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QUASI-G_δ-DIAGONALS AND WEAK σ-SPACES IN GO-SPACES

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1. Definitions and Preliminaries

In [5], topological spaces with various types of diagonals and weak σ-spaces are studied. In this paper these notions are extended and reflected upon GO-spaces.

Recall that a LOTS (=linearly ordered topological space) is a topological space whose topology agrees with the topology induced by some linear ordering $<$. A GO-space (=generalized ordered space) is a subspace of a LOTS. An equivalent way to obtain a GO-space X is to start with a linearly ordered set $(Y, <)$ and equip Y with any topology that contains the order topology and has a base consisting of open sets each of which are order-convex (where A is order-convex if given $x, y\in A$ with $x<y$, then $\{z: x<z<y\}\subset A$). It is then convenient to write

$$X=\mathrm{GO}_Y(R, E, I, L),$$

where

(i) $I=\{x\in X: \{x\}$ is open$\}$,

(ii) $R=\{x\in X-I: [x, \rightarrow)$ is open$\}$,

(iii) $L=\{x\in X-I: (\leftarrow, x]$ is open$\}$, and

(iv) $E=X-(R\cup L\cup I)$

(where $[x, \rightarrow)=\{z\in X: x\leqq z\}$ and $(\leftarrow, x]$ is similarly defined). The subscript Y will usually be omitted since it will be clear what it is.

Let Z^+ denote the set of natural numbers and let $\boldsymbol{R}$ denote the real line with the usual order topology.

DEFINITION 1. Let $\mathcal{G}=\{\mathcal{G}_i: i\in Z^+\}$ be a set of collections of open subsets of a topological space X. If, for any distinct $x, y\in X$, there exists $n\in Z^+$ such that

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(i) $x\in \mathrm{St}(x, \mathcal{G}_n)\subset X-\{y\}$, then $\mathcal{G}$ is a *quasi-G_δ-diagonal*,

(ii) $x\in \mathrm{St}(x, \mathcal{G}_n)\subset X-\{y\}$ and $ord(x, \mathcal{G}_n)$ is finite, then $\mathcal{G}$ is a *θ-quasi-G_δ-diagonal*. Furthermore,

(iii) if, for $x\in U$, where U is open in X, there exists $n\in Z^+$ such that $x\in \mathrm{St}(x, \mathcal{G}_n)\subset U$, then $\mathcal{G}$ is a *quasi-development* for X and X is said to be *quasi-developable*.

It is clear that a quasi-developable space has a quasi-G_δ-diagonal and, in [1] it is shown that a quasi-developable space has a θ-quasi-G_δ-diagonal.

THEOREM 1 [6]. *If a LOTS X has a quasi-G_δ-diagonal, then X is quasi-developable.*

This theorem does not extend to the more general setting of a GO-space.

EXAMPLE 1. The Sorgenfrey Line $S=GO_R(\boldsymbol{R}, \phi, \phi, \phi)$ is a hereditarily Lindelöf GO-space that is not quasi-developable. Since S has a weaker metric topology, it has a G_δ-diagonal.

DEFINITION 2. A collection $\mathcal{C}$ of sets is *coherent* if whenever $\mathcal{B}$ is a non-void proper subcollection of $\mathcal{C}$, then $(\cup\mathcal{B})\cap(\cup(\mathcal{C}-\mathcal{B}))\neq\phi$. A collection $\mathcal{C}$ is a *maximal coherent* subcollection of $\mathcal{A}$ if $\mathcal{C}$ is coherent and not a proper subcollection of another coherent subcollection of $\mathcal{A}$.

The following theorem is an adaption to GO-spaces of a result in [3]. See also [7, 3, 9]-

THEOREM 2. *If $\mathcal{G}$ is a coherent collection of convex open subsets of a GO-space X, then there exist sequences $\langle x_i : i\in Z^+\rangle$ and $\langle y_i : i\in Z^+\rangle$ (each possibly finite) in $\cup\mathcal{G}$ such that*

(i) $x_1=y_1$,

(ii) *if $x_{k+1}(y_{k+1})$ is defined, then $x_k<x_{k+1}$ $(y_{k+1}<y_k)$ and $x_{k+1}\notin \mathrm{St}(x_k, \mathcal{G})$ $(y_{k+1}\notin \mathrm{St}(y_k, \mathcal{G}))$,*

(iii) $\mathrm{St}(x_{k+1}, \mathcal{G})\cap \mathrm{St}(x_k, \mathcal{G})\neq\phi$ $(\mathrm{St}(y_{k+1}, \mathcal{G})\cap \mathrm{St}(y_k, \mathcal{G})\neq\phi)$, *and*

(iv) $(\cup\{\mathrm{St}(x_k, \mathcal{G}) : k\in Z^+\})\cup(\cup\{\mathrm{St}(y_k, \mathcal{G}) : k\in Z^+\})=\cup\mathcal{G}$.

This result can be extended in a useful fashion.

THEOREM 3. *If $\mathcal{G}$ is a coherent collection of convex open subsets of a GO-space X, then there is a collection $\mathcal{H}$ of convex open subsets of X such that*

(i) *$\mathcal{H}$ is countable, and*

(ii) *for all* $x \in \cup \mathcal{G}$, $x \in \mathrm{St}(x, \mathcal{H}) \subset \mathrm{St}(x, \mathcal{G})$.

PROOF. Let $\langle x_i \rangle$ and $\langle y_i \rangle$ be the sequences guaranteed by Theorem 2. If $\langle x_i \rangle$ is infinite, then choose A_i, $B_{i+1} \in \mathcal{G}$ such that $x_i \in A_i$, $x_{i+1} \in B_{i+1}$ and $A_i \cap B_{i+1} \neq \phi$. Let $\mathcal{H}^+ = \{A_i : i \in Z^+\} \cup \{B_{i+1} : i \in Z^+\}$. If $\langle x_i \rangle$ is finite, say $\langle x_i \rangle = \langle x_1, \cdots, x_n \rangle$, then choose A_i, B_{i+1} for $i=1, \cdots, n-1$ as above. Let $C_n = (x_n, \rightarrow) \cap (\cup \mathcal{G})$. Then let $\mathcal{H}^+ = \{A_i : i=1, \cdots, n-1\} \cup \{B_{i+1} : i=1, \cdots, n-1\} \cup \{C_n\}$. Argue using $\langle y_i \rangle$ in a similar fashion to obtain $\mathcal{H}^-$. Let $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$. It is clear that $\mathcal{H}$ has the advertised properties.

2. Quasi-G_δ-diagonals in GO-spaces

THEOREM 4. *If a GO-space X has a quasi-G_δ-diagonal, then X has a quasi-G_δ-diagonal $\mathcal{K} = \{\mathcal{K}_n : n \in Z^+\}$ such that the elements of each $\mathcal{K}_n$ are order-convex and $\mathcal{K}_n$ is a pairwise disjoint collection.*

PROOF. Let $\mathcal{G} = \{\mathcal{G}_n : n \in Z^+\}$ be a quasi-G_δ-diagonal for X. No generality is lost if it is assumed that the elements of each $\mathcal{G}_n$ are order-convex. For each $n \in Z^+$, let $\{\mathcal{G}(n, \alpha) : \alpha \in \lambda_n\}$ be the set of maximal coherent subcollections of $\mathcal{G}_n$. For each $\alpha \in \lambda_n$, let $\mathcal{H}(n, \alpha) = \{H(n, \alpha, i) : i \in Z^+\}$ be the collection obtained from $\mathcal{G}(n, \alpha)$ guaranteed by Theorem 3. Let $\mathcal{K}(n, i) = \{H(n, \alpha, i) : \alpha \in \lambda_n\}$. Then $\mathcal{K} = \{\mathcal{K}(n, i) : (n, i) \in Z^+ \times Z^+\}$ is the desired quasi-G_δ-diagonal.

COROLLARY 4.1. *A GO-space with a quasi-G_δ-diagonal has a θ-quasi-G_δ-diagonal.*

The following theorem extends a result in [6].

THEOREM 5. *If X is a GO-space with a quasi-G_δ-diagonal, then X is a first-countable space. Furthermore, X is quasi-developable at $E \cup I$.*

PROOF. Let $\{\mathcal{H}_n : n \in Z^+\}$ be a quasi-G_δ-diagonal guaranteed by Theorem 4 for the GO-space $X = GO(R, E, I, L)$. If $x \in R$, let

$$U(x, n) = \mathrm{St}(x, \mathcal{H}_n) \cap [x, \rightarrow) \qquad \text{for } n \in Z^+ .$$

It follows that $\{U(x, n) : n \in Z^+\}$ is a countable local base at x. A countable local base at y, $y \in L$ is similarly defined.

For each $n, m \in Z^+$, let

$$\mathcal{G}(n, m)=\{H\cap H' : H\in\mathcal{H}_n,\ H'\in\mathcal{H}_m\}.$$

For each $x\in E$, $\{\mathrm{St}(x, \mathcal{G}(n, m)) : (n, m)\in Z^+\times Z^+\}$ is a countable local base at x. If $\mathcal{I}=\{\{x\} : x\in I\}$, then $\mathcal{I}\cup\{\mathcal{G}(n, m) : (n, m)\in Z^+\times Z^+\}$ is a quasi-development for $E\cup I$.

Notice that the Sorgenfrey Line $S=GO(\boldsymbol{R}, \phi, \phi, \phi)$ is a GO-space with a G_δ-diagonal that is not quasi-developable. Thus, Theorem 5 cannot be extended.

THEOREM 6. *Let X be a GO-space with a quasi-G_δ-diagonal. If X is either connected or compact, then X is homeomorphic to a subspace of $\boldsymbol{R}$.*

PROOF. In either case X is a LOTS [6] and, hence, quasi-developable by Theorem 1. If X is connected, then the result follows from [3]. If X is compact, then X is a metrizable space [2]. Then it is a second-countable LOTS. Hence X has at most countably many jumps and no gaps. Extend X by filling each jump with an open unit interval and extending the given order in the natural fashion to obtain Y. It follows that Y is connected metrizable LOTS. Thus Y is a homeomorphic to a subspace of $\boldsymbol{R}$. Since $X\subseteq Y$, it follows that X is homeomorphic to a subspace of $\boldsymbol{R}$.

3. Main theorem and its proof

DEFINITION 3. A space X is said to be a *weak σ-space* if X has a σ-disjoint network $\mathcal{M}=\cup\{\mathcal{M}_n : n\in Z^+\}$ such that each $\mathcal{M}_n$ is a discrete collection of subsets of X in some open set containing $\cup\mathcal{M}_n$ (equivalently $\mathcal{M}_n$ is discrete in $\cup\mathcal{M}_n$).

If X is a quasi-developable space, then X is a weak σ-space [1].

EXAMPLE 2. Let $Y=[0, \omega_1]$ be equipped with the usual order topology and $X=GO_Y(\phi, \phi, [0, \omega_1]-\{\omega_1\}, \{\omega_1\})$. If $\mathcal{M}_1=\{\{\alpha\} : \alpha<\omega_1\}$ and $\mathcal{M}_2=\{\{\omega_1\}\}$, then the network $\{\mathcal{M}_1, \mathcal{M}_2\}$ insures that X is a weak σ-space. Notice that X is not first-countable at ω_1.

This example demonstrates the limits of the next theorem.

Let X^+ denote the order completion (the Dedekind compactification) of a GO-space X [8].

THEOREM 7. *If X is a first-countable GO-space, then X is quasi-developable if and only if X is a weak σ-space.*

PROOF. Let $\mathcal{M}=\cup\{\mathcal{M}_n : n\in Z^+\}$ be a σ-disjoint network for $X=GO(R, E, I, L)$ such that each $\mathcal{M}_n$ is discrete in an open set B_n. For each $x\in X$, let $\{U(x, k): k\in Z^+\}$ be a local base at x. For each $n\in Z^+$, let

$$R_n=\{x\in R: x\in M\subset[x, \to) \text{ for some } M\in\mathcal{M}_n\}.$$

It follows that R_n is discrete in B_n and $R=\cup R_n$. If $x\in R_n$, let $g'(x, n)$ be the order-convex component of $B_n-\{y\in R_n: y\neq x\}$ that contains x. Clearly $g'(x, n)$ is open. Let

$$g(x, n, k)=g'(x, n)\cap U(x, k)\cap[x, \to).$$

For each $n, k\in Z^+$, let

$$\mathcal{R}(n, k)=\{g(x, n, k): x\in R_n\}.$$

Since R_n is discrete in B_n, it easily follows that $\mathcal{R}(n, k)$ is a pairwise disjoint collection of open sets and that $\mathcal{R}=\{\mathcal{R}(n, k): (n, k)\in Z^+\times Z^+\}$ is a quasi-development for R. Similarly define $\mathcal{L}$ to be a quasi-development for L. Clearly $\mathcal{I}=\{\{x\}: x\in I\}$ is a quasi-development for I.

To construct a quasi-development for E, let $\mathcal{M}_n=\{M(n, \alpha): \alpha\in\lambda(n)\}$ be discrete in the open set B_n. For each $\alpha\in\lambda(n)$, let

$$\mathcal{C}(n, \alpha)=\{C(n, \alpha, \gamma): \gamma\in\lambda(n, \alpha)\}$$

be the collection of order-convex components of

$$B_n-\cup\{\overline{M(n, \beta)}: \beta\in\lambda(n), \beta\neq\alpha\}$$

that intersect $M(n, \alpha)$. For each $\gamma\in\lambda(n, \alpha)$, let

$$M(n, \alpha, \gamma)=M(n, \alpha)\cap C(n, \alpha, \gamma).$$

For each $\gamma\in\lambda(n, \alpha)$,

(i) if $\inf_{X^+} M(n, \alpha, \gamma)=a^+\notin M(n, \alpha, \gamma)\cap E$, then let

$$l(M(n, \alpha, \gamma), k)=(a^+, \to)\cap C(n, \alpha, \gamma) \quad \text{for each } k\in Z^+.$$

(ii) if $\inf_{X^+} M(n, \alpha, \gamma)=a\in M(n, \alpha, \gamma)\cap E$, then, by the first-countability of X, there is an increasing sequence $\{x(n, \alpha, \gamma, k): k\in Z^+\}$ of elements of $C(n, \alpha, \gamma)$ that converges to a. Let

$$l(M(n, \alpha, \gamma), k)=(x(n, \alpha, \gamma, k), \to)\cap C(n, \alpha, \gamma) \quad \text{for each } k\in Z^+.$$

(iii) if $\sup_{X^+} M(n, \alpha, \gamma)=e^+\notin M(n, \alpha, \gamma)\cap E$, then let

$$r(M(n, \alpha, \gamma), k)=(\leftarrow, e^+)\cap C(n, \alpha, \gamma) \quad \text{for each } k\in Z^+.$$

(iv) if $\sup_{X^+} M(n, \alpha, \gamma)=e\in M(n, \alpha, \gamma)\cap E$, then choose a decreasing sequence $\{y(n, \alpha, \gamma, k): k\in Z^+\}$ of elements of $C(n, \alpha, \gamma)$ that converges to e. Let

$$r(M(n, \alpha, \gamma), k)=(\leftarrow, y(n, \alpha, \gamma, k))\cap C(n, \alpha, \gamma) \quad \text{for each } k\in Z^+.$$

Finally, let

$$g(M(n, \alpha, \gamma), k)=l(M(n, \alpha, \gamma), k)\cap r(M(n, \alpha, \gamma), k) \quad \text{for each } k\in Z^+.$$

Notice that

(a) $C(n, \alpha, \gamma)\cap(\cup\mathcal{M}_n)=M(n, \alpha, \gamma)$,

(b) $g(M(n, \alpha, \gamma), k)\subset C(n, \alpha, \gamma)$ for all $k\in Z^+$, and

(c) $g(M(n, \alpha, \gamma), k)$ is an open order-convex set.

Let $\mathcal{G}(n, k)=\{g(M(n, \alpha, \gamma), k): \alpha\in\lambda(n), \gamma\in\lambda(n, \alpha)\}$ and $\mathcal{G}=\{\mathcal{G}(n, k): (n, k)\in Z^+\times Z^+\}$. It follows that $\mathcal{G}$ is a quasi-development for E. To see this, let $p\in E\cap U$, where U is an open set in X. There exist s and $t\in X$ such that $p\in(s, t)\subset U$. Since $\mathcal{M}$ is a network, there exist $n\in Z^+$ and $\alpha\in\lambda(n)$ such that $p\in M(n, \alpha)\subset(s, t)$. There exists $\gamma\in\lambda(n, \alpha)$ such that $p\in M(n, \alpha, \gamma)\subset(s, t)$. Notice that $M(n, \alpha, \gamma)\subset C(n, \alpha, \gamma)\cap(s, t)$. Hence, depending on the endpoints of $M(n, \alpha, \gamma)$ in X^+, there exists $k\in Z^+$ such that

$$\mathrm{St}(p, \mathcal{G}(n, k))=g(M(n, \alpha, \gamma), k)\subset(s, t).$$

Hence, $\mathcal{R}\cup\mathcal{L}\cup\mathcal{I}\cup\mathcal{G}$ is a quasi-development for X.

COROLLARY 7.1. *A GO-space is quasi-developable if and only if it is a weak σ-space with a quasi-G_δ-diagonal.*

4. Some other results

THEOREM 8. *If a GO-space X is a weak σ-space, then X is hereditarily paracompact.*

PROOF. Since a subspace of a GO-space is a GO-space and a subspace of a weak σ-space is a weak σ-space, it suffices to show that X is paracompact. Let $\mathcal{U}=\{U_\alpha: \alpha\in\lambda\}$ be an open cover of X and let $\mathcal{M}=\cup\{\mathcal{M}_n: n\in Z^+\}$ be a weak σ-structure on X. Let

$$\mathcal{C}_n=\{M: M\in\mathcal{M}_n \text{ and } M\subset U_\alpha \text{ for some } U_\alpha\in\mathcal{U}.\}.$$

Then $\{\mathcal{C}_n: n\in Z^+\}$ is a discrete collection of subsets of the subspace $\cup\mathcal{C}_n$.

Therefore X is weakly θ-refinable [4] and hence paracompact.

DEFINITION 4. A space X is a *$w\Delta$-space* if there is a set $\{\mathcal{G}_n : n\in Z^+\}$ of open covers of X such that, for each p in X, if $x_n \in \mathrm{St}(p, \mathcal{G}_n)$ for $n=1, 2, \cdots$, then the sequence $\langle x_n \rangle$ has a cluster point.

COROLLARY 8.1. *If a first-countable GO-space X is a weak σ-space and a $w\Delta$-space, then X is metrizable.*

PROOF. It follows from Theorem 7 that X is quasi-developable. Since X is paracompact by Theorem 8, the result follows from [2].

THEOREM 9. *Let X be a GO-space which is a weak σ-space. If X is either compact or connected, then X can be embedded in $\boldsymbol{R}$.*

PROOF. Since X is either compact or connected, it follows from Theorem 8 that X is first-countable. Hence X is quasi-developable by Theorem 7. Therefore, the result follows from Theorem 6.

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