TSUKUBA J. MATH.
Vol. 18 No. 2 (1994), 483—492

ON REAL HYPERSURFACES OF TYPE A IN A COMPLEX SPACE FORM (I)

By

Yong-Soo PYO

§1. Introduction.

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC, according as $c>0$, $c=0$ or $c<0$.

Now, let M be a real hypersurface of an n-dimensional complex space form $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kähler metric and the almost complex structure of $M_n(c)$. Okumura [7] and Montiel and Romero [6] proved the following

THEOREM A. *Let M be a real hypersurface of P_nC, $n\geqq 2$. If it satisfies*

$$A\phi-\phi A=0\,, \tag{1.1}$$

then M is locally a tube of radius r over one of the following Kähler submanifolds:

(A_1) *a hyperplane $P_{n-1}C$, where $0<r<\pi/2$,*

(A_2) *a totally geodesic P_kC $(1\leqq k\leqq n-2)$, where $0<r<\pi/2$,*

where A is the shape operator in the direction of the unit normal C on M.

THEOREM B. *Let M be a real hypersurface of H_nC, $n\geqq 2$. If it satisfies* (1.1), *then M is locally one of the following hypersurfaces:*

(A_0) *a horosphere in H_nC, i.e., a Montiel tube,*

(A_1) *a tube of a totally geodesic hyperplane $H_{n-1}C$,*

(A_2) *a tube of a totally geodesic H_kC $(1\leqq k\leqq n-2)$.*

On the other hand, the following theorem is proved by Maeda and Udagawa [4] under that the structure vector ξ is principal and then recently by Kimura

This research was partially supported by KOSEF (1992-1993).
Received February 1, 1993, revised March 15, 1993.

and Maeda [3] and Ki, Kim and Lee [1] without the above assumption.

THEOREM C. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 2$. If it satisfies*

$$\nabla_\xi A=0\,,\qquad g(A\xi,\,\xi)\neq 0\,,\tag{1.2}$$

then M is locally of type A, where ∇ is the Riemannian connection on M.

The purpose of this article is to prove the following generalized property of Theorem C.

THEOREM. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$, $n\geqq 2$. If it satisfies*

$$\nabla_\xi A=a(A\phi-\phi A)\,,\qquad 2a\neq -g(A\xi,\,\xi)\tag{1.3}$$

for some non-zero constant a, then M is locally of type A.

The author would like to thank Professors U-H. Ki and H. Nakagawa for his valuble suggestions and encouragement during the preparation of this paper.

§2. Preliminaries.

First of all, we recall fundamental properties about real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood in M. We denote by J the almost complex structure of $M_n(c)$. For a local vector field X on the neighborhood in M, the images of X and C under the linear transformation J can be represented as

$$JX=\phi X+\eta(X)C\,,\qquad JC=-\xi\,,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighborhood in M, respectively. Then it is seen that $g(\xi,\,X)=\eta(X)$, where g denotes the Riemannian metric tensor on M induced from the metric tensor on $M_n(c)$. The set of tensors $(\phi,\,\xi,\,\eta,\,g)$ is called an *almost contact metric structure* on M. They satisfy the following properties:

$$\phi^2=-I+\eta\otimes\xi,\qquad \phi\xi=0,\qquad \eta(\xi)=1,$$

where I denotes the identity transformation. Furthermore, the covariant derivatives of the structure tensors are given by

$$\nabla_X \xi = \phi AX, \qquad \nabla_X \phi(Y) = \eta(Y)AX - g(AX, Y)\xi \tag{2.1}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection on M and A denotes the shape operator of M in the direction of C.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively obtained:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.2}$$

$$\nabla_X A(Y) - \nabla_Y A(X) = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

Next, we suppose that the structure vector field ξ is principal with corresponding principal curvature α. Then it is seen in [2] and [5] that α is constant on M and it satisfies

$$A\phi A = \frac{c}{4}\phi + \frac{1}{2}\alpha(A\phi + \phi A). \tag{2.4}$$

§ 3. Proof of Theorem.

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 2$. In this section, we shall give a sufficient condition for the structure vector field ξ to be principal. First, we assume that ξ is principal, i.e., $A\xi = \alpha\xi$, where α is constant. Then, by (2.1) and (2.4), we get

$$\nabla_X A(\xi) = -\frac{c}{4}\phi X - \frac{1}{2}\alpha(A\phi - \phi A)X, \tag{3.1}$$

from which together with (2.3) it follows that

$$\nabla_\xi A = -\frac{1}{2}\alpha(A\phi - \phi A).$$

Taking account of this property and the assumption of Theorems A and B, we shall assert the following

PROPOSITION 3.1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 2$. If it satisfies*

$$\nabla_\xi A = a(A\phi - \phi A) \tag{3.2}$$

for some non-zero constant a, then ξ is principal.

By the assumption (3.2) and (2.3), it turns out to be

$$\nabla_Y A(\xi)=a(A\phi-\phi A)Y-\frac{c}{4}\phi Y .$$

Differentiating this equation with respect to X covariantly and taking account of (2.1), we get

$$\begin{aligned}(3.3)\qquad \nabla_X\nabla_Y A(\xi)=&-\nabla_Y A(\phi AX)\\ &+a\{\nabla_X A(\phi Y)+g(Y,\xi)A^2X-g(AX,Y)A\xi\\ &-g(AY,\xi)AX+g(AX,AY)\xi-\phi\nabla_X A(Y)\}\\ &-\frac{c}{4}\{g(Y,\xi)AX-g(AX,Y)\xi\}\end{aligned}$$

for any vector fields X and Y. Since the Ricci formula for the shape operator A is given by

$$(3.4)\qquad \nabla_X\nabla_Y A(Z)-\nabla_X\nabla_Y A(Z)=R(X,Y)(AZ)-A(R(X,Y)Z),$$

we obtain by (2.2), (2.3) and (3.3)

$$\begin{aligned}(3.5)\qquad &\nabla_X A(\phi AY)-\nabla_Y A(\phi AX)+a\{\nabla_X A(\phi Y)-\nabla_Y A(\phi X)\}\\ &=-\{ag(Y,\xi)+g(AY,\xi)\}A^2X+\{ag(X,\xi)+g(AX,\xi)\}A^2Y\\ &\quad+\{ag(AY,\xi)+g(A^2Y,\xi)\}AX-\{ag(AX,\xi)+g(A^2X,\xi)\}AY\\ &\quad+\frac{c}{4}[\{ag(Y,\xi)+g(AY,\xi)\}X-\{ag(X,\xi)+g(AX,\xi)\}Y]\\ &\quad+\frac{c}{4}\{g(A\phi Y,\xi)\phi X-g(A\phi X,\xi)\phi Y\}-\frac{c}{2}g(\phi X,Y)\phi A\xi\end{aligned}$$

for any vector fields X and Y.

Now, in order to prove the proposition, we shall express (3.5) with the simpler form. The inner product of (3.5) and ξ, combining with (2.3) and (3.2), implies

$$\begin{aligned}(3.6)\qquad &ag((A\phi A\phi-\phi A\phi A)X,Y)\\ &\quad+a^2\{g(X,\xi)g(AY,\xi)-g(Y,\xi)g(AX,\xi)\}\\ &\quad+a\{g(X,\xi)g(A^2Y,\xi)-g(Y,\xi)g(A^2X,\xi)\}\\ &\quad+2\{g(AX,\xi)g(A^2Y,\xi)-g(AY,\xi)g(A^2X,\xi)\}\\ &=0\end{aligned}$$

for any vector fields X and Y. Since Y is any vector fields, we get

$$\begin{aligned}(3.7)\qquad a(A\phi A\phi-\phi A\phi A)X+\{ag(X, \xi)+2g(AX, \xi)\}A^2\xi \\ +\{a^2g(X, \xi)-2g(A^2X, \xi)\}A\xi \\ -a\{ag(AX, \xi)+g(A^2X, \xi)\}\xi \\ =0\end{aligned}$$

for any vector field X. On the other hand, taking account of (2.1) and the skew-symmetry of the transformation ϕ, we have

$$g((A\phi A\phi-\phi A\phi A)X, \phi X)=g(X, \xi)g(A\phi AX, \xi).$$

Putting $Y=\phi X$ in (3.6) and applying the above property, we get

$$\begin{aligned}(3.8)\qquad ag(X, \xi)\{g(A\phi AX, \xi)+ag(A\phi X, \xi)+g(A^2\phi X, \xi)\} \\ +2\{g(AX, \xi)g(A^2\phi X, \xi)-g(A\phi X, \xi)g(A^2X, \xi)\} \\ =0.\end{aligned}$$

Let T_0 be a distribution defined by the subspace $T_0(x)=\{u\in T_xM: g(u, \xi(x))=0\}$ of the tangent space T_xM of M at any point x, which is called the *holomorphic distribution*. For any vector field X belonging to T_0, (3.8) is simplified as

$$g(AX, \xi)g(A^2\phi X, \xi)-g(A\phi X, \xi)g(A^2X, \xi)=0.$$

Furthermore, this equation holds for any vector field X. By polarization, we have

$$\begin{aligned}g(AX, \xi)g(A^2\phi Y, \xi)-g(A\phi X, \xi)g(A^2Y, \xi) \\ +g(AY, \xi)g(A^2\phi X, \xi)-g(A\phi Y, \xi)g(A^2X, \xi) \\ =0\end{aligned}$$

for any vector fields X and Y. Hence we have

$$\begin{aligned}(3.9)\qquad g(AX, \xi)\phi A^2\xi+g(A\phi X, \xi)A^2\xi \\ -g(A^2\phi X, \xi)A\xi-g(A^2X, \xi)\phi A\xi \\ =0.\end{aligned}$$

Now, suppose that the structure vector field ξ is not principal. Then we can put $A\xi=\alpha\xi+\beta U$, where U is a unit vector field in the holomorphic distribution T_0, and α and β are smooth functions on M. So we may consider that the function β does not vanish identically on M. Let M_0 be the non-empty open subset of M consisting of points x at which $\beta(x)\neq 0$. And we put $AU=$

$\beta\xi+\gamma U+\delta V$, where U and V are orthonormal vector fields in the holomorphic distribution T_0, and γ and δ are smooth functions on M_0.

First, we shall assert the following

LEMMA 3.2.

$$AU=\beta\xi+\gamma U \quad on\ M_0 . \tag{3.10}$$

PROOF. By the forms $A\xi=\alpha\xi+\beta U$ and $AU=\beta\xi+\gamma U+\delta V$, it turns out to be

$$A^2\xi=(\alpha^2+\beta^2)\xi+\beta(\alpha+\gamma)U+\beta\delta V .$$

Thus we can rewrite (3.9) as

$$\begin{aligned}&\{\alpha g(A^2\phi X, \xi)-(\alpha^2+\beta^2)g(A\phi X, \xi)\}\xi\\ &\quad+\beta\{g(A^2\phi X, \xi)-(\alpha-\gamma)g(A\phi X, \xi)\}U-\beta\delta g(A\phi X, \xi)V\\ &\quad+\beta\{g(A^2X, \xi)-(\alpha+\gamma)g(AX, \xi)\}\phi U-\beta\delta g(AX, \xi)\phi V\\ &=0\end{aligned} \tag{3.11}$$

for any vector field X. The inner product of (3.11) and ϕU implies

$$g(A^2X, \xi)-(\alpha+\gamma)g(AX, \xi)-\delta g(A\phi X, \xi)g(V, \phi U)=0 .$$

Putting $X=V$ in this equation and calculating directly, we have

$$\delta\{1+g(V, \phi U)^2\}=0 .$$

Accordingly it turns out to be $\delta=0$. This completes the proof. □

Furthermore, by the above proof, we also get

$$A^2\xi=(\alpha+\gamma)A\xi , \qquad \beta^2=\alpha\gamma . \tag{3.12}$$

By polarization in (3.8), we have

$$\begin{aligned}&ag(X, \xi)\{g(A\phi AY, \xi)+ag(A\phi Y, \xi)+g(A^2\phi Y, \xi)\}\\ &\quad+ag(Y, \xi)\{g(A\phi AX, \xi)+ag(A\phi X, \xi)+g(A^2\phi X, \xi)\}\\ &\quad+2\{g(AX, \xi)g(A^2\phi Y, \xi)-g(A\phi X, \xi)g(A^2Y, \xi)\}\\ &\quad+2\{g(AY, \xi)g(A^2\phi X, \xi)-g(A\phi Y, \xi)g(A^2X, \xi)\}\\ &=0 .\end{aligned}$$

Putting $Y=\xi$, we see

$$a\{g(A\phi AX,\ \xi)+ag(A\phi X,\ \xi)+g(A^2\phi X,\ \xi)\}$$
$$+2\{g(A\xi,\ \xi)g(A^2\phi X,\ \xi)-g(A\phi X,\ \xi)g(A^2\xi,\ \xi)\}$$
$$=0$$

for any vector fild X because $A\phi A\xi$ is orthogonal to ξ. Consequently

$$aA\phi A\xi+(a+2\alpha)\phi A^2\xi+(a^2-2\alpha^2-2\beta^2)\phi A\xi=0\,.$$

By (3.12), we get

$$A\phi U+\lambda\phi U=0\,,\qquad \lambda=a+\alpha+\gamma\,. \tag{3.13}$$

We remark here that the property $a\neq 0$ is essential to derive the above first equation.

Next, we give the following

LEMMA 3.3. *Assume that $A^2\xi+kA\xi=0$, where k is constant. Then it satisfies*

$$a\lambda^2+\left(4a\gamma-2k\gamma+\frac{c}{4}\right)\lambda-a^2\gamma-\frac{c}{4}(2k+2\alpha+\gamma)=0 \qquad \text{on } M_0\,. \tag{3.14}$$

PROOF. Differentiating our assumption $A^2\xi+kA\xi=0$ with resect to X and taking account of (2.1), (2.3) and (3.2), we get

$$\nabla_X A(A\xi)+aA(A\phi-\phi A)X+ak(A\phi-\phi A)X$$
$$+A^2\phi AX+kA\phi AX-\frac{c}{4}A\phi X-\frac{c}{4}k\phi X$$
$$=0$$

for any vector field X. The inner product of this equation with any vector field Y implies

$$g(\nabla_X A(Y),\ A\xi)+ag(A(A\phi-\phi A)X,\ Y)+akg((A\phi-\phi A)X,\ Y)$$
$$+g(A^2\phi AX,\ Y)+kg(A\phi AX,\ Y)=\frac{c}{4}g(A\phi X,\ Y)-\frac{c}{4}kg(\phi X,\ Y)$$
$$=0\,.$$

Exchanging X and Y in the above equation and substituting the second one from the first one, we have

$$g(\nabla_X A(Y)-\nabla_Y A(X),\ A\xi)+ag((A^2\phi-2A\phi A+\phi A^2)X,\ Y)$$
$$+g((A^2\phi A+A\phi A^2)X,\ Y)+2kg(A\phi AX,\ Y)$$
$$-\frac{c}{4}g((A\phi+\phi A)X,\ Y)-\frac{c}{2}kg(\phi X,\ Y)$$
$$=0$$

for any vector fields X and Y. Putting $X=U$ and $Y=\phi U$ in this equation and taking account of (3.10), (3.12) and (3.13), we can easily show the equation (3.14). □

Now, we are in position to prove Proposition 3.1.

PROOF OF PROPOSITION 3.1. By the form $A\xi=\alpha\xi+\beta U$ and (2.1), we have

$$\nabla_\xi A(\xi)=d\alpha(\xi)\xi+\alpha\beta\phi U+d\beta(\xi)U-\beta A\phi U+\beta\nabla_\xi U\,.$$

This, combining with the assumption (3.2), implies

$$d\alpha(\xi)\xi+d\beta(\xi)U+\beta(a+\alpha)\phi U-\beta A\phi U+\beta\nabla_\xi U=0\,.$$

From the inner product of ξ and U respectively, we get $d\alpha(\xi)=0$ and $d\beta(\xi)=0$, where we have used that $g(\nabla_\xi U, \xi)=0$, $g(A\phi U, \xi)=0$ and $g(A\phi U, U)=0$. Hence

$$(a+\alpha)\phi U-A\phi U+\nabla_\xi U=0\,. \tag{3.15}$$

By (3.13) and the above equation, we find

$$\begin{cases} \nabla_\xi U=-(2a+2\alpha+\gamma)\phi U\,, \\ d\alpha(\xi)=0\,, \qquad d\beta(\xi)=0\,. \end{cases} \tag{3.16}$$

On the other hand, by making use of (3.2) and (3.10), $\gamma=g(AU, U)$ gives us to

$$d\gamma(\xi)=0\,. \tag{3.17}$$

Furthermore, from (3.13) and (3.16), we get $d\lambda(\xi)=0$. Differentiating (3.13) with respect to ξ covariantly and taking account of (2.1) and the above property, we get

$$\nabla_\xi A(\phi U)-g(AU, \xi)A\xi+A\phi(\nabla_\xi U)+\lambda\{-g(AU, \xi)\xi+\phi\nabla_\xi U\}=0.$$

By (3.2), (3.12), (3.13) and the first equation of (3.16), the above equation gives the following

$$a+\alpha+\gamma=0 \quad \text{or} \quad a+2\alpha+2\gamma=0\,. \tag{3.18}$$

Since $a\neq 0$, $\alpha+\gamma\neq 0$ by the above equation.

Now, we consider the first case $a+\alpha+\gamma=0$ of (3.18). By (3.13) and (3,15), we get

$$A\phi U=0\,, \qquad \nabla_\xi U=\gamma\phi U\,. \tag{3.19}$$

By (2.1), we have $\nabla_U\xi=\phi AU=\gamma\phi U$. This implies $[\xi, U]=0$ by the second equation of (3.19). On the other hand, by (2.1), (3.10) and (3.17), we get

$$\nabla_U \nabla_\xi \xi = d\beta(U)\phi U - \beta\gamma\xi + \beta\phi\nabla_U U\,,$$
$$\nabla_\xi \nabla_U \xi = -\beta\gamma\xi - \gamma^2 U\,.$$

Accordingly, by the Riemannian curvature tensor $R(\xi, U)\xi$ and (2.2), we have

$$\left(\frac{c}{4}-\gamma^2\right)U - d\beta(U)\phi U - \beta\phi\nabla_U U = 0\,,$$

where we have used (3.12). The inner product of the above equation and ϕU yields $d\beta(U)=0$. Thus

$$\left(\frac{c}{4}-\gamma^2\right)U - \beta\phi\nabla_U U = 0\,,$$

from which we get

$$\beta\nabla_U U = \left(\gamma^2 - \frac{c}{4}\right)\phi U\,, \qquad d\beta(U)=0\,. \tag{3.20}$$

Differentiating $A\xi = \alpha\xi + \beta U$ with respect to any vector field X covariantly and taking account of (3.2), we get

$$a(A\phi - \phi A)X - \frac{c}{4}\phi X + A\phi AX - d\alpha(X)\xi - \alpha\phi AX - d\beta(X)U - \beta\nabla_X U = 0.$$

By taking the inner product of this equation with ξ and U respectively, we get

$$d\alpha(X) = a\beta g(\phi X, U)\,, \tag{3.21}$$

$$d\beta(X) = \left(a\gamma - \frac{c}{4}\right)g(\phi X, U)\,, \tag{3.22}$$

where we have used (3.10) and the first equation of (3·19). Because of $\beta^2 = \alpha\gamma$, it is easily seen that

$$2\beta d\beta(X) = \gamma d\alpha(X) + \alpha d\gamma(X)\,,$$

from which together with (3.21) and (3.22) it turns out to be

$$2\left(a\gamma - \frac{c}{4}\right)g(\phi X, U) = a(\gamma - \alpha)g(\phi X, U)$$

for any vector field X. This implies $2a^2 + c = 0$. Hence, by (3.14), we get $\gamma = 0$, where we have used that $\lambda = a + \alpha + \gamma = 0$ and $k = a$. Thus we have $\beta = 0$ by (3.12), a contradiction.

Lastly, we suppose that $a + 2\alpha + 2\gamma = 0$.

On the other hand, putting $X = \xi$ and $Y = U$ in (3.5) and from the inner product of ξ and U respectively, we obtain

$$\begin{cases} \beta g(\phi\nabla_U U, U) = (a+\gamma)(a+\alpha+\gamma) + \gamma(a+\alpha) + \dfrac{c}{4}\,, \\ \beta(a+\alpha+2\gamma)g(\phi\nabla_U U, U) = a(a+2\gamma)(a+\alpha+\gamma) + \gamma^2(a+\alpha) - \dfrac{c}{4}(a+\alpha)\,, \end{cases}$$

where we have used (3.2), (3.10), (3.12), (3.13), (3.16) and (3.17). Combining of the above two equations, we get

$$(a+\alpha+\gamma)\left(a\alpha+2a\gamma+2\alpha\gamma+2\gamma^2+\frac{c}{2}\right)=0 .$$

By our assumption, we have $a^2=c$. Therefore, by (3.14), we obtain $\alpha=0$, where we have used that $a+2\alpha+2\gamma=0$ and $k=\lambda=a/2$. Hence $\beta=0$, a contradiction.

These mean that the subset M_0 is empty and hence the structure vector field ξ is principal. □

REMARK. The equation (3.2) is equivalent to

$$\mathcal{L}_\xi(h+ag)=0 ,$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ and $h(X, Y)=g(AX, Y)$ for any vector fields X and Y.

The main theorem is proved by Proposition 3.1, the remark stated first in this section and Theorems A and B.

References

[1] U.-H. Ki, S.-J. Kim and S.-B. Lee, Some characterizations of a real hypersurfaces of type A, Kyungpook Math. J. **31** (1991), 73-82.

[2] U.-H. Ki and Y. J. Suh, On real hypersurfaces of a complex space form, Math. J. Okayama **32** (1990), 207-221.

[3] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space II, Tsukuba J. Math. **15** (1991), 547-561.

[4] S. Maeda and S. Udagawa, Real hypersurfaces of a complex projective space in term of holomorphic distribution, Tsukuba J. Math. **14** (1990), 39-52.

[5] Y. Maeda, On real hypersurfaces of a complex projective space, J. Math. Soc. Japan **28** (1976), 529-540.

[6] S. Montiel and A. Romero, On some real hypersurfaces of a complex hyperbolic space, Geometriae Dedicata **20** (1986), 245-261.

[7] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. **212** (1975), 355-364.

Institute of Mathematics
University of Tsukuba
350 Ibaraki, Japan

Department of Applied Mathematics
National Fisheries University of Pusan
Pusan 608-737, Korea