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ANOTHER PROOF OF THE REPRESENTATION FORMULA OF THE SCATTERING KERNEL FOR THE ELASTIC WAVE EQUATION

By

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§ 0. Introduction

In 1977, Majda [6] proved a representation formula of the scattering kernel for the scalar-valued wave equation. Melrose [7] and Soga [9] obtained the equivalent representation formula. This formula was very useful to investigate the inverse scattering problems (cf. Majda [6], Soga [9], [10]). For the elastic wave equation, Shibata and Soga [8] recently have given us the scattering theory by the same conception as in Lax and Phillips [4] and a representation formula has been proved by Soga [11]. Since he uses the same approach as in the case of the scalar-valued wave equation (cf. Soga [9]) it is necessary to get the leading terms of integrals $\int_{S^{n-1}}(J_{\pm}k)(t\varphi(\omega), \omega)d\omega$ as $|t|\to\infty$, where $J_{\pm}$ is a pseudo-differential operator with a homogeneous symbol of order $(n-1)/2$ (for the precise definition of $J_{\pm}$ see § 1). This caused the difficulty in his strategy and the necessity of the convexity of each slowness surface.

In the present paper, we give a proof of the representation formula of the scattering kernel for the elastic wave equation without a convexity assumption of the slowness surfaces. Our proof is based on a kernel representation for the Fourier transform of the scattering kernel (cf. Theorem 1.2 in § 1). Since the Fourier transformation changes the operator $J_{\pm}$ into a multiplication operator, in the proof of that kernel representation we do not meet the difficulty in Soga [11] stated above. Furthermore, we do not need the convexity assumption of the slowness surfaces to obtain that kernel representation. This is one of the main parts in the present paper. Thus, our proof gives us not only the simplicity but also the removal of the convexity assumption of the slowness surfaces.

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Let Ω be an exterior domain in $\boldsymbol{R}^n$ $(n\geqq 3)$ with smooth and compact boundary. We set

$$A(\partial_x)u=\sum_{i,j=1}^{n}\partial_{x_i}(a_{ij}\partial_{x_j}u), \qquad u={}^t(u_1, u_2, \cdots, u_n),$$

where $a_{ij}=(a_{ipjq}|_{q\to 1,\cdots,n}^{p\downarrow 1,\cdots,n})$ are $n\times n$ matrices and each a_{ipjq} is constant. We consider the elastic wave equation with the Dirichlet or the Neumann boundary condition

$$\text{(0.1)} \qquad \begin{cases} (\partial_t^2-A(\partial_x))u(t, x)=0 & \text{in } \boldsymbol{R}\times\Omega, \\ B(\partial_x)u(t, x)=0 & \text{on } \boldsymbol{R}\times\partial\Omega, \\ u(0, x)=f_1(x), \quad \partial_t u(0, x)=f_2(x) & \text{on } \Omega. \end{cases}$$

Here the boundary operator is of the form $B(\partial_x)u=u|_{\partial\Omega}$ (for the Dirichlet condition), $B(\partial_x)u=\sum_{i,j=1}^{n}\nu_i(x)a_{ij}\partial_{x_j}u|_{\partial\Omega}$ (for the Neumann condition), where $\nu(x)={}^t(\nu_1(x), \nu_2(x), \cdots, \nu_n(x))$ is the unit outer normal to Ω at $x\in\partial\Omega$.

We assume that

$$\text{(A.1)} \qquad a_{ipjq}=a_{pijq}=a_{jqip},$$

$$\text{(A.2)} \qquad \sum_{i,p,j,q=1}^{n} a_{ipjq}\varepsilon_{jq}\bar{\varepsilon}_{ip}\geqq\delta_1\sum_{i,p=1}^{n}|\varepsilon_{ip}|^2,$$

$$\text{(A.3)} \qquad A(\xi)=\sum_{i,j=1}^{n}a_{ij}\xi_i\xi_j \text{ has } d \text{ characteristic roots of constant multiplicity for any } \xi\in\boldsymbol{R}^n\setminus\{0\},$$

where (ε_{jq}) is any $n\times n$ symmetric matrix and δ_1 is some positive constant independent of (ε_{jq}).

Under the assumptions (A.1)~(A.3), Shibata and Soga [8] formulate the scattering theory which is analogous to the theory of Lax and Phillips [4]. Let $k_-(s, \omega)$ and $k_+(s, \omega)\in L^2(\boldsymbol{R}\times S^{n-1})=\{L^2(\boldsymbol{R}\times S^{n-1})\}^n$ be the incoming and outgoing translation representations of the initial data $f={}^t(f_1, f_2)$ respectively. The mapping $S: k_-\to k_+$ is called the scattering operator, which is a unitary operator from $L^2(\boldsymbol{R}\times S^{n-1})$ to itself. The scattering operator S has a temperate distribution kernel called the scattering kernel, and S is of the form

$$(Sk)(s, \theta)=\kappa_n(-D_s)k(s, \theta)+\int_{\boldsymbol{R}\times S^{n-1}}S_0(s-s', \theta, \omega)k(s', \omega)ds'd\omega,$$

where $\kappa_n(-D_s)$ is a pseudo-differential operator with symbol $\kappa_n(-\sigma)$ defined as $\kappa_n(\sigma)=1$ (for odd n) and $\kappa_n(\sigma)=-\sigma/|\sigma|$ (for even n), and $S_0(s, \theta, \omega)$ is a temperate distribution kernel.

The representation formula of $S_0(s, \theta, \omega)$ is given by use of the solution

$w_j(t, x; \omega)$ of the problem

$$\begin{cases} (\partial_t^2 - A(\partial_x))w_j = 0 & \text{in } \boldsymbol{R}\times\Omega, \\ B(\partial_x)w_j = -2^{-1}(-2\pi\sqrt{-1})^{1-n}\lambda_j(\omega)^{-n/4} & \\ \qquad\qquad B(\partial_x)\{\delta(t-\lambda_j(\omega)^{-1/2}\omega\cdot x)P_j(\omega)\} & \text{on } \boldsymbol{R}\times\partial\Omega, \\ w_j = 0 \quad \text{if } t \text{ is small enough.} & \end{cases}$$

In the above, $\{\lambda_j(\xi)\}_{j=1,\dots,d}$ $(0<\lambda_1(\xi)<\cdots<\lambda_d(\xi))$ are the eigenvalues of $A(\xi)$, and each $P_j(\xi)$ is the eigenprojector of the eigenvalue $\lambda_j(\xi)$. From the assumptions (A.1)$\sim$(A.3) it follows that each $\lambda_j(\xi)$ and $P_j(\xi)$ is a smooth function in $\xi\in\boldsymbol{R}^n\setminus\{0\}$. Note that $w_j(t, x; \omega)$ is an $n\times n$ matrix of smooth functions in $x\in\bar{\Omega}$ and $\omega\in S^{n-1}$ with the value of temperate distributions in $t\in\boldsymbol{R}$.

THEOREM 0.1. *If we assume* (A.1)$\sim$(A.3), *then the temperate distribution* $S_0(s, \theta, \omega)$ *stated above is of the form*

$$S_0(s, \theta, \omega) = \sum_{i,j=1}^{d} \lambda_i(\theta)^{-n/4}\int_{\partial\Omega}\{P_i(\theta)(\partial_t^{n-2}N(\partial_y)w_j)(\lambda_i(\theta)^{-1/2}y\cdot\theta - s, y; \omega)$$
$$-\lambda_i(\theta)^{-1/2}P_i(\theta)^t(N(\partial_y)(\theta\cdot y))(\partial_t^{n-1}w_j)(\lambda_i(\theta)^{-1/2}y\cdot\theta - s, y; \omega)\}\, dS_y,$$

where $N(\partial_x)u = \sum_{i,j=1}^{n}\nu_i(x)a_{ij}\partial_{x_j}u\,|_{\partial\Omega}$.

Note that the above integral means the Riemann integral of smooth functions with the value of temperate distributions.

Soga [11] obtains the same result as Theorem 0.1 with an additional assumption that every slowness surface $\{\theta\in\boldsymbol{R}^n \mid \lambda_j(\theta)=1\}$ is strictly convex (cf. Theorem 1 in [11]). Thus, Theorem 0.1 is an improvement of Theorem 1 in [11].

We do not prove Theorem 0.1 directly. In our approach, we first obtain a representation formula of the Fourier transform of $S_0(s, \theta, \omega)$ by the outgoing scattered plane waves. That representation formula is stated in § 1 as Theorem 1.2, which is proved in § 2$\sim$§ 4. Theorem 0.1 is derived from Theorem 1.2 by the Fourier inversion formula (cf. § 1).

§ 1. A representation formula of the modified scattering matrix.

In this section, we review the scattering theory obtained by Shibata and Soga [8] and the definition of the modified scattering matrix in Lax and Phillips [5]. Next, we state a representation formula of the modified scattering matrix by the outgoing scattered waves as Theorem 1.2, which gives us Theorem 0.1

by the Fourier transformation.

We denote H by the Hilbert space defined as the completion of $\{f={}^t(f_1, f_2)| B(\partial_x)f_1=0\}$ with the energy norm

$$\|f\|_H^2=\frac{1}{2}\int_\Omega\left\{\sum_{i,j,p,q=1}^{n} a_{ipjq}\partial_{x_j}f_{1q}(x)\overline{\partial_{x_i}f_{1p}(x)}+|f_2(x)|^2\right\}dx\,.$$

The mapping $f\mapsto{}^t(u(t, \cdot), \partial_t u(t, \cdot))$ becomes a group of unitary operators $\{U(t)\}_{t\in\boldsymbol{R}}$ on H, where $u(t, x)$ is a solution of problem (0.1) with initial data $f={}^t(f_1, f_2)$. In the free space case (i.e. $\Omega=\boldsymbol{R}^n$), we denote by H_0 the Hilbert space with the energy norm $\|f\|_{H_0}$, and by $\{U_0(t)\}_{t\in\boldsymbol{R}}$ a group of unitary operators on H_0 which is a solution operator of the free space problem.

The free space translation representation $T_0^{\pm}: H_0\to L^2(\boldsymbol{R}\times S^{n-1})$ has the representation

$$T_0^{\pm}f(s, \omega)=\sum_{j=1}^{d}\lambda_j(\omega)^{1/4}P_j(\omega)(J_{\pm}\mathcal{R}_jf)(\lambda_j(\omega)^{1/2}s, \omega) \quad \text{for any } f\in C_0^\infty(\boldsymbol{R}^n)\,,$$

where

$$\mathcal{R}_jf(s, \omega)=-\lambda_j(\omega)^{1/2}\partial_s\tilde{f}_1(s, \omega)+\tilde{f}_2(s, \omega) \qquad (j=1, 2, \cdots, d)\,,$$

$$\tilde{f}_j(s, \omega)=\int_{x\cdot\omega=s}f_j(x)dS_x \qquad (j=1, 2) \text{ (the Radon transform)},$$

$J_{\pm}=(-\partial_s)^{(n-1)/2}$ for odd n and $J_{\pm}=(-\partial_s)^{(n/2)-1}\lambda_{\pm}(D_s)$ for even n with

$$\lambda_{\pm}(\sigma)=\begin{cases}\dfrac{1-\sqrt{-1}}{\sqrt{2}}\sigma^{1/2} & (\text{for } \sigma\geqq 0)\,,\\[2mm] \pm\dfrac{1+\sqrt{-1}}{\sqrt{2}}|\sigma|^{1/2} & (\text{for } \sigma<0)\,.\end{cases}$$

We fix a constant $\rho>0$ with $\partial\Omega\subset B_\rho$, where $B_\rho=\{x\in\boldsymbol{R}^n\,|\,|x|<\rho\}$. We define the outgoing subspace $D_{\pm}^{\rho}$ as

$$D_{\pm}^{\rho}=U_0(\pm C_{\min}^{-1}\rho)D_{\pm}^{0}\,,$$

where $D_{\pm}^0=\{f\in H_0\,|\,T_0^{\pm}f(s, \omega)=0 \text{ in } \pm s<0\}=\{f\in H_0\,|\,U_0(t)f=0 \text{ in } |x|<\pm C_{\min}t\}$ and $C_{\min}=\min_{j=1,\cdots,d}\inf_{\omega\in S^{n-1}}\{\lambda_j(\omega)^{1/2}\}>0$. The outgoing subspace $D_{\pm}^{\rho}$ is the closed subspace in H_0 and H.

The scattering operator S introduced in §0 is represented as $S=T_0^{+}W_{+}^{-1}W_{-}(T_0^{-})^{-1}$, where the wave operators from H_0 to H

$$W_{\pm}f=s-\lim_{t\to\infty}U(-t)U_0(t)f$$

are well-defined and complete (cf. §3 of [8]). We define unitary operators $\mathcal{T}_{\pm}: H\to L^2(\boldsymbol{R}\times S^{n-1})$ and $\mathcal{T}_{\pm}^0: H_0\to L^2(\boldsymbol{R}\times S^{n-1})$ as $\mathcal{T}_{\pm}=F^{-1}T_0^{\pm}W_{\pm}^{-1}$ and $\mathcal{T}_{\pm}^0=F^{-1}T_0^{\pm}$, where $Fk(\sigma, \omega)=\int_{-\infty}^{\infty}\exp(-\sqrt{-1}\sigma\cdot s)k(s, \omega)ds$ is the Fourier transformation with respect to $s\in\boldsymbol{R}$. The operators $\mathcal{T}_{+}$ and $\mathcal{T}_{-}$ (resp. $\mathcal{T}_{+}^{0}$ and $\mathcal{T}_{-}^{0}$) are called the outgoing and incoming spectral representation of $\{U(t)\}$ (resp. $\{U_0(t)\}$) respectively. They satisfy

$$\mathcal{T}_{\pm}U(t)=e^{\sqrt{-1}\sigma t}\mathcal{T}_{\pm} \quad \text{for any } t\in\boldsymbol{R}, \tag{1.1}$$

$$(\text{resp. } \mathcal{T}_{\pm}^{0}U_0(t)=e^{\sqrt{-1}\sigma t}\mathcal{T}_{\pm}^{0} \quad \text{for any } t\in\boldsymbol{R}).$$

Now, wet set $\mathcal{S}=F^{-1}SF$. Using the outgoing and incoming spectral representations, we can express the operators $\mathcal{S}$ as $\mathcal{S}=\mathcal{T}_{+}\mathcal{T}_{-}^{-1}$. Hence, the operator $\mathcal{S}$ has the following properties:

(i) $\mathcal{S}$ is unitary on $L^2(\boldsymbol{R}\times S^{n-1})$,

(ii) $\mathcal{S}$ commutes with multiplication by bounded measureable complex-valued functions.

Then, by Corollary 4.2 in Chap. II of Lax and Phillips [4], we have the following Proposition.

PROPOSITION 1.1. *There is a $B(L^2(S^{n-1}), L^2(S^{n-1}))$-valued function $\mathcal{S}(\sigma)$ on $\sigma\in\boldsymbol{R}$ called the modified scattering matrix satisfying that $\mathcal{S}(\sigma)$ is unitary for almost all $\sigma\in\boldsymbol{R}$, and for any $k\in L^2(\boldsymbol{R}\times S^{n-1})$ we have*

$$\mathcal{S}k(\sigma, \theta)=(\mathcal{S}(\sigma)k(\sigma, \cdot))(\theta) \quad \textit{for almost all } \sigma\in\boldsymbol{R} \textit{ and } \theta\in S^{n-1}.$$

Note that for odd n, $\mathcal{S}(\sigma)$ is the same as the scattering matrix in Lax and Phillips [4].

We denote $v_{+}^{(j)}(x;\sigma, \omega)\in C^{\infty}(\bar{\Omega}\times(\boldsymbol{R}\setminus\{0\})\times S^{n-1})$ by the outgoing solution of problem

$$\begin{cases} (A(\partial_x)+\sigma^2)v_{+}^{(j)}(x;\sigma, \omega)=0 & \text{in } \Omega, \\ B(\partial_x)v_{+}^{(j)}(x;\sigma, \omega)=-\lambda_j(\omega)^{-n/4} & \\ \qquad\cdot B(\partial_x)\{e^{-\sqrt{-1}\lambda_j(\omega)^{-1/2}\sigma\omega\cdot x}P_j(\omega)\} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where outgoing means that $v_{+}^{(j)}(x;\sigma, \omega)$ is the analytic continuation of the $L^2(\Omega)$-valued solution of problem (1.2) with $\operatorname{Im}\sigma<0$. Note that $v_{+}^{(j)}$ is an $n\times n$ matrix of smooth functions in $x\in\bar{\Omega}$ and $\omega\in S^{n-1}$ with the value of temperate distributions in $\sigma\in\boldsymbol{R}$, and satisfies $(Fw_j)(x;\sigma, \omega)=2^{-1}(-2\pi\sqrt{-1})^{1-n}v_{+}^{(j)}(x;\sigma, \omega)$ for each $j=1, 2, \cdots, d$.

THEOREM 1.2. *If we assume* (A.1)$\sim$(A.3), *then the modified scattering matrix*

$\mathcal{S}(\sigma)$ is represented as

$$(\mathcal{S}(\sigma)k(\sigma,\ \cdot))(\theta)=\kappa_n(\sigma)k(\sigma,\ \theta)+\int_{S^{n-1}}K(\sigma,\ \theta,\ \omega)k(\sigma,\ \omega)d\omega$$

$$a.e.\ \sigma \text{ and } \theta \text{ for any } k\in C_0^\infty(\boldsymbol{R}\times S^{n-1}),$$

where $K(\sigma,\ \theta,\ \omega)\in C^\infty((\boldsymbol{R}\setminus\{0\})\times S^{n-1}\times S^{n-1})$ is of the form

$$K(\sigma,\ \theta,\ \omega)=\frac{(\sqrt{-1}\sigma)^{n-2}}{2(-2\pi\sqrt{-1})^{n-1}}\sum_{i,j=1}^{d}\lambda_i(\theta)^{-n/4}$$

$$\int_{\partial\Omega}e^{\sqrt{-1}\sigma\lambda_i(\theta)^{-1/2}\theta\cdot y}\{P_i(\theta)(N(\partial_y)v_+^{(j)})(y\ ;\ \sigma,\ \omega)$$

$$-\sqrt{-1}\sigma\lambda_i(\theta)^{-1/2}P_i(\theta)^t(N(\partial_y)(\theta\cdot y))v_+^{(j)}(y\ ;\ \sigma,\ \omega)\}\,dS_y\,.$$

Now, we prove Theorem 0.1 by use of Theorem 1.2. We denote $\langle\ ,\ \rangle$ by the pairing of temperate distributions. From $S=F\mathcal{S}F^{-1}$ and Theorem 1.2 it follows that

$$\text{(1.3)}\qquad \langle Sk,\ h\rangle-\langle\kappa_n(-D_s)k,\ h\rangle$$

$$=\int_{\boldsymbol{R}\times S^{n-1}}\int_{S^{n-1}}K(\sigma,\ \theta,\ \omega)F^{-1}k(\sigma,\ \omega)d\omega\cdot Fh(\sigma,\ \theta)d\theta d\sigma$$

for any $k,\ h\in\mathcal{S}(\boldsymbol{R}\times S^{n-1})$ with $F^{-1}k,\ Fh\in C_0^\infty(\boldsymbol{R}\times S^{n-1})$, which yields the right-hand side of (1.3) is of the form

$$\int_{S^{n-1}}d\theta\int_{S^{n-1}}d\omega\int_{\partial\Omega}dS_y\int_{-\infty}^{\infty}d\sigma\sum_{i,j=1}^{d}\lambda_i(\theta)^{-n/4}e^{\sqrt{-1}\sigma\lambda_i(\theta)^{-1/2}\theta\cdot y}$$

$$\Big\{P_i(\theta)\frac{(\sqrt{-1}\sigma)^{n-2}}{2(-2\pi\sqrt{-1})^{n-1}}((N(\partial_y)v_+^{(j)})(y\ ;\ \sigma,\ \omega)F^{-1}k(\sigma,\ \omega))\cdot Fh(\sigma,\ \theta)$$

$$-\lambda_i(\theta)^{-1/2}P_i(\theta)^t(N(\partial_y)(y\cdot\theta))$$

$$\frac{(\sqrt{-1}\sigma)^{n-1}}{2(-2\pi\sqrt{-1})^{n-1}}(v_+^{(j)}(y\ ;\ \sigma,\ \omega)F^{-1}k(\sigma,\ \omega))\cdot Fh(\sigma,\ \theta)\Big\}\,.$$

In fact, since from the assumptions (A.1)$\sim$(A.3) it follows that each $v_+^{(j)}(x\ ;\ \sigma,\ \omega)$ is locally uniformly bounded in $\bar{\Omega}\times\boldsymbol{R}\times S^{n-1}$ (cf. the proof of Theorem 1.2 in Iwashita and Shibata [3]), the integrated function of the right-hand side is absolutely integrable. Hence, by the Fourier inversion formula we obtain Theorem 0.1.

Thus, our purpose in the rest of the present paper is to prove Theorem 1.2. In the case of the scalar-valued wave equation Lax and Phillips [5] give a kernel representation of the modified scattering matrix, however, they use the asymptotic behaviour of the fundamental solutions for the free space problem as $|x|\to\infty$. Hence, the lack of the assumption about the convexity of the slowness surfaces does not allow us to use their approach completely. Nevertheless we can prove Theorem 1.2 because the result about the analytic continuation of resolvent obtained by Iwashita [2] and Iwashita and Shibata [3] gives us a characterization of outgoing and incoming solutions.

In § 2, we give the representation of $\mathcal{T}_{\pm}$ by the distorted plane waves. Theorem 1.2 is proved in § 3 by means of the explicit form of the difference between the outgoing and incoming fundamental solutions for the free space problem. It is one of the crucial fact to obtain Theorem 1.2 as is pointed out in Lax and Phillips [5]. In our case, we can not write the fundamental solutions by use of special functions, however, we can get the explicit form stated above by looking at the construction of that solutions carefully (cf. § 4).

§ 2. The spectral representations.

In § 1, we see $\mathcal{S}(\sigma)=\mathcal{T}_{+}\mathcal{T}_{-}^{-1}$. Hence, to prove Theorem 1.2, we have to represent $\mathcal{T}_{+}$ (resp. $\mathcal{T}_{-}$) by the outgoing (resp. incoming) distorted plane wave, which is our purpose in this section.

We set

$$w_0(x\,;\,\sigma,\,\omega)=\sum_{j=1}^{d}\lambda_j(\omega)^{-n/4}e^{-\sqrt{-1}\sigma\lambda_j(\omega)^{-1/2}\omega\cdot x}P_j(\omega)\,,$$

$$\phi_{\pm}(x\,;\,\sigma,\,\omega)=c_{n,\pm}(\sigma)w_0(x\,;\,\sigma,\,\omega)\{1,\ \sqrt{-1}\sigma\}\,,$$

where

$$c_{n,\pm}(\sigma)=\begin{cases}(-1)^{(n-1)/2}2(2\pi)^{-1}(\sqrt{-1}\sigma)^{(n-3)/2} & \text{(for odd } n),\\ \mp(-1)^{(n/2)-1}2(2\pi)^{-1}(\sqrt{-1}\sigma)^{(n/2)-2}\lambda_{\mp}(\sigma) & \text{(for even } n).\end{cases}$$

Since $\mathcal{T}_{\pm}^{0}=F^{-1}T_0^{\pm}$ we have the following representation

(2.1) $$(\mathcal{T}_{\pm}^{0}f)_l(\sigma,\,\omega)=(f,\,\phi_{\mp}(\cdot\,;\,\sigma,\,\omega)e_l)_{H_0}$$

for any $f\in S(\boldsymbol{R}^n)$ and $l=1,\,\cdots,\,n$,

where $(\mathcal{T}_{\pm}^{0}f)_l$ is the l-th component of $\mathcal{T}_{\pm}^{0}f$ and $e_l={}^t(0,\,\cdots,\,1,\,\cdots,\,0)$ (i.e. the l-th column is 1 and other columns are 0).

Next, we define the distorted plane wave $\psi_{\pm}$ as

$$\psi_{\pm}(x\,;\,\sigma,\,\omega)=c_{n,\pm}(\sigma)\{w_0(x\,;\,\sigma,\,\omega)+v_{\pm}(x\,;\,\sigma,\,\omega)\}\,\{1,\ \sqrt{-1}\sigma\}$$

where $v_{\pm}(x\,;\,\sigma,\,\omega)=\sum_{j=1}^{d}v_{\pm}^{(j)}(x\,;\,\sigma,\,\omega)$, and $v_{+}^{(j)}(x\,;\,\sigma,\,\omega)$ is defined in § 1, and $v_{-}^{(j)}(x\,;\,\sigma,\,\omega)$ is the incoming solution satisfying (1.2). In the above, incoming means that $v_{-}^{(j)}(x\,;\,\sigma,\,\omega)$ is the analytic continuation of the $L^2(\Omega)$-valued solution of problem (1.2) with $\mathrm{Im}\,\sigma>0$. Note that $v_{\pm}(x\,;\,\sigma,\,\omega)\in C^{\infty}(\bar{\Omega}\times(\boldsymbol{R}\setminus\{0\})\times S^{n-1})$.

To represent $\mathcal{S}_{\pm}$ by using the distorted plane waves we need the asymptotic behaviour of $v_{\pm}^{(j)}(x;\sigma,\omega)$ as $|x|\to\infty$. To get this we introduce the solution operator $W^{\pm}(z)$ of problem

$$(2.2)\qquad \begin{cases} (A(\partial_x)+z^2)v(x;z)=0 & \text{in } \Omega, \\ B(\partial_x)v(x;z)=g(x) & \text{on } \partial\Omega. \end{cases}$$

For any $m\geqq 0$, the operator $W^{\pm}(z)$ is a $B(H^{m+\gamma_0}(\partial\Omega),\ H^{m+2}(\Omega))$-valued holomorphic function in $\pm\operatorname{Im} z<0$, and a $B(H^{m+\gamma_0}(\partial\Omega),\ H^{m+2}(\Omega_a))$-valued continuous function in $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$, where $\gamma_0=3/2$ in the case of the Dirichlet boundary condition, $\gamma_0=1/2$ in the case of the Neumann boundary condition and $\Omega_a=\Omega\cap B_a$ (cf. Iwashita [2] and Iwashita and Shibata [3]). As for the asymptotic behaviour, we have the following Lemma.

LEMMA 2.1. *Under the assumptions* (A.1)$\sim$(A.3) *for any* $s>1/2$ *and* $|\alpha|\leqq 2$, *the operator* $\langle x\rangle^{-s}\partial_x^{\alpha}W^{\pm}(z)$ *is a* $B(H^{\gamma_0}(\partial\Omega),\ L^2(\Omega))$-*valued continuous function in* $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$, *where* $\langle x\rangle=(1+|x|^2)^{1/2}$.

PROOF OF LEMMA 2.1. For $\rho>0$ stated in §1 we take a cutoff function $\chi\in C^{\infty}(\boldsymbol{R}^n)$ such that $\chi(x)=1$ in $|x|>2\rho$, $\chi(x)=0$ in $|x|<\rho$. Then the function $u(x;z)=\chi(x)W^{\pm}(z)g(x)$ satisfies

$$(A(\partial_x)+z^2)u(x;z)=[A(\partial_x),\ \chi]W^{\pm}(z)g(x)\qquad \text{in } \boldsymbol{R}^n.$$

We denote by $R_0^{\pm}(z)$ the solution operator for the free space problem. By the uniqueness of L^2-solution of the free space problem and a continunity of $R_0^{\pm}(z)$ and $W^{\pm}(z)$ in $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$, we have

$$\chi(x)W^{\pm}(z)g(x)=R_0^{\pm}(z)\{[A(\partial_x),\ \chi]W^{\pm}(z)g\}(x)$$
$$\text{for any } \pm\operatorname{Im} z\leqq 0,\ z\neq 0 \text{ and } g\in H^{\gamma_0}(\partial\Omega).$$

Since for $s>1/2$ and $|\alpha|\leqq 2$, $\langle x\rangle^{-s}\partial_x^{\alpha}R_0^{\pm}(z)\langle x\rangle^{-s}$ is a $B(L^2(\boldsymbol{R}^n),\ L^2(\boldsymbol{R}^n))$-valued continuous function in $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$ (cf. Yajima [13]) and $\langle x\rangle^{s}[A(\partial_x),\ \chi]W^{\pm}(z)$ is a $B(H^{\gamma_0}(\partial\Omega),\ L^2(\boldsymbol{R}^n))$-valued continuous function in $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$, the operator $g\mapsto\langle x\rangle^{-s}\partial_x^{\alpha}(\chi(x)W^{\pm}(z)g(x))$ is a $B(H^{\gamma_0}(\partial\Omega),\ L^2(\Omega))$-valued continuous function in $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$. This completes the proof of Lemma 2.1.

Using the operator $W^{\pm}(z)$ we can write

$$(2.3)\qquad v_{\pm}^{(j)}(x;\sigma,\omega)=W^{\pm}(\sigma)[-\lambda_j(\omega)^{-n/4}B(\partial_x)\{e^{-\sqrt{-1}\lambda_j(\omega)^{-1/2}\sigma\omega\cdot x}P_j(\omega)\}](x)$$
$$\text{for any } \pm\operatorname{Im}\sigma\leqq 0,\ \sigma\neq 0 \text{ and } \omega\in S^{n-1},$$

which implies that for any $s>1/2$ and $|\alpha|\leqq 2$, $\langle x\rangle^{-s}\partial_x^{\alpha}v_{\pm}^{(j)}(x;\sigma,\omega)$ is a $L^2(\Omega)$-

valued continuous function in $\pm\operatorname{Im}\sigma\leqq 0$, $\sigma\neq 0$ and $\omega\in S^{n-1}$. Hence, by the same methods as in Lemma 4.3 and Theorem 5.2 in Lax and Phillips [5] we can get following Lemma.

LEMMA 2.2. *For any positive integer m we set*

$$V_m=\{f\in H\,|\,\langle x\rangle^m\partial_x f_1(x)\in L^2(\Omega)\ \ and\ \ \langle x\rangle^m f_2(x)\in L^2(\Omega)\},$$

and denote A by a generator of $\{U(t)\}_{t\in\boldsymbol{R}}$. *If we assume* (A.1)$\sim$(A.3), *then we have*

(1) $V_m\cap D_\pm^\rho\cap D(A)$ *is dense in* $D_\pm^\rho$,

(2) *for any* $f\in V_m$ *with* $m>n/2+1$, $(f,\ \psi_\mp(\cdot\,;\sigma,\omega)e_l)_H$ *is well-defined and a continuous in* $\pm\operatorname{Im}\sigma\leqq 0$, $\sigma\neq 0$ *and* $\omega\in S^{n-1}$,

(3) *any* $f\in V_m\cap D_\pm^\rho$ *with* $m>n/2+1$ *is orthogonal to* $\psi_\mp-\phi_\mp$, *that is* $(f,\ (\psi_\mp(\cdot\,;\sigma,\omega)-\phi_\mp(\cdot\,;\sigma,\omega))e_l)_H=0$ *for any* $\sigma\in\boldsymbol{R}\setminus\{0\}$, $\omega\in S^{n-1}$, *and* $l=1, 2, \cdots, n$.

Lax and Phillips [5] construct the spectral representation $\mathcal{T}_\pm$ by using Lemma 2.2 and

$$U(t)V_m\subset V_m\qquad\text{for any } t\in\boldsymbol{R}, \tag{2.4}$$

for any fixed integer $m\geqq 0$. The fact (2.4) is used implicitely in [5], however, they do not give the proof of (2.4). Furthermore, we need the fact (2.4) to get the representation of $\mathcal{T}_\pm$ by using the distorted plane waves. Hence, we prove it.

Proof of (2.4). It is sufficient to prove that there is a constant $c_0>0$ independent of m and a constant $C_m>0$ such that

$$\|U(t)f\|_{m,\Omega}\leqq C_m e^{c_0|t|}\|f\|_{m,\Omega}\qquad\text{for any } t\in\boldsymbol{R} \text{ and } f\in V_m, \tag{2.5}$$

where $\|f\|_{m,D}^2=\int_D\langle x\rangle^{2m}\{|\partial_x f_1(x)|^2+|f_2(x)|^2\}dx$ and D is a domain in $\boldsymbol{R}^n$.

We set

$$V_{m,0}=\{f\in H_0\,|\,\langle x\rangle^m\partial_x f_1(x),\ \langle x\rangle^m f_2(x)\in L^2(\boldsymbol{R}^n)\}.$$

Note that V_m (resp. $V_{m,0}$) is a Banach space with norm $\|\cdot\|_{m,\Omega}$ (resp. $\|\cdot\|_{m,\boldsymbol{R}^n}$). Since $S(\boldsymbol{R}^n)$ is dence in $V_{m,0}$ from the argument to prove $U_0(t)\in\mathcal{L}(S(\boldsymbol{R}^n), S(\boldsymbol{R}^n))$ we have

$$\|U_0(t)f\|_{m,\boldsymbol{R}^n}\leqq C_m e^{c_0|t|}\|f\|_{m,\boldsymbol{R}^n}\qquad\text{for any } t\in\boldsymbol{R} \text{ and } f\in V_{m,0}. \tag{2.6}$$

For the cutoff function $\chi(x)$ taken in the proof of Lemma 2.1 we can prove that

$$\chi U(t)f=U_0(t)(\chi f)+\int_0^t U_0(t-s)QU(s)f\,ds \qquad \text{for any } f\in H,$$

where Q is defined as $Qg={}^t(0, [\chi, A(\partial_x)]g_1)$. Since the operator $Q: H\to H_0$ is bounded and $\operatorname{supp} Qg\subset B_{3\rho}\cap\Omega$ for any $g\in H$, from (2.6) it follows that

$$\|\chi U(t)f\|_{m,\Omega}\leqq C_m e^{c_0|t|}\{\|\chi f\|_{m,R^n}+\|f\|_H\} \qquad \text{for any } t\in \boldsymbol{R} \text{ and } f\in V_m.$$

Noting that the estimate $\|f_1\|_{L^2(\Omega\cap B_{2\rho})}\leqq C\|\partial_x f_1\|_{L^2(\Omega)}$ (cf. Shibata and Soga [8]) yields that

$$\|\chi f\|_{m,R^n}\leqq C\|f\|_{m,\Omega} \qquad \text{for any } f\in V_m.$$

Combining the above estimates with the estimate $\|(1-\chi)U(t)f\|_{m,\Omega}\leqq C\|f\|_H$, we obtain (2.5). This completes the proof of (2.4).

Now, we state a representation of $\mathcal{T}_\pm$ which is indispensable to obtain Theorem 1.2.

PROPOSITION 2.3. *We assume* (A.1)$\sim$(A.3) *and take an integer* $m>n/2+1$. *Then, for any* $f\in V_m$ *we have*

$$(\mathcal{T}_\pm f)_l=(f, \phi_\mp(\cdot\,;\sigma,\omega)e_l)_H \qquad \textit{for any } l=1, 2, \cdots, n.$$

REMARK. The same procedure as the construction of $\mathcal{T}_\pm$ in [5] (cf. Theorem 5.3 in [5]) implies that for any $m>n/2+1$ we have

$$(2.7) \qquad (\mathcal{T}_\pm f)_l(\sigma,\omega)=(f, \phi_\mp(\cdot\,;\sigma,\omega)e_l)_H$$

$$\text{for any } f\in \bigcup_{t\in R} U(t)(V_m\cap D_\pm^\rho\cap D(A)) \text{ and } l=1, 2, \cdots, n.$$

Thus, Proposition 2.3 is stronger than (2.7), and to get Proposition 2.3 we need an additional consideration.

PROOF OF PROPOSITION 2.3. Since to obtain Proposition 2.3 we need (2.7) we start with the proof of (2.7).

For any $f\in V_m$ we define the l-th component of $\tilde{\mathcal{T}}_\pm$ as

$$(\tilde{\mathcal{T}}_\pm f)_l=(f, \phi_\mp(\cdot\,;\sigma,\omega)e_l)_H \qquad \text{for } l=1, 2, \cdots, n.$$

Note that from (3) in Lemma 2.2, (2.1) and the fact that $\mathcal{T}_\pm f=\mathcal{T}_\pm^0 f$ for any $f\in D_\pm^\rho$, which is derived from $W_\pm f=f$ for any $f\in D_\pm^\rho$, it follows that $\mathcal{T}_\pm f=\tilde{\mathcal{T}}_\pm f$ for any $f\in D_\pm^\rho\cap V_m$. Hence, by (1.1) the representation (2.7) is equivalent to

$$(2.8) \qquad \tilde{\mathcal{T}}_\pm(U(t)f)=e^{\sqrt{-1}\sigma\cdot t}\tilde{\mathcal{T}}_\pm f \qquad \text{for any } f\in V_m\cap D_\pm^\rho\cap D(A).$$

We take $f\in V_m\cap D_\pm^\rho\cap D(A)$. There is a sequence $f^{(j)}\in C_0^\infty(\Omega)$ such that $f^{(j)}\to f$ in V_m as $j\to\infty$. We differentiate $\tilde{\mathcal{T}}_\pm(U(t)f^{(j)})$ by t and use the integration by parts, that is,

$$\frac{d}{dt}(\mathfrak{T}_{\pm}(U(t)f^{(j)}))_l=(AU(t)f^{(j)},\ \psi_{\mp}(\cdot\ ;\ \sigma,\ \omega)e_l)_H$$
$$=-(U(t)f^{(j)},\ A\psi_{\mp}(\cdot\ ;\ \sigma,\ \omega)e_l)_H$$
$$=\sqrt{-1}\sigma\tilde{\mathfrak{T}}_{\pm}(U(t)f^{(j)}))_l\ .$$

Note that it is possible to do the above reductions since from the finiteness of the proagation speed of the solution of the elastic wave equation (0.1) it follows that $U(t)f^{(j)}$ has compact support. Thus, we obtain $\tilde{\mathfrak{T}}_{\pm}(U(t)f^{(j)})=e^{\sqrt{-1}\sigma\cdot t}\tilde{\mathfrak{T}}_{\pm}f^{(j)}$. From (2.5) it follows that $U(t)f^{(j)}\to U(t)f$ in V_m as $j\to\infty$ for any fixed t. Hence, taking the limit as $j\to\infty$ we have (2.7).

Next, we prove the representation of $T_{\pm}$ is valid for any $f\in V_m$. To show it we need the following Lemma.

LEMMA 2.4. *If we assume* (A.1)~(A.3) *and fix an integer* $m>n/2+1$, *then for any* $f\in V_m$ *we have* $\tilde{\mathfrak{T}}_{\pm}f\in L^2(\boldsymbol{R}\times S^{n-1})$ *and*

$$\|\tilde{\mathfrak{T}}_{\pm}f\|_{L^2(\boldsymbol{R}\times S^{n-1})}\leqq 2\sqrt{2}(2\pi)^{(n-2)/2}\|f\|_H\ .$$

The proof of Lemma 2.4 is postponed and we continue the proof of Proposition 2.3. For any $f\in V_m$ there is a sequence $f^{(j)}\in\bigcup_{t\in\boldsymbol{R}}U(t)(V_m\cap D_{\pm}^{\rho}\cap D(A))$ $(j=1,\ 2,\ \cdots)$ such that $f^{(j)}\to f$ in H as $j\to\infty$. Since (2.7) and (2.5) imply that $\tilde{\mathfrak{T}}_{\pm}f^{(j)}$ is well-defined and $\mathfrak{T}_{\pm}f^{(j)}=\tilde{\mathfrak{T}}_{\pm}f^{(j)}$, from Lemma 2.4 we have $\lim_{t\to\infty}\mathfrak{T}_{\pm}f^{(j)}=\tilde{\mathfrak{T}}_{\pm}f$ in $L^2(\boldsymbol{R}\times S^{n-1})$. Hence, it follows that $\mathfrak{T}_{\pm}f=\tilde{\mathfrak{T}}_{\pm}f$, which completes the proof of Proposition 2.3.

Now, we prove Lemma 2.4. We start with a preparation to obtain Lemma 2.4.

LEMMA 2.5. *Under the assumptions* (A.1)~(A.3), *for any fixed integer* $m>n/2+1$ *we have*

$$\|\tilde{\mathfrak{T}}_{+}f\|^2_{L^2(\boldsymbol{R}\times S^{n-1})}+\|\tilde{\mathfrak{T}}_{-}f\|^2_{L^2(\boldsymbol{R}\times S^{n-1})}=8(2\pi)^{n-2}\|f\|_H^2\qquad \text{for any } f\in D_{vox}(A),$$

where $D_{vox}(A)=\{f\in D(A)|\ \operatorname{supp} f$ *is compact*$\}$.

PROOF OF LEMMA 2.5. We set

$$w_{\pm}(x\ ;\ \sigma,\ \omega)=w_0(x\ ;\ \sigma,\ \omega)+v_{\pm}(x\ ;\ \sigma,\ \omega),$$

and for any $g\in L^2_{vox}(\Omega)$, we define the l-th component of $F_{\pm}g$ as

$$(F_{\pm}g)_l(\xi)=\int_{\Omega}g(x)\cdot\overline{w_{\mp}\left(x\ ;\ |\xi|,\ \frac{\xi}{|\xi|}\right)e_l}dx\qquad \text{for } l=1,\ 2,\ \cdots,\ n,$$

where $L^2_{vox}(\Omega)=\{g\in L^2(\Omega)\mid \operatorname{supp} g$ is compact$\}$. By the same argument as for Theorem 6.7 and Lemma 6.9 in Wilcox [12], we can prove that $F_\pm g\in L^2(\boldsymbol{R}^n)$ for any $g\in L^2_{vox}(\Omega)$ and the following equality holds:

(2.8) $$\|F_\pm g\|_{L^2(\boldsymbol{R}^n)}=(2\pi)^n\|g\|_{L^2(\Omega)} \quad \text{for any } g\in L^2_{vox}(\Omega).$$

For any $f\in D_{vox}(A)$, integration by parts implies that

$$(\mathfrak{T}_\pm f)_l(\sigma,\omega)=-\frac{\sqrt{-1}\sigma}{2}\overline{c_{n,\mp}(\sigma)}\Big\{\sqrt{-1}\sigma\int_\Omega f_1(x)\cdot\overline{w_\mp(x;\sigma,\omega)e_l}\,dx$$
$$+\int_\Omega f_2(x)\cdot\overline{w_\mp(x;\sigma,\omega)e_l}\,dx\Big\}.$$

Since $\lambda_j(\omega)$ and $P_j(\omega)$ $(j=1,2,\cdots,n)$ are even functions and $W^-(z)=W^+(e^{-\pi\sqrt{-1}}z)$ for any $\operatorname{Im} z>0$ we have $w_\pm(x;-\sigma,-\omega)=w_\mp(x;\sigma,\omega)$ for any $\sigma\in\boldsymbol{R}\setminus\{0\}$ and $\omega\in S^{n-1}$, which yields

$$(F_\mp g)_l(\sigma\omega)=\int_\Omega g(x)\cdot\overline{w_\mp(x;\sigma,\omega)e_l}\,dx \qquad \text{for any } \sigma>0,$$

$$(F_\pm g)_l(\sigma\omega)=\int_\Omega g(x)\cdot\overline{w_\pm(x;\sigma,\omega)e_l}\,dx \qquad \text{for any } \sigma<0.$$

Hence, we get

$$|\mathfrak{T}_+f(\sigma,\omega)|^2+|\mathfrak{T}_-f(\sigma,\omega)|^2$$
$$=(2\pi)^{-2}|\sigma|^{n-1}\sum_{\alpha=+,-}\{|\sigma|^2|F_\alpha f_1(\sigma\omega)|^2+|F_\alpha f_2(\sigma\omega)|^2$$
$$+2\operatorname{Re}(\sqrt{-1}\sigma F_\alpha f_1(\sigma\omega)\cdot F_\alpha f_2(\sigma\omega))\}.$$

Since integration by parts gives us

$$\sigma^2(F_\pm f_1)(\sigma\omega)=(F_\pm(-A(\partial_x)f_1)(\sigma\omega),$$

from (2.8) it follows that $\mathfrak{T}_\pm f\in L^2(\boldsymbol{R}\times S^{n-1})$ and

$$\|\mathfrak{T}_+f\|^2_{L^2(\boldsymbol{R}\times S^{n-1})}+\|\mathfrak{T}_-f\|^2_{L^2(\boldsymbol{R}\times S^{n-1})}$$
$$=4(2\pi)^{n-2}\{-(A(\partial_x)f_1,f_1)_{L^2(\Omega)}+\|f_2\|^2_{L^2(\Omega)}\}.$$

This completes the proof of Lemma 2.5 by use of integration by parts for the term $-(A(\partial_x)f_1,f_1)_{L^2(\Omega)}$.

Last, we prove Lemma 2.4. For any $f\in V_m$, there is a sequence $f^{(j)}\in D_{vox}(A)$ $(j=1,2,\cdots)$ satisfying $f^{(j)}\to f$ in V_m as $j\to\infty$. From Lemma 2.5 it follows that $\lim_{j\to\infty}\mathfrak{T}_\pm f^{(j)}$ exists in $L^2(\boldsymbol{R}\times S^{n-1})$. Since $\mathfrak{T}_\pm f^{(j)}\to\mathfrak{T}_\pm f$ in $C((\boldsymbol{R}\setminus\{0\})\times S^{n-1})$, we have $\lim_{j\to\infty}\mathfrak{T}_\pm f^{(j)}=\mathfrak{T}_\pm f$ in $L^2(\boldsymbol{R}\times S^{n-1})$. This completes the proof of Lemma 2.4.

§ 3. Proof of Theorem 1.2.

In this section, we give the proof of Theorem 1.2. First, we state the existence of the integral kernel of the adjoint operator $\mathcal{S}_1(\sigma)^*$ of the operator $\mathcal{S}_1(\sigma)$ defined as $\mathcal{S}_1(\sigma)=\mathcal{S}(\sigma)-\kappa_n(\sigma)$.

PROPOSITION 3.1. *Under the assumptions* (A.1)~(A.3), *there exists a smooth function* $\tilde{K}(\sigma, \theta, \omega)$ *in* $\sigma\in \boldsymbol{R}\setminus\{0\}$, θ *and* $\omega\in S^{n-1}$ *such that*

$$(3.1) \qquad (\mathcal{S}_1(\sigma)^* k(\sigma, \cdot))(\theta)=\int_{S^{n-1}}\tilde{K}(\sigma, \theta, \omega)k(\sigma, \omega)d\omega$$

a.e. in σ *and* θ *for any* $k\in C_0^\infty(\boldsymbol{R}\times S^{n-1})$,

where $\tilde{K}(\sigma, \theta, \omega)^*={}^t\overline{\tilde{K}(\sigma, \theta, \omega)}$ *is of the form*

$$(3.2) \qquad \tilde{K}(\sigma, \theta, \omega)^*=\frac{\sqrt{-1}}{4\pi}\left(\frac{\sigma}{-2\pi}\right)^{n-2}\sum_{j=1}^{d}\lambda_j(\omega)^{-n/4}$$

$$\cdot\int_{\partial\Omega}e^{\sqrt{-1}\sigma\lambda_j(\omega)^{-1/2}\omega\cdot y}\{P_j(\omega)(N(\partial_y)v_+)(y\,;\sigma, \theta)$$

$$-\sqrt{-1}\sigma\lambda_j(\omega)^{-1/2}P_j(\omega)^t(N(\partial_y)(\omega\cdot y))v_+(y\,;\sigma, \theta)\}\,dS_y.$$

Since the assertion of Proposition 3.1 implies that $\mathcal{S}_1(\sigma)$ is an integral operator with kernel $\tilde{K}(\sigma, \omega, \theta)^*$ we can get Theorem 1.2 by Proposition 3.1.

PROOF OF PROPOSITION 3.1. We fix arbitrary $\sigma\in\boldsymbol{R}\setminus\{0\}$. As in the proof of Theorem 6.2 in [5] we begin with the assumption that $\mathcal{S}_1(\sigma)^*$ can be expressed as an integral operator with kernel $\tilde{K}(\sigma, \theta, \omega)$. In this case, $\mathcal{S}_1(\sigma)^*$ can be expressed as the form (3.1), and $\mathcal{S}_1(\sigma)^*$ is continuous from H to $L^2_{loc}((\boldsymbol{R}\setminus\{0\})\times S^{n-1})$. From the unitarity of $\mathcal{T}_\pm$ it follows that $\mathcal{S}(\sigma)^*=(\mathcal{T}_+\mathcal{T}_-^{-1})^* =\mathcal{T}_-\mathcal{T}_+^{-1}$. Hence, we have

$$\mathcal{T}_-f(\sigma, \theta)=\kappa_n(\sigma)\mathcal{T}_+f(\sigma, \theta)+\int_{S^{n-1}}\tilde{K}(\sigma, \theta, \omega)\mathcal{T}_+f(\sigma, \omega)d\omega$$

a.e. in σ and θ for any $f\in H$.

From Proposition 2.3, the definition of the distorted plane waves $\psi_\pm$, and $\kappa_n(\sigma)\lambda_-(\sigma)=-\lambda_+(\sigma)$ for even n, it suffices to show that there exists $\tilde{K}(\sigma, \theta, \omega)$ such that

$$\varphi(x\,;\sigma, \theta)=v_+(x\,;\sigma, \theta)-v_-(x\,;\sigma, \theta)$$

$$-\kappa_n(\sigma)\int_{S^{n-1}}\{w_0(x\,;\sigma, \omega)+v_-(x\,;\sigma, \omega)\}\,\tilde{K}(\sigma, \theta, \omega)^*d\omega$$

vanishes for all $\theta\in S^{n-1}$. Note that $\varphi(x\,;\sigma, \theta)$ satisfies

$$\begin{cases} (A(\partial_x)+\sigma^2)\varphi(x\,;\,\sigma,\,\theta)=0 & \text{in } \Omega, \\ B(\partial_x)\varphi(x\,;\,\sigma,\,\theta)=0 & \text{on } \partial\Omega. \end{cases}$$

We denote by $G^+(x\,;\,z)$ and $G^-(x\,;\,z)$ the outgoing and incoming fundamental solution for the free space problem respectively. The fundamental solutions $G^\pm(x\,;\,z)$ are $S'(\boldsymbol{R}_x^n)$-valued continuous functions in $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$ and satisfy the following properties:

(3.3) $G^\pm(x\,;\,z)$ is continuous in $x\neq 0$ and $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$,

(3.4) ${}^tG^\pm(-x\,;\,z)=G^\pm(x\,;\,z)$ in $S'(\boldsymbol{R}_x^n)$ for any $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$.

For any $g\in C_0^\infty(\boldsymbol{R}^n)$, we set $V^\pm(x\,;\,z)=\int_{\boldsymbol{R}^n}G^\pm(x-y\,;\,z)g(y)dy$, where the integral means the temperate distribution sense. Then, we have

(3.5) For any integer $m\geqq 0$, we can extend the operator $g\mapsto V^\pm(x\,;\,z)$ as a $B(H^m(\boldsymbol{R}^n),\ H^{m+2}(\boldsymbol{R}^n))$-valued holomorphic function in $\pm\operatorname{Im} z<0$.

(3.6) For any $a>0$ and integer $m\geqq 0$, we can extend the operator $g\mapsto V^\pm(\cdot,\,z)$ as a $B(H_a^m(\boldsymbol{R}^n),\ H^{m+2}(B_a))$-valued continuous function in $\pm\operatorname{Im} z\leqq 0$, $z\neq 0$,

where $H_a^m(\boldsymbol{R}^n)=\{f\in H^m(\boldsymbol{R}^n)\,|\,\operatorname{supp} f\subset B_a\}$.

Integration by parts and (3.3) and the continunity of $v_+(x\,;\,\sigma,\,\omega)$ yield

$$v_+(x\,;\,z,\,\theta)=\int_{\partial\Omega}\{{}^t(N(\partial_y)G^+(y-x\,;\,z))v_+(y\,;\,z,\,\theta) - {}^tG^+(y-x\,;\,z)N(\partial_y)v_+(y\,;\,z,\,\theta)\}\,dS_y \tag{3.7}$$

for any $\operatorname{Im} z\leqq 0$, $z\neq 0$, $\theta\in S^{n-1}$ and $x\in\Omega$.

Now, we note that the following representation of the difference between the outgoing and incoming fundamental solutions which is proved in §4;

$$G^+(x\,;\,\sigma)-G^-(x\,;\,\sigma)=\frac{\sigma}{|\sigma|}\frac{\sqrt{-1}}{4\pi}\Big(\frac{|\sigma|}{2\pi}\Big)^{n-2} \cdot\sum_{j=1}^{d}\int_{S^{n-1}}\lambda_j(\omega)^{-n/2}e^{\sqrt{-1}|\sigma|\lambda_j(\omega)^{-1/2}\omega\cdot x}P_j(\omega)d\omega, \quad \text{for any } \sigma\in\boldsymbol{R}\setminus\{0\}.$$

Since each $\lambda_j(\omega)$ is an even function and each $P_j(\omega)$ is an $n\times n$ matrix of even functions, we obtain

$$G^+(x\,;\,\sigma)-G^-(x\,;\,\sigma)=-\kappa_n(\sigma)\frac{\sqrt{-1}}{4\pi}\Big(\frac{\sigma}{-2\pi}\Big)^{n-2} \cdot\sum_{j=1}^{d}\int_{S^{n-1}}\lambda_j(\omega)^{-n/2}e^{\sqrt{-1}\sigma\lambda_j(\omega)^{-1/2}\omega\cdot x}P_j(\omega)d\omega \quad \text{for any } \sigma\in\boldsymbol{R}\setminus\{0\}. \tag{3.8}$$

From (3.7) and (3.8), it follows that

$$\varphi(x;\sigma,\theta)=\phi(x;\sigma,\theta)-\kappa_n(\sigma)\int_{S^{n-1}}\sum_{j=1}^{d}\lambda_j(\omega)^{-n/4}e^{-\sqrt{-1}\sigma\lambda_j(\omega)^{-1/2}\omega\cdot x}$$

$$\cdot\Big[P_j(\omega)\tilde{K}(\sigma,\theta,\omega)^*-\frac{\sqrt{-1}}{4\pi}\Big(\frac{\sigma}{-2\pi}\Big)^{n-2}\lambda_j(\omega)^{-n/4}$$

$$\cdot\int_{\partial\Omega}e^{\sqrt{-1}\sigma\lambda_j(\omega)^{-1/2}\omega\cdot y}\{{}^tP_j(\omega)(N(\partial_y)v_+)(y;\sigma,\theta)$$

$$-\sqrt{-1}\sigma\lambda_j(\omega)^{-1/2}{}^t(N(\partial_y)(\omega\cdot y)P_j(\omega))v_+(y;\sigma,\theta)\}\,dS_y\Big]d\omega,$$

where $\phi(x;z,\theta)=\phi_1(x;z,\theta)+\phi_2(x;z,\theta)$ and

$$\phi_1(x;z,\theta)=-v_-(x;z,\theta)-\kappa_n(\sigma)\int_{S^{n-1}}v_-(x;z,\omega)\tilde{K}(\sigma,\theta,\omega)^*d\omega,$$

$$\phi_2(x;z,\theta)=\int_{\partial\Omega}\{{}^t(N(\partial_y)G^-(y-x;z))v_+(y;\sigma,\theta)$$

$$-{}^tG^-(y-x;z)N(\partial_y)v_+(y;\sigma,\theta)\}\,dS,\quad \text{for any } \operatorname{Im}z\geqq 0,\ z\neq 0.$$

Now, we define $\tilde{K}(\sigma,\theta,\omega)$ as (3.2). Then noting that each $P_j(\omega)$ is symmetric and $P_j(\omega)^2=P_j(\omega)$ for any $\omega\in S^{n-1}$ and $j=1, 2, \cdots, n$ we have $\varphi(x;\sigma,\omega)=\phi(x;\sigma,\omega)$. Hence, to get Proposition 3.1 it suffices to prove

$$\phi(x;\sigma,\theta)=0\quad \text{for any } x \text{ and } \theta. \tag{3.9}$$

By (3.4) we have

$$(A(\partial_x)+z^2)\phi(x;z,\theta)=0\quad \text{in } \Omega \text{ for any } \operatorname{Im}z\geqq 0,\ z\neq 0,$$

and since $\varphi(x;\sigma,\theta)=\phi(x;\sigma,\theta)$ in $\bar{\Omega}$ it follows that

$$B(\partial_x)\phi(x;\sigma,\theta)=0\quad \text{on } \partial\Omega. \tag{3.10}$$

LEMMA 3.2. *For any fixed $\theta\in S^{n-1}$ the function ϕ is a $H^2(\Omega)$-valued holomorphic function in* $\operatorname{Im}z>0$ *and a $H^2(\Omega_a)$-valued continuous function in* $\operatorname{Im}z\geqq 0$, $z\neq 0$ *for any $a>0$ satisfying $\partial\Omega\subset B_a$.*

We postpone the proof of Lemma 3.2, which is given later.

Using Lemma 3.2, we prove (3.9). We set $h(x;z,\theta)=B(\partial_x)\phi(x;z,\theta)$. The uniqueness of $L^2(\Omega)$-valued solution of problem (2.2) implies

$$\phi(x;z,\theta)=(W^-(z)h(\cdot;z,\theta))(x)\quad \text{for any } \operatorname{Im}z>0. \tag{3.11}$$

Since each side of (3.11) is $H^2(\Omega_a)$-valued continuous function in $\operatorname{Im}z\geqq 0$, $z\neq 0$ (cf. Lemma 3.2), the equality (3.11) holds in the region $\operatorname{Im}z\geqq 0$, $z\neq 0$. Hence, it follows that $\phi(x;\sigma,\theta)=(W^-(\sigma)h(\cdot;\sigma,\theta))(x)$. Noting that (3.10), we have

$h(x;\sigma,\theta)=0$, which completes the proof of (3.9). Thus, we get Proposition 3.1.

Now, we begin to prove Lemma 3.2. The construction of $v_-^{(j)}(x;\sigma,\omega)$ (cf. (2.3)) and the properties of $W^-(z)$ stated in § 2 imply that the function $\psi_1(x;z,\theta)$ has the same property as that in Lemma 3.2. Thus, we have only to show $\psi_2(x;z,\theta)$ has the properties stated in Lemma 3.2. We take $\chi_0\in C_0^\infty(\boldsymbol{R}^n)$ satisfying $0\leqq\chi_0\leqq 1$, $\chi_0(x)=1$ near $\partial\Omega$, and set $w(x;\theta)=\chi_0(x)v_+(x;\sigma,\theta)$. Note that we omit the variable σ because we fix $\sigma\in\boldsymbol{R}\setminus\{0\}$. The function $w(x;\theta)$ satisfies

$$\begin{cases} (A(\partial_x)+z^2)w(x;\theta)=g(x;z,\theta) & \text{in } \Omega, \\ \quad B(\partial_x)w(x;\theta)=B(\partial_x)v_+(x;\sigma,\theta) & \text{on } \partial\Omega, \end{cases}$$

where $g(x;z,\theta)=[A(\partial_x),\chi_0]v_+(x;\sigma,\theta)+(z^2-\sigma^2)w(x;\theta)\in C_0^\infty(\bar{\Omega})$.

For any $\chi\in C_0^\infty(\Omega)$ and $z\in\boldsymbol{C}$ with $\operatorname{Im} z>0$, we set $V^-(x;z)=\int_{\boldsymbol{R}^n}G^-(x-y;z)\chi(y)dy\in H^2(\boldsymbol{R}^n)$. Integration by parts yields

$$\int_{\partial\Omega}\{N(\partial_x)V^-(x;z)\cdot w(x;\theta)-V^-(x;z)\cdot N(\partial_x)w(x;\theta)\}\,dS_x$$
$$=\int_\Omega\{\chi(x)\cdot w(x;\theta)-V^-(x;z)\cdot g(x;z,\theta)\}\,dx\,,$$

where $\cdot$ means the inner product of $\boldsymbol{C}^n$. Hence, we have

$$\int_{\boldsymbol{R}^n}\chi(y)\cdot\left\{\int_\Omega {}^tG^-(x-y;z)g(x;z,\theta)dx+\psi_2(y;z,\theta)-w(y;\theta)\right\}dy=0$$
$$\text{for any } \operatorname{Im} z>0\,.$$

For any fixed $x_0\in\Omega$, we replace $\chi(y)$ by $\varepsilon^{-n}\chi((y-x_0)/\varepsilon)$ with $\chi\in C_0^\infty(\boldsymbol{R}^n)$ satisfy $\int_{\boldsymbol{R}^n}\chi(y)dy=1$, $\operatorname{supp}\chi\subset\{x\in\boldsymbol{R}^n\,|\,|x|<1\}$. Taking the limit as $\varepsilon\downarrow 0$, we have

$$\psi_2(x;z,\theta)=\chi_0(x)v_+(x;\sigma,\theta)-\int_\Omega {}^tG^-(y-x;z)g(y;z,\theta)dy \tag{3.12}$$
$$\text{for any } \operatorname{Im} z>0 \text{ and } x\in\Omega.$$

In fact, from (3.4) and (3.5) it follows that $\int_\Omega {}^tG^-(y-x;z)g(y;z,\theta)dy\in C^\infty(\Omega)\cap H^2(\Omega)$ for any $\operatorname{Im} z>0$.

Note that (3.3) yields $\psi_2(x;z,\theta)$ is continuous in $x\in\Omega$, $\theta\in S^{n-1}$ and $\operatorname{Im} z\geqq 0$, $z\neq 0$. Thus, the equality (3.12) holds in the region $\operatorname{Im} z\geqq 0$, $z\neq 0$. From (3.4), (3.5) and (3.6), it follows that the right-hand side of (3.12) has the same property as that stated in Lemma 3.2. This completes the proof of Lemma 3.2.

§ 4. The difference between the outgoing and incoming fundamental solutions.

This section is devoted to prove the representation formula of $G^+(x;z)-G^-(x;z)$ which is used in § 3 to obtain Theorem 1.2.

PROPOSITION 4.1. *Under the assumptions* (A.1)~(A.3), *we have*

$$G^+(x;\sigma)-G^-(x;\sigma)=\frac{\sigma}{|\sigma|}\frac{\sqrt{-1}}{4\pi}\left(\frac{|\sigma|}{2\pi}\right)^{n-2}$$
$$\cdot\sum_{j=1}^{d}\int_{S^{n-1}}\lambda_j(\omega)^{-n/2}e^{\sqrt{-1}|\sigma|\lambda_j(\omega)^{-1/2}\omega\cdot x}P_j(\omega)d\omega,\qquad \textit{for any } \sigma\in\boldsymbol{R}\setminus\{0\}.$$

PROOF OF PROPOSITION 4.1. Since $G^\pm(x;\sigma)=G^\mp(x;|\sigma|)$ for any $\sigma<0$ it is sufficient to prove Proposition 4.1 for $\sigma>0$. We fix $\sigma>0$, and take $\delta>0$ such that $\sigma^2-2\delta>0$. We denote by $\varphi_l(\lambda)\in C^\infty(\boldsymbol{R})$ $(l=1, 2, 3)$ cutoff functions satisfying $\operatorname{supp}\varphi_1\subset(-\infty, \sigma^2-\delta)$, $\operatorname{supp}\varphi_2\subset(\sigma^2-2\delta, \sigma^2+2\delta)$, $\operatorname{supp}\varphi_3\subset(\sigma^2+\delta, \infty)$, and $\sum_{l=1}^{3}\varphi_l(\lambda)=1$ for any $\lambda\in\boldsymbol{R}$. By definition of $G^\pm(x;z)$ for $\pm\operatorname{Im}z<0$, we can divide $G^\pm(x;z)$ of the form

$$G^\pm(x;z)=\sum_{j=1}^{d}\sum_{l=1}^{3}I_{j,l}^\pm(x;z),$$

where each $I_{j,l}^\pm(x;z)$ is defined as

$$I_{j,l}^\pm(x;z)=(2\pi)^{-n}\int_{\boldsymbol{R}^n}e^{\sqrt{-1}\xi\cdot x}\frac{\varphi_l(\lambda_j(\xi))P_j(\xi)}{z^2-\lambda_j(\xi)}d\xi\,.$$

Note that the above integral means the Fourier transformation of temperate distributions.

Now, for $l=1$ and $l=3$ we prove

$$\lim_{\varepsilon\downarrow 0}I_{j,l}^+(x;\sigma-\varepsilon)=\lim_{\varepsilon\downarrow 0}I_{j,l}^-(x;\sigma+\varepsilon)\qquad \text{in } S'(\boldsymbol{R}^n)\,. \tag{4.1}$$

Since we have

$$\left|\frac{\varphi_1(\lambda_j(\xi))P_j(\xi)}{(\sigma\mp\sqrt{-1}\varepsilon)^2-\lambda_j(\xi)}\right|\leqq C_j\quad \text{for any } \xi\in\boldsymbol{R}^n \text{ and } 0<\varepsilon<\sqrt{\delta/2}\,,$$

the equality (4.1) is obvious for $l=1$. In the case $l=3$, we note that

$$\left|\frac{\varphi_3(\lambda_j(\xi))P_j(\xi)}{(\sigma\mp\sqrt{-1}\varepsilon)^2-\lambda_j(\xi)}\right|\leqq C_j\qquad \text{for any } \xi\in\boldsymbol{R}^n \text{ and } 0<\varepsilon. \tag{4.2}$$

In fact, $|\lambda_j(\xi)-(\sigma\mp\sqrt{-1}\varepsilon)^2|\geqq\delta$ holds for any $\xi\in\boldsymbol{R}^n$ with $\varphi_3(\lambda_j(\xi))\neq 0$ and $\varepsilon>0$. From (4.2) it follows that the equality (4.1) holds for $l=3$.

As for $I_{j,2}^\pm(x;z)$, using the polar coordinate, we have

$$I_{j,2}^{\pm}(x\,;z)=(2\pi)^{-n}\int_0^\infty\int_{S^{n-1}} e^{\sqrt{-1}r\omega\cdot x}\frac{\varphi_2(r^2\lambda_j(\omega))P_j(\omega)}{z^2-r^2\lambda_j(\omega)}r^{n-1}d\omega dr\,.$$

Now, we set $\mu=\lambda_j(\omega)^{1/2}r$, and a change of variables gives

$$I_{j,2}^{\pm}(x\,;z)=\int_0^\infty\frac{1}{\mu^2-z^2}\psi_j(\mu)d\mu,$$

where

$$\psi_j(\mu)=-(2\pi)^{-n}\mu^{n-1}\int_{S^{n-1}}\varphi_2(\mu^2)\lambda_j(\omega)^{-n/2}e^{\sqrt{-1}\mu\lambda_j(\omega)^{-1/2}\omega\cdot x}P_j(\omega)d\omega\,.$$

This implies

$$\begin{aligned}&I_{j,2}^{+}(x\,;\sigma-\sqrt{-1}\varepsilon)-I_{j,2}^{-}(x\,;\sigma+\sqrt{-1}\varepsilon)\\&=\int_0^\infty\frac{\varepsilon}{(\mu-\sigma)^2+\varepsilon^2}\cdot\frac{-4\sqrt{-1}\sigma\psi_j(\mu)}{(\mu+\sigma)^2+\varepsilon^2}d\mu\qquad\text{for any } \varepsilon>0\,.\end{aligned}$$

Since (4.1) holds for $l=1$ and $l=3$ from the fact that $\psi_j\in C_0^\infty((0,\infty))$ and

$$\left|\frac{-4\sqrt{-1}\sigma\psi_j(\mu)}{(\mu+\sigma)^2+\varepsilon^2}-\frac{-4\sqrt{-1}\sigma\psi_j(\mu)}{(\mu+\sigma)^2}\right|\leqq\varepsilon^2\sup_{\mu\geq 0}\left|\frac{4\sigma\psi_j(\mu)}{(\mu+\sigma)^4}\right|,$$

we obtain

$$\begin{aligned}G^+(x\,;\sigma)-G^-(x\,;\sigma)&=\lim_{\varepsilon\downarrow 0}\sum_{j=1}^d\int_{-\infty}^\infty\frac{\varepsilon}{(\mu-\sigma)^2+\varepsilon^2}\cdot\frac{-4\sqrt{-1}\sigma\psi_j(\mu)}{(\mu+\sigma)^2}d\mu\\&=-\frac{\pi\sqrt{-1}}{\sigma}\sum_{j=1}^d\psi_j(\sigma),\end{aligned}$$

where we use a well-known property of the Poisson kernel. This completes the proof of Proposition 4.1.

References

[1] P.G. Ciarlet, "Mathematical Elasticity, Volume 1: Three-dimensional Elasticity," North-Holland, New York, 1988.

[2] H. Iwashita, A remark on the analyticity of spectral functions for some exterior boundary value problems, Sci. Rep. of Niigata Univ. **24** (1988), 25-31.

[3] H. Iwashita and Y. Shibata, On the analyticity of spectral functions for some exterior boundary value problems, Glasnik Matematicki **23** (43) (1988), 291-313.

[4] P.D. Lax and R.S. Phillips, "Scattering Theory Revised Edition," Academic Press, New York, 1989.

[5] P.D. Lax and R.S. Phillips, Scattering theory for the acoustic equation in an even number of space dimensions, Indiana Univ. Math. J. **22** (1972), 101-134.

[6] A. Majda, A representation formula for the scattering operator and the inverse problem for arbitrary bodies, Comm. Pure Appl. Math. **30** (1977), 165-194.

[7] R.B. Melrose, Forward scattering by a convex obstacle, Comm. Pure Appl. Math. **33** (1980), 461-499.

[8] Y. Shibata and H. Soga, Scattering theory for the elastic wave equation, Publ. RIMS Kyoto Univ. **25** (1989), 861-887.

[9] H. Soga, Singularities of the scattering kernel for convex obstacles, J. Math. Kyoto Univ. **22** (1983), 729–765.
[10] H. Soga, Conditions against rapid decrease of oscillatory integrals and their applications to inverse scattering problems, Osaka J. Math. **23** (1986), 441–456.
[11] H. Soga, Representation of the scattering kernel for the elastic wave equation and singularities of the back-scattering, Osaka J. Math. **29** (1992).
[12] C.H. Wilcox, "Scattering Theory for the d'Alembert Equation in Exterior Domain," Springer, Berlin, Heidelberg, New York, 1975.
[13] K. Yajima, The limiting absorption principle for uniformly propagative systems, J. Fac. Science Univ. Tokyo Sect. IA **21** (1974), 119–131.

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