^n.$$

To prove that  $\overline{\bigcup \mathcal{A}'} = \bigcup \mathcal{A}'$ , let  $g \in C(X, Y) \setminus \bigcup \mathcal{A}'$ . For each  $k=1, \dots, m$ , let  $p_k : (\mathcal{B}_n)^m \rightarrow \mathcal{B}_n$  be the projection defined by  $p_k(B_1, \dots, B_m) = B_k$  and

$$\mathcal{B}'(k) = \{B \in p_k(\mathcal{B}') \mid g(C_k) \notin B\} \subset \mathcal{B}_n.$$

In case  $\mathcal{B}'(k)=\emptyset$ , let  $M_k=C(X, Y)$ . In case  $\mathcal{B}'(k)\neq\emptyset$ , we can write

$$\mathcal{B}'(k)|g(C_k)=\{G_{k,1}, \dots, G_{k,m_k}\},$$

because  $\mathcal{B}_n$  is CF. Note that  $g(C_k)\setminus G_{k,i}\neq\emptyset$  for each  $i=1, \dots, m_k$ . We can choose points  $x_{k,1}, \dots, x_{k,m_k}\in C_k$  such that  $g(x_{k,i})\in g(C_k)\setminus G_{k,i}$ . Then

$$V_{k,i}=Y\setminus\bigcup\{B\in\mathcal{B}_n|g(x_{k,i})\notin B\}$$

is an open neighborhood of  $g(x_{k,i})$  in  $Y$  because  $\mathcal{B}_n$  is CP. Let

$$M_k=M(\{x_{k,1}\}, \dots, \{x_{k,m_k}\}; V_{k,1}, \dots, V_{k,m_k}).$$

Then  $M(g)=\bigcap_{k=1}^m M_k$  is an open neighborhood of  $g$  in  $C(X, Y)$ . And moreover  $M(g)\cap(\bigcup\mathcal{A}')=\emptyset$ . In fact, for any  $(B_1, \dots, B_m)\in\mathcal{B}'$ ,

$$g\notin M(C_1, \dots, C_m; B_1, \dots, B_m),$$

whence  $g(C_k)\not\subset B_k$ , i. e.,  $B_k\in\mathcal{B}'(k)$  for some  $k\leq m$ . Then  $B_k\cap g(C_k)=G_{k,i}$  for some  $i\leq m_k$ , which implies that

$$g(x_{k,i})\in V_{k,i}\cap(g(C_k)\setminus B_k).$$

By the definition of  $V_{k,i}$ , we have  $V_{k,i}\cap B_k=\emptyset$ . Hence

$$M(g)\cap M(C_1, \dots, C_m; B_1, \dots, B_m)=\emptyset.$$

Thus  $g\notin\overline{\bigcup\mathcal{A}'}$ .  $\square$

REMARKS. In the above proof,

$$\mathcal{M}=\{M(C_1, \dots, C_n; B_1, \dots, B_n)|n\in\mathbf{N},$$

$$C_i\in\mathfrak{R}(X), B_i\in\mathcal{B} \quad \text{for } i=1, \dots, n\}$$

is a quasi-base for  $C(X, Y)$  by Lemma 2.2. Since  $\mathcal{B}=\bigcup_{k=1}^{\infty}\mathcal{B}_k$ ,  $\mathcal{M}=\bigcup_{k=1}^{\infty}\mathcal{M}_k$ , where

$$\mathcal{M}_k=\{M(C_1, \dots, C_n; B_1, \dots, B_n)|C_i\in\mathfrak{R}(X), B_i\in\mathcal{B}_k \text{ and } n\in\mathbf{N}\}.$$

Although one might expect that each  $\mathcal{M}_k$  is CP, this is not true. In fact, let  $X=\{0\}\cup\{1/n|n\in\mathbf{N}\}$  and  $Y=[0, 1]$ . We inductively define families  $\mathcal{B}_n$  of closed sets in  $Y$  as follows:  $\mathcal{B}_1=\{[0, 1/2], [1/2, 1]\}$  and

$$\mathcal{B}_n=\mathcal{B}_{n-1}\cup\left\{\left[\frac{i-1}{n+1}, \frac{i}{n+1}\right]\middle| i=1, \dots, n+1\right\}$$

for each  $n>1$ . Clearly  $\mathcal{B}_1\subset\mathcal{B}_2\subset\dots$  are CP and CF in  $Y$  and  $\mathcal{B}=\bigcup_{n\in\mathbf{N}}\mathcal{B}_n$  is a quasi-base of  $Y$ . To see that  $\mathcal{M}_k$  is not CP in  $C(X, Y)$ , let

$$\mathcal{M}'_k=\left\{M\left(\left\{\frac{1}{n}\right\}, \left[\frac{1}{k+1}, \frac{2}{k+1}\right]\right)\middle| n\in\mathbf{N}\right\}\subset\mathcal{M}_k$$

and define  $f \in C(X, Y)$  by

$$f(x) = \frac{1}{k+1}(1-x) \quad \text{for each } x \in X.$$

It is easy to see that  $f \notin \bigcup \mathcal{M}'_k = \overline{\bigcup \mathcal{M}'_k}$ . We show that  $f \in \overline{\bigcup \mathcal{M}'_k}$ . To this end, let  $\mathfrak{U} = \bigcap_{i=1}^l M(C_i, U_i)$  be any basic open neighborhood of  $f$  in  $C(X, Y)$ , where  $C_i \in \mathfrak{R}(X)$  and  $U_i \in \mathfrak{D}(Y)$ . In the case  $0 \notin \bigcup_{i=1}^l C_i$ , there exist  $N \in \mathbf{N}$  such that  $1/m \notin \bigcup_{i=1}^l C_i$  for each  $m \geq N$ . Then we have

$$M\left(\left\{\frac{1}{2N}\right\}, \left[\frac{1}{k+1}, \frac{2}{k+1}\right]\right) \cap \mathfrak{U} \neq \emptyset,$$

whence  $(\bigcup \mathcal{M}'_k) \cap \mathfrak{U} \neq \emptyset$ . In the case  $0 \in C_j$  for some  $j \leq l$ ,  $1/(k+1) = f(0) \in U_j$ . Let

$$U = \bigcap \{U_j \mid 0 \in C_j, j=1, \dots, l\} (\neq \emptyset).$$

Since  $U$  is open in  $Y$  and  $1/(k+1) \in U$ , we can choose some  $m \in \mathbf{N}$  such that

$$\left[\frac{1}{k+1} - \frac{1}{m}, \frac{1}{k+1} + \frac{1}{m}\right] \subset U \quad \text{and} \quad \left[0, \frac{2}{m}\right] \cap C_i = \emptyset \quad \text{if } 0 \notin C_i.$$

We define  $g \in C(X, Y)$  by

$$g(x) = \begin{cases} \frac{1}{k+1} & \text{if } x \leq \frac{1}{m}, \\ \frac{1}{k+1} \left(1 + \frac{2}{m} - 2x\right) & \text{if } \frac{1}{m} \leq x \leq \frac{2}{m}, \\ f(x) & \text{if } x \geq \frac{2}{m}. \end{cases}$$

Then  $g \in M(\{1/m\}, [1/(k+1), 2/(k+1)]) \cap \mathfrak{U}$ , whence  $(\bigcup \mathcal{M}'_k) \cap \mathfrak{U} \neq \emptyset$ .

In fact, if  $C_i \cap [0, 2/m] = \emptyset$  then  $g(C_i) = f(C_i) \subset U_i$ . If  $C_i \cap [0, 2/m] \neq \emptyset$ , we have  $0 \in C_i$ , whence  $U \subset U_i$ . Then

$$\begin{aligned} g(C_i) &= g\left(C_i \cap \left[0, \frac{2}{m}\right]\right) \cup g\left(C_i \setminus \left[0, \frac{2}{m}\right]\right) \\ &\subset \left[\frac{1}{k+1} - \frac{1}{m}, \frac{1}{k+1} + \frac{1}{m}\right] \cup f\left(C_i \setminus \left[0, \frac{2}{m}\right]\right) \\ &\subset U_i. \end{aligned}$$

Therefore  $f \in \overline{\bigcup \mathcal{M}'_k}$ , that is,  $\mathcal{M}_k$  is not CP in  $C(X, Y)$ .

By  $\mathcal{S}$ , we denote the class of stratifiable spaces. It is known that a stratifiable space is an ANR( $\mathcal{S}$ ) iff it is an ANE( $\mathcal{S}$ ). In Theorem 2.1, if  $Y$  is an ANR( $\mathcal{S}$ ) then  $C(X, Y)$  is an ANE( $\mathcal{S}$ ), hence an ANR( $\mathcal{S}$ ). In fact, let  $A$  be a closed set in a stratifiable space  $Z$  and  $\varphi \in C(A, C(X, Y))$ . We define  $\tilde{\varphi} : A \times X$

$\rightarrow Y$  by  $\tilde{\varphi}(a, x) = \varphi(a)(x)$ . By the compactness of  $X$ , we have  $\tilde{\varphi}$  is continuous. Since  $Y$  is an ANE( $\mathcal{S}$ ) and  $Z \times X$  is stratifiable, there exists a neighborhood  $W$  of  $A \times X$  in  $Z \times X$  and  $\tilde{\Phi} \in C(W, Y)$  such that  $\tilde{\Phi}|_{A \times X} = \tilde{\varphi}$ . Since  $X$  is compact,  $A$  have a neighborhood  $U$  in  $Y$  such that  $A \times X \subset U \times X \subset W$ . We define  $\Phi : U \rightarrow C(X, Y)$  by

$$\Phi(z)(x) = \tilde{\Phi}(z, x) \quad (x \in X)$$

for each  $z \in U$ . Then  $\Phi$  is an extension of  $\varphi$  on  $U$ . Thus we have the following result.

**COROLLARY 2.5.** *Let  $X$  be a compact metric space and  $Y$  an ANR( $\mathcal{S}$ ) which has a  $\sigma$ -CP-CF quasi-base consisting of closed sets. Then  $C(X, Y)$  is an ANR( $\mathcal{S}$ ).*

In Theorem 2.1, it is a problem whether metrizability of  $X$  is necessary or not, that is,

**PROBLEM 2.6.** *Is Theorem 2.1 true for a non-metrizable compact space  $X$ ?*

### 3. A Counterexample.

In this section, we show that  $C(X, Y)$  is not stratifiable for  $X = \{0\} \cup \{1/n | n \in \mathbf{N}\}$  and the stratifiable space  $Y$  which is constructed by Mizokami in [Mi, Example 2.1] (indeed,  $Y$  is a countable Lašnev space). First we show the following:

**LEMMA 3.1.** *Let  $X$  be compact,  $y_0 \in Y$  and  $\mathcal{A}$  a neighborhood base of  $y_0$  in  $Y$ . Then  $\{M(X, A) | A \in \mathcal{A}\}$  is a neighborhood base of the constant map  $f_0$  with  $f_0(X) = \{y_0\}$  in  $C(X, Y)$ .*

**PROOF.** For each neighborhood  $\mathcal{N}$  of  $f_0$  in  $C(X, Y)$ , there exist  $C_1, \dots, C_n \in \mathfrak{R}(X)$  and  $U_1, \dots, U_n \in \mathfrak{O}(Y)$  such that

$$f_0 \in M(C_1, \dots, C_n; U_1, \dots, U_n) \subset \mathcal{N}.$$

Since each  $U_i$  is an open neighborhood of  $y_0$  in  $Y$ , there is  $A \in \mathcal{A}$  such that  $A \subset \bigcap_{i=1}^n U_i$ , whence

$$f_0 \in M(X, A) \subset M(C_1, \dots, C_n; U_1, \dots, U_n) \subset \mathcal{N}. \quad \square$$

**EXAMPLE 3.2.** *Let  $X = \{0\} \cup \{1/n | n \in \mathbf{N}\} \subset \mathbf{R}$  be the space of a convergent sequence. There exists a countable Lašnev space  $Y$  such that  $C(X, Y)$  is not stratifiable.*



PROOF. Let  $Y$  be the space of [Mi, Example 2.1], namely  $Y = Y'/A$ , where

$$Y' = \left[ (\mathbf{Q} \cap (0, 1)) \setminus \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\} \right] \times \left[ \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbf{N} \right\} \right]$$

is a subspace of  $\mathbf{R}^2$  and  $A = \{(x, 0) \mid (x, 0) \in Y'\}$ . Let  $p: Y' \rightarrow Y$  be the quotient map and  $y_0 = p(A) \in Y$ . We shall show that  $C(X, Y)$  is not stratifiable. For each  $k \in \mathbf{N}$ , let

$$N_k = p\left( \left( \left( \frac{1}{k+1}, \frac{1}{k} \right) \times \left[ 0, \frac{1}{k} \right] \right) \cap Y' \right)$$

and  $\tilde{N}_k = \bigcup_{i \geq k} N_i$ . For simplicity, we write  $N = \tilde{N}_1$ . Note that  $y_0 \in N$  and  $N$  has the weak topology with respect to  $\{N_k\}_{k \in \mathbf{N}}$ . For each  $(y_1, \dots, y_n) \in N^n$ , we define  $f_{(y_1, \dots, y_n)} \in C(X, N)$  by

$$f_{(y_1, \dots, y_n)}(x) = \begin{cases} y_i & \text{if } x = \frac{1}{i} \geq \frac{1}{n}, \\ y_0 & \text{otherwise.} \end{cases}$$

In case  $y_1 = \dots = y_n = y_0$ ,  $f_{(y_0, \dots, y_0)}$  is the constant map, which is simply denoted by  $f_0$ .

To see that  $C(X, Y)$  is not stratifiable, it suffices to show that  $C(X, N)$  is not stratifiable. On the contrary, assume that  $C(X, N)$  is stratifiable. Then  $f_0$  has a CP neighborhood base  $\mathfrak{B}$  consisting of closed sets in  $C(X, N)$  (see [Ce, Lemma 7.3]). For each  $B^* \in \mathfrak{B}$ , we define a subset  $O(B^*)$  of  $N$  by

$$O(B^*) = \bigcup \{f(X) \mid y_0 \in f(X) \in \mathfrak{F}(N) \text{ and } f \in \text{Int } B^*\}.$$

Then we have

LEMMA 3.3.  $O(\mathfrak{B}) = \{O(B^*) \mid B^* \in \mathfrak{B}\}$  is a neighborhood base of  $y_0$  in  $N$ .

PROOF. For each neighborhood  $V$  of  $y_0$  in  $N$ , there exists  $B^* \in \mathfrak{B}$  such that  $B^* \subset M(X, V)$ , whence  $O(B^*) \subset \bigcup \{f(X) \mid f \in \text{Int } B^*\} \subset V$ .

Next we show that  $O(B^*)$  is a neighborhood of  $y_0$  in  $N$  for each  $B^* \in \mathfrak{B}$ . Since each  $B^*$  is a neighborhood of  $f_0$  in  $C(X, N)$ , there are  $C_1, \dots, C_n \in \mathfrak{R}(X)$  and  $U_1, \dots, U_n \in \mathfrak{O}(N)$  such that

$$f_0 \in M(C_1, \dots, C_n; U_1, \dots, U_n) \subset \text{Int } B^*.$$

Since each  $U_i$  is a neighborhood of  $y_0$  in  $N$ ,  $U = \bigcap_{i=1}^n U_i$  is a neighborhood of  $y_0$  in  $N$ . Observe that for each  $y \in U$ ,

$$f_y \in M(X, U) \subset M(C_1, \dots, C_n; U_1, \dots, U_n) \subset \text{Int } B^*.$$

Then it follows that  $U = \bigcup_{y \in U} f_y(X) \subset O(B^*)$ . Thus  $O(B^*)$  is a neighborhood of  $y_0$  in  $Y$ .  $\square$

Next, for each  $(y_1, \dots, y_n) \in N^n$ , we define

$$\mathfrak{B}(y_1, \dots, y_n) = \{B^* \in \mathfrak{B} \mid f_{(y_1, \dots, y_i)} \in \text{Int } B^* \text{ for each } i=1, \dots, n\}.$$

LEMMA 3.4. For any neighborhood  $V_k$  of  $y_0$  in  $N_k$ ,

$$\mathfrak{B} = \cup \{\mathfrak{B}(y) \mid y \in V_k \setminus \{y_0\}\}.$$

And for any  $(y_1, \dots, y_n) \in N^n$  and any neighborhood  $V_k$  of  $y_0$  in  $N_k$ ,

$$\mathfrak{B}(y_1, \dots, y_n) = \cup \{\mathfrak{B}(y_1, \dots, y_n, y) \mid y \in V_k \setminus \{y_0\}\}.$$

PROOF. Because of similarity, we show only the second statement. From the definition of  $\mathfrak{B}(y_1, \dots, y_n, y)$ ,

$$\cup \{\mathfrak{B}(y_1, \dots, y_n, y) \mid y \in V_k \setminus \{y_0\}\} \subset \mathfrak{B}(y_1, \dots, y_n).$$

Conversely let  $B^* \in \mathfrak{B}(y_1, \dots, y_n)$ . Since  $f_{(y_1, \dots, y_n)} \in \text{Int } B^*$ , we have  $C_1, \dots, C_l \in \mathfrak{R}(X)$  and  $U_1, \dots, U_l \in \mathfrak{O}(N)$  such that

$$f_{(y_1, \dots, y_n)} \in M(C_1, \dots, C_l; U_1, \dots, U_l) \subset \text{Int } B^*.$$

Then  $O_k = V_k \cap \cap \{U_i \mid y_0 \in U_i\}$  is an open neighborhood of  $y_0$  in  $N_k$ . For  $y' \in O_k \setminus \{y_0\} \subset V_k \setminus \{y_0\}$ ,  $B^* \in \mathfrak{B}(y_0, \dots, y_n, y')$ . In fact, if  $1/(n+1) \notin C_i$  then

$$f_{(y_1, \dots, y_n, y')}(C_i) = f_{(y_1, \dots, y_n)}(C_i) \subset U_i.$$

If  $1/(n+1) \in C_i$  then  $y_0 \in f_{(y_1, \dots, y_n)}(C_i) \subset U_i$ , which  $O_k \subset U_i$ . Hence

$$f_{(y_1, \dots, y_n, y')}(C_i) \subset f_{(y_1, \dots, y_n)}(C_i) \cup \{y'\} \subset U_i \cup O_k \subset U_i.$$

Therefore  $f_{(y_1, \dots, y_n, y')} \in M(C_1, \dots, C_l; U_1, \dots, U_l) \subset \text{Int } B^*$ .  $\square$

LEMMA 3.5. There exist  $1 = k_0 < k_1 < \dots \in \mathbf{N}$ , open neighborhoods  $N = W_0 \supset W_1 \supset \dots$  of  $y_0$  in  $N$ ,  $y_n \in (W_{n-1} \setminus W_n) \cap N_{k_{n-1}}$  and  $B_n^* \in \mathfrak{B}_{n-1} = \mathfrak{B}(y_1, \dots, y_{n-1})$  where  $\mathfrak{B}_0 = \mathfrak{B}$  such that

- (1)<sub>n</sub>  $O(\mathfrak{B}_n) \mid \tilde{N}_k$  is a neighborhood base of  $y_0$  in  $\tilde{N}_k$  for each  $k \geq k_n$ ,
- (2)<sub>n</sub>  $f_n = f_{(y_1, \dots, y_n)} \in M_n(W_n)$  and  $M_n(W_n) \cap B_n^* = \emptyset$ , where

$$M_n(W_n) = M\left(\{1\}, \dots, \left\{\frac{1}{n}\right\}; \{y_1\}, \dots, \{y_n\}\right) \cap M\left(X \setminus \left\{1, \dots, \frac{1}{n}\right\}, W_n\right).$$

PROOF. Note that  $k_0 = 1$  and  $O(\mathfrak{B}_0) = O(\mathfrak{B})$  satisfy (1)<sub>0</sub> by Lemma 3.3. Supposing that  $\{k_0, \dots, k_{n-1}\}$ ,  $\{W_0, \dots, W_{n-1}\}$ ,  $\{y_1, \dots, y_{n-1}\}$  and  $\{B_1^*, \dots, B_{n-1}^*\}$  have been obtained, we find  $k_n, W_n, y_n$  and  $B_n^*$ .

First assume that no  $y \in (W_{n-1} \cap N_{k_{n-1}}) \setminus \{y_0\}$  and no  $k > k_{n-1}$  satisfy (1)<sub>n</sub>, that is, for each  $y \in (W_{n-1} \cap N_{k_{n-1}}) \setminus \{y_0\}$ ,  $O(\mathfrak{B}(y_1, \dots, y_{n-1}, y)) \mid \tilde{N}_m$  is not a

neighborhood base of  $y_0$  in  $\tilde{N}_m$  for infinitely many  $m > k_{n-1}$ . Since  $(W_{n-1} \cap N_{k_{n-1}}) \setminus \{y_0\}$  is countable, we can write

$$(W_{n-1} \cap N_{k_{n-1}}) \setminus \{y_0\} = \{z_i \mid i \in \mathbf{N}\} (\subset N),$$

where  $z_i \neq z_j$  if  $i \neq j$ . Then we can inductively choose  $k_{n-1} < m_1 < m_2 < \dots$  and neighborhoods  $V_{m_i}$  of  $y_0$  in  $\tilde{N}_{m_i}$  such that

$$O(B^*) \cap \tilde{N}_{m_i} \not\subset V_{m_i} \quad \text{for each } B^* \in \mathfrak{B}(y_1, \dots, y_{n-1}, z_i).$$

Without loss of generality, we can assume  $\tilde{N}_{k_{n-1}} = V_{m_0} \supset V_{m_1} \supset \dots$  and define  $V = \bigcup_{i=0}^{\infty} (V_{m_i} \setminus \tilde{N}_{m_{i+1}})$ . Then  $V$  is a neighborhood of  $y_0$  in  $\tilde{N}_{k_{n-1}}$ . By  $(1)_{n-1}$ ,  $O(B^*) \cap \tilde{N}_{k_{n-1}} \subset V$  for some  $B^* \in \mathfrak{B}_{n-1}$ , whence

$$O(B^*) \cap \tilde{N}_{m_i} \subset V \cap \tilde{N}_{m_i} = V_{m_i} \quad \text{for each } i \in \mathbf{N}$$

On the other hand,

$$\mathfrak{B}_{n-1} = \mathfrak{B}(y_1, \dots, y_{n-1}) = \bigcup_{i \in \mathbf{N}} \mathfrak{B}(y_1, \dots, y_{n-1}, z_i)$$

by Lemma 3.4, whence  $B^* \in \mathfrak{B}(y_1, \dots, y_{n-1}, z_i)$  for some  $i \in \mathbf{N}$ . This is a contradiction. Therefore we have  $k_n \in \mathbf{N}$  and  $y_n \in (W_{n-1} \cap N_{k_{n-1}}) \setminus \{y_0\}$  satisfying  $(1)_n$ .

Note that  $(W_{n-1} \cap N_{k_{n-1}}) \setminus \{y_n\}$  is a neighborhood of  $y_0$  in  $N_{k_{n-1}}$ . By  $(1)_{n-1}$ , we have  $B_n^* \in \mathfrak{B}_{n-1}$  such that  $O(B_n^*) \cap N_{k_{n-1}} \subset (W_{n-1} \cap N_{k_{n-1}}) \setminus \{y_n\}$ . Hence  $f = f_{(y_1, \dots, y_n)} \notin B_n^*$ . Since  $B_n^*$  is closed in  $C(X, N)$ , there exist  $C_1, \dots, C_l \in \mathfrak{R}(X)$  and  $U_1, \dots, U_l \in \mathfrak{O}(N)$  such that

$$f_n \in M(C_1, \dots, C_l; U_1, \dots, U_l) \subset C(X, N) \setminus B_n^*.$$

Let

$$W_n = (W_{n-1} \cap (\bigcap \{U_i \mid y_0 \in U_i\})) \setminus \{y_n\} \subset W_{n-1}.$$

Then  $W_n$  is an open neighborhood of  $y_0$  in  $N$ ,  $y_n \in (W_{n-1} \setminus W_n) \cap N_{k_{n-1}}$  and  $f \in M_n(W_n)$ . To see that  $M_n(W_n) \cap B_n^* = \emptyset$ , it suffices to show that

$$M_n(W_n) \subset M(C_1, \dots, C_l; U_1, \dots, U_l).$$

Let  $g \in M_n(W_n)$ . If  $C_i \subset \{1, \dots, 1/n\}$  then  $g(C_i) = f_n(C_i) \subset U_i$ . If  $C_i \setminus \{1, \dots, 1/n\} \neq \emptyset$  then

$$\begin{aligned} g(C_i) &\subset g\left(X \setminus \left\{1, \dots, \frac{1}{n}\right\}\right) \cup g\left(C_i \cap \left\{1, \dots, \frac{1}{n}\right\}\right) \\ &\subset W_n \cup f_n(C_i) \subset U_i, \end{aligned}$$

because  $y_0 \in f_n(C_i) \subset U_i$ . Thus  $W_n$  and  $B_n^*$  satisfy  $(2)_n$ .  $\square$

To complete the proof of Example 3.2, let  $\{k_n \mid n \in \mathbf{N}\}$ ,  $\{y_n \mid n \in \mathbf{N}\}$ ,  $\{W_n \mid n \in \mathbf{N}\}$

and  $\{B_n^* | n \in \mathbf{N}\}$  be obtained in Lemma 3.5. We define  $f \in C(X, Y)$  by

$$f(x) = \begin{cases} y_n & \text{if } x = \frac{1}{n}, \\ y_0 & \text{if } x = 0. \end{cases}$$

Then  $f_n = f_{(y_1, \dots, y_n)}$  converges to  $f$  in  $C(X, N)$  if  $n \rightarrow \infty$ . In fact, let  $U^* = M(C_1, \dots, C_l; U_1, \dots, U_l)$  be a basic neighborhood of  $f$  in  $C(X, N)$ , where  $C_i \in \mathfrak{R}(X)$  and  $U_i \in \mathfrak{O}(N)$ . Without loss of generality, we can assume  $C_1 = \{1\}$ . And let

$$n_0 = \max \left\{ n \mid \frac{1}{n} \in \bigcup \{C_i \mid 0 \notin C_i\} \right\}.$$

For each  $n \geq n_0$ ,  $f_n(C_i) = f(C_i) \subset U_i$  if  $0 \notin C_i$  and  $f_n(C_i) \subset f(C_i) \subset U_i$  if  $0 \in C_i$ , whence  $f_n \in U^*$ .

Since  $f \in M_n(W_n)$  by the definition,  $f \notin B_n^* = \text{cl } B_n^*$  for each  $n \in \mathbf{N}$ , whence  $f \notin \text{cl}(\bigcup \{B_n^* | n \in \mathbf{N}\})$  because  $\mathfrak{B}$  is CP. Then  $f$  has a neighborhood  $V^*$  in  $C(X, N)$  such that  $V^* \cap B_n^* = \emptyset$  for each  $n \in \mathbf{N}$ . Choose  $m \in \mathbf{N}$  so that  $f_{k_m} \in V^*$ . Then  $B_{k_m+1}^* \in \mathfrak{B}_{k_m} = \mathfrak{B}(y_1, \dots, y_{k_m})$ . From the definition of  $\mathfrak{B}(y_1, \dots, y_{k_m})$ , it follows that

$$f_{k_m} = f_{(y_1, \dots, y_m)} \in \text{Int } B_{k_m+1}^* \subset B_{k_m+1}^*.$$

Hence  $f_{k_m} \in V^* \cap B_{k_m+1}^*$ . This is a contradiction. The proof is completed.  $\square$

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