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# 3-DIMENSIONAL SPACE-LIKE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF AN INDEFINITE SPACE FORM II

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## Introduction.

For an $n$-dimensional complete space-like submanifold $M$ with parallel mean curvature vector of $S_{p}^{n+p}(c)$, it is seen by Cheng [3] that $M$ is totally umbilic if $n=2$ and $H^{2}\leqq c$ or if $n>2$ and $n^{2}H^{2}<4(n-1)c$, where $H$ denotes the mean curvature, i.e., the norm of the mean curvature vector $\boldsymbol{h}$. On the other hand, Aiyama and Cheng [1] prove the following

THEOREM A. *Let $M$ be a complete space-like hypersurface in a Lorentz space form $M_{1}^{4}(c)$ with parallel mean curvature vector. If the Ricci curvature is bounded from above by some number less than* $3(c-H^{2})$, *then $c$ must be positive and $M$ is congruent to a Riemannian 3-sphere $\boldsymbol{S}^{3}(c')$.*

Recentely we verified the following which is Theorem 1 in [2] as a high codimensional version of Theorem A.

THEOREM B. *Let $M$ be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector $\boldsymbol{h}$ of an indefinite space form $\boldsymbol{S}_{p}^{3+p}(c)$, $p\geqq 2$. If it satisfies*

$$(1.1)\qquad \frac{8}{9}c\leqq H^{2}\leqq c \quad \textit{and} \quad Ric(M)\leqq\delta_{1}<3(c-H^{2}),$$

*then $M$ is totally umbilic.*

However, we get a more natural extending theorem. In this paper, we verify the following theorem.

THEOREM. *Let $M$ be a complete 3-dimensional space-like submanifold in an*

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*indefinite space form $M_p^{3+p}(c)$ with parallel mean curvature vector. If the Ricci curvature is bounded from above by some number less than $3(c-H^2)$, then $c$ must be positive and $M$ is congruent to a Riemannian 3-sphere $S^3(c')$.*

Theorem B is included in the above theorem.

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## 2. Preliminaries.

Throughout this paper all manifolds are assumed to be smooth, connected without boundary. We discuss in smooth category. Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional indefinite Riemannian manifold of index $p$ and with constant curvature $c$, which is called *an indefinite space form of index p*. According to $c>0$, $c=0$ or $c<0$ it is denoted by $\boldsymbol{S}_p^{n+p}(c)$, $\boldsymbol{R}_p^{n+p}$ or $\boldsymbol{H}_p^{n+p}(c)$. A submanifold $M$ of an indefinite space form $M_p^{n+p}(c)$ is said to be *space-like* if the induced metric on $M$ from that of the ambient space is positive definite.

Let $M$ be an $n$-dimensional space-like submanifold of $M_p^{n+p}(c)$. We choose a local field of orthonormal frames $e_1, \cdots, e_{n+p}$ adapted to the indefinite Riemannian metric of $M_p^{n+p}(c)$ and the dual coframe $\omega_1, \cdots, \omega_{n+p}$ in such a way that, restricted to the submanifold $M$, $e_1, \cdots, e_n$ are tangent to $M$. In the sequel, the following convention on the range of indices is used, unless otherwise stated:

$$1 \leqq i,\ j,\ \cdots \leqq n\,; \qquad n+1 \leqq \alpha,\ \beta,\ \cdots \leqq n+p\,.$$

We denote the second fundamental form $\alpha$ of $M$ by

$$\alpha = -\sum_{\alpha,i,j} h_{ij}^{\alpha}\omega_i\omega_j e_\alpha\,.$$

The mean curvature vector $\boldsymbol{h}$ and the mean curvature $H$ are defined by

$$\text{(2.1)} \qquad \boldsymbol{h} = -\frac{1}{n}\sum_{\alpha}(\sum_i h_{ii}^{\alpha})e_\alpha, \qquad H = |\boldsymbol{h}| = \frac{1}{n}\sqrt{\sum_{\alpha}(\sum_i h_{ii}^{\alpha})^2}.$$

Let $S=\sum(h_{ij}^{\alpha})^2$ denote the squared norm of the second fundamental form $\alpha$ of $M$. From the Gauss equation, the components of the Ricci curvature *Ric* is given by

$$\text{(2.2)} \qquad R_{jk} = (n-1)c\delta_{jk} - \sum_{\alpha,i} h_{ii}^{\alpha}h_{jk}^{\alpha} + \sum_{\alpha,i} h_{ik}^{\alpha}h_{ij}^{\alpha}\,.$$

## 3. Proof of Theorem.

In order to prove Theorem, the following facts are needed.

PROPOSITION 1. *Let $M$ be a 3-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $M_p^{3+p}(c)$. If the Ricci curvature of $M$ is bounded from above by some number less than $3(c-H^2)$, then $M$ is pseudo-umbilic.*

This is proved as Proposition 4.1 in [2].

PROPOSITION 2. *Let $M$ be an n-dimensional complete space-like submanifold with non-zero parallel mean curvature vector of $M_p^{n+p}(c)$. If $M$ is a pseudo-umbilical submanifold, then $M$ is a maximal submanifold of a totally umbilical hypersurface $M_{p-1}^{n+p-1}(\tilde{c})$ of $M_p^{n+p}(c)$.*

The generalized case of Proposition 2 was proved by Chen in [4].

LEMMA 3. *Let $M$ be a 3-dimensional complete maximal space-like submanifold of $M_q^{3+q}(\tilde{c})$. If the Ricci curvature of $M$ is bounded from above by $3\tilde{c}$, then $\tilde{c}>0$ and $M$ is congruent to $S^3(\tilde{c})$.*

PROOF. By using (2.2) and the assumpution that $M$ is maximal, the diagonal components of the Ricci curvature are given by

$$R_{ii}=2\tilde{c}-\sum_{\alpha,j} h_{jj}^{\alpha}h_{ii}^{\alpha}+\sum_{\alpha,j} h_{ij}^{\alpha}h_{ij}^{\alpha}=2\tilde{c}+S\geqq 2\tilde{c}\ .$$

This combines with the condition $R_{ii}<3\tilde{c}$ to be $\tilde{c}>0$. By means of Ishihara's theorem [5], $M$ is concluded to be a totally geodesic submanifold of $S_q^{3+q}(\tilde{c})$. Thus $M$ is congruent to $S^3(\tilde{c})$. ■

Now, let us prove Theorem.

PROOF OF THEOREM. First of all, we consider the case which the mean curvature vector $\boldsymbol{h}$ is zero. In this case, by virtue of Lemma 3, we can prove that $c>0$ and $M$ is a totally geodesic submanifold $S^3(c)$ of $S_p^{3+p}(c)$. Next we suppose that $\boldsymbol{h}\neq 0$. Combining Proposition 1 and Proposition 2, we have concluded that $M$ is a maximal submanifold of a totally umbilic hypersurface $M_{p-1}^{3+p-1}(c_1)$ of $M_p^{3+p}(c)$, where $c_1=c-H^2$ and then $R_{ii}<3c_1$. Thus it follows from Lemma 3 that $c>H^2$ and $M$ is congruent to $S^3(c_1)$. Now we have completed the proof of Theorem. ■

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