TSUKUBA J. MATH.
Vol. 18 No. 1 (1994), 223—246

ASYMPTOTIC BEHAVIOUR OF DENSITIES OF MULTI-DIMENSIONAL STABLE DISTRIBUTIONS

By

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Abstract. In one-dimension asymptotic behaviour of densities of stable distributions is well-known. However, in multi-dimensional cases it is very difficult to investigate asymptotic behaviour of densities of non-degenerate stable distributions in general. In the present paper we give the following two results: If the Lévy measure of the stable distribution has mass at a half-line, then the density decreases along the half-line with the same order as in one-dimensional case. If the Lévy measure is supported only on finitely many halflines, then we can determine asymptotic behaviour along each direction starting at 0.

Keywords: multi-dimensional stable distribution, Lévy-Ito decomposition of Lévy processes.

1. Introduction and results

Let $\mu(dx)$ be a *stable distribution* on $\boldsymbol{R}^d$ with exponent $0<\alpha<2$. Then its log-characteristic function $\Psi(z)$ is given as follows: For $z=|z|\xi$, $\xi\in\boldsymbol{S}^{d-1}=\{x\in\boldsymbol{R}^d : |x|=1\}$,

$$\Psi(z)=-|z|^{\alpha}\int_{S^{d-1}}|\langle\xi,\theta\rangle|^{\alpha}\left[1-i\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle\xi,\theta\rangle\right]\lambda(d\theta)+i\langle z,b\rangle \quad \text{if } \alpha\neq 1,$$

$$-|z|\int_{S^{d-1}}|\langle\xi,\theta\rangle|\left[1+i\frac{2}{\pi}\operatorname{sgn}\langle\xi,\theta\rangle\log|\langle z,\theta\rangle|\right]\lambda(d\theta)+i\langle z,b\rangle \quad \text{if } \alpha=1,$$

where $\lambda(d\theta)$ is a finite measure on $\boldsymbol{S}^{d-1}$ and $b\in\boldsymbol{R}^d$. If $b=0$ $(\alpha\neq 1)$ or $\int\theta\lambda(d\theta)=0$ $(\alpha=1)$, then μ satisfies the *scaling property*: $\mu^{t*}(dx)=t^{-d/\alpha}\mu(t^{-1/\alpha}dx)$, in this case μ is called *strictly stable*. Note that the Lévy measure $n(dx)$ of μ is given by

Received January 27, 1993

$$n(dx)=\int_{S^{d-1}}\lambda(d\theta)\int_0^\infty 1_{dx}(r\theta)r^{-1-\alpha}dr \qquad \text{on} \quad \boldsymbol{R}^d\setminus\{0\}.$$

We say that μ is non-degenerate if the support of μ spans $\boldsymbol{R}^d$, or equivalently the support of λ spans $\boldsymbol{R}^d$. Write this condition **Span Spt** $\lambda=\boldsymbol{R}^d$.

Throughout the present paper we always assume that μ is a non-degenerate stable distribution on $\boldsymbol{R}^d$. It is then well-known that $\mu(dx)$ is absolutely continuous and has a density $p(x)$ with respect to the Lebesgue measure dx on $\boldsymbol{R}^d$, which is expressed as

$$p(x)=\frac{1}{(2\pi)^d}\int_{R^d}\exp[-i\langle x, z\rangle+\Psi(z)]dz. \tag{1.1}$$

Furthermore $p(x)$ is a C^∞-function with derivatives of all orders vanishing at infinity (cf. [6], [7], [8] and [9]).

If we write $p(x)=p(x;b)$, then $p(x;b)=p(x-b;0)$. Henceforth, we assume $b=0$. Then note that μ is strictly stable except $\alpha=1$.

We are concerned with asymptotic behaviour of the density function $p(x)$ as $|x|\to+\infty$. In one-dimension it is well-known that $p(x)$ decreases like $|x|^{-1-\alpha}$ as $x\to+\infty$ if λ has mass at $\{+1\}$. In addition, if λ has no mass at $\{-1\}$, then $p(x)=0$ for $x<0$ when $0<\alpha<1$, and $p(x)>0$ for $x<0$ and decreases exponentially fast as $x\to-\infty$ when $1\leqq\alpha<2$ (see § 2). In multi-dimensional cases Pruitt and Taylor [6] give an upper estimate $p(x)\leqq K|x|^{-1-\alpha}$ for a strictly stable density. When λ is absolutely continuous and has a continuous density with respect to the uniform measure on $\boldsymbol{S}^{d-1}$, Dziubanski [2] investigates an asymptotic behaviour $p(r\sigma)\sim cr^{-d-\alpha}$ as $r\to+\infty$, where $\sigma\in\boldsymbol{S}^{d-1}$, $c=c(\sigma)\geqq0$ and $a\sim b$ means that $a/b\to1$. Furthermore Arkhipov [1] gives an asymptotic expansion of $p(r\sigma)$ under some additional regularity condition on the density of λ. On the other hand one can easily deduce that if λ is supported only on the orthonormal basis of $\boldsymbol{R}^d$, then $p(x)=\prod_{j=1}^d p_j(x_j)$, where $x=(x_1,\cdots,x_d)$ and p_j is a one-dimensional density corresponding to e_j. Therefore if $\sigma\in\boldsymbol{S}^{d-1}\cap\{x_j>0,\ j=1,\cdots,d\}$, then we have $p(r\sigma)\sim cr^{-d(1+\alpha)}$ as $r\uparrow+\infty$, where $c=c(\sigma)>0$.

From these results it would be expected that a general α-stable density $p(r\sigma)$ on $\boldsymbol{R}^d$ has the following asymptotic property: For each $\sigma\in\boldsymbol{S}^{d-1}$ there exist $c=c(\sigma)>0$ and $k=k(\alpha,\sigma)\geqq1+\alpha$ such that

$$p(r\sigma)\sim cr^{-k} \qquad \text{as} \quad r\to+\infty.$$

In this paper we first discuss a lower estimate for a general stable density $p(r\sigma)$ and we show that a lower estimate coincides with that of the upper estimate when λ has mass at σ. Furthermore we show that the above asymptotic relation is valid when λ is a discrete measure whose support consists of only finitely many points in $\boldsymbol{S}^{d-1}$.

Our first result is the following: Let μ be a non-degenerate stable distribution on $\boldsymbol{R}^d$ and **Con Spt** λ be the smallest convex hull in $\boldsymbol{S}^{d-1}$ containing all elements of **Spt** λ, and **Int** S denotes the interior of a set S in $\boldsymbol{S}^{d-1}$. Recall that $b=0$.

THEOREM 1. *Suppose that* λ *has mass at* $\sigma_0 \in \boldsymbol{S}^{d-1}$, *i.e.,* $\lambda(\{\sigma_0\})>0$. *If* $0<\alpha<1$ *and* $\sigma_0 \in$ **Int** (**Con Spt** λ), *or if* $1 \leqq \alpha < 2$, *then there exist positive constants* $C_1=C_1(\alpha, \sigma_0)$ *and* $r_0=r_0(\alpha, \sigma_0)$ *such that* $0<C_1 \leqq r^{1+\alpha} p(r\sigma_0)$ *for all* $r \geqq r_0$, *where* C_1 *is independent of* $r \geqq r_0$.

REMARK 1. By the result of [6], assuming thst $\int \theta \lambda(d\theta)=0$ when $\alpha=1$, it holds that $0<C_1 \leqq r^{1+\alpha} p(r\sigma_0) \leqq C_2 < \infty$ for all $r \geqq r_0$ where the constant C_2 is independent of $r \geqq 0$ and σ_0 (the upper estimate seems valid without the restriction $\int \theta \lambda(d\theta)=0$ when $\alpha=1$, but we have no proof for it).

Now we assume that λ has mass at only finitely many points in $\boldsymbol{S}^{d-1}$ (of course we also assume that $b=0$ and **Span Spt** $\lambda=\boldsymbol{R}^d$). To state the next result we define the following subsets of $\boldsymbol{R}^d$: For each $1 \leqq k \leqq d$

(i) $S^0(k)$ is a union of closed convex cones with the origin as vertex, the cones which are subtended by every linearly independent k-elements of **Spt** λ,

(ii) $S(k)=S^0(k) \cap \boldsymbol{S}^{d-1}$, $S(0)=\emptyset$ and $T(k)=S(k)-S(k-1)$.

Now our main result in the present paper is the following:

THEOREM 2. *Let* $d \leqq 3$. *Suppose that* **Spt** λ *is a finite set of* $\boldsymbol{S}^{d-1}$. *Let* $\sigma \in \boldsymbol{S}^{d-1}$.

a) *Let* $0<\alpha<1$.

If $\sigma \in T(k) \cap$ **Int** $S(d)$ *for some* $1 \leqq k \leqq d$, *then* $p(r\sigma) \sim c_1 r^{-k(1+\alpha)}$ *as* $r \to +\infty$.

If $\sigma \notin$ **Int** $S(d)$, *then* $p(r\sigma)=0$.

b) *Let* $1 \leqq \alpha < 2$

If $\sigma \in T(k)$ *for some* $1 \leqq k \leqq d$, *then* $p(r\sigma) \sim c_2 r^{-k(1+\alpha)}$ *as* $r \to +\infty$.

If $\sigma \notin S(d)$, then $p(r\sigma)$ decreases faster than any negative order of r, that is, $p(r\sigma)$ is a rapidly decreasing function of $r \geqq 0$.

Here constants $c_1, c_2>0$ are independent of r and can be determined explicitly by the expression of $\Psi(z)$.

For $d \geqq 4$ this theorem could be also proved in a similar way to our proof. However, it seems to be so tedious to describe the proof in general. So we treat the case of $d=2$ and 3. This theorem is proved by using the rotation of contour of integration as is similar to the one-dimensional case. Lemmas 2 and 4 are essential to the proof of this theorem (see § 3).

In the first cases of (a) and (b) in Theorem 2 we can give more concrete information. We say that λ *has mass at* $(m+1)$-*directions* $\sigma_j \in S^{d-1}$, $j=0, 1, 2, \cdots, m$, if λ has mass at σ_j and/or $-\sigma_j$ for each $j=0, 1, 2, \cdots, m$ (of course we assume $\sigma_j \neq \sigma_k$ if $j \neq k$). Now suppose that λ has mass at only $(m+1)$-directions σ_j, $j=0, 1, 2, \cdots, m$. When $\sigma \in T(k)$ for some $1 \leqq k \leqq d$, we define a vertex set $V_k(\sigma)$ of $\{\sigma_j, j=0, 1, \cdots, m\}$ and an index set $I_k(\sigma)$ as follows;

$\{\sigma_{j_1}, \cdots, \sigma_{j_k}\} \in V_k(\sigma)$ if $\{\sigma_{j_1}, \cdots, \sigma_{j_k}\}$ is linearly independent and σ is contained in the interior of **Span** $\{\sigma_{j_1}, \cdots, \sigma_{j_k}\}$,

$$j(k) \equiv \{j_1, \cdots, j_k\} \in I_k(\sigma) \quad \text{if} \quad \{\sigma_{j_1}, \cdots, \sigma_{j_k}\} \in V_k(\sigma).$$

Moreover for $j(k) \in I_k(\sigma)$ set $H_{j(k)} = \mathbf{Span}\, \{\sigma_{j_1}, \cdots, \sigma_{j_k}\}$ and fix an orthonormal basis $\{e_{j_1}, \cdots, e_{j_k}\}$ of $H_{j(k)}$. Now let

(i) $p_{j(k)}$ be a k-dimensional density on $H_{j(k)}$ with a log-characteristic function $\Psi|_{H_{j(k)}}$,

(ii) $p_{j(k)}^{\perp}$ be a $(d-k)$-dimensional density on $H_{j(k)}^{\perp}$ with a log-characteristic function $\Psi|_{H_{j(k)}^{\perp}}$ (if $k=d$, set $p_{j(k)}^{\perp}=1$).

In particular we write $p_j = p_{j(1)}$: a one-dimensional density on $H_{j(1)}$, when $j(1) = \{j\}$.

THEOREM 3. *Let* $d \leqq 3$. *Suppose that* $\sigma \in T(k) \cap \mathbf{Int}\, S(d)$ *in case of* $0<\alpha<1$ *and that* $\sigma \in T(k)$ *in case of* $1 \leqq \alpha < 2$ *for some* $1 \leqq k \leqq d$. *Then*

$$p(r\sigma) \sim \sum_{j(k) \in I_k(\sigma)} p_{j(k)}(r\sigma(j(k)))\, p_{j(k)}^{\perp}(0) \qquad \text{as} \quad r \to +\infty,$$

$$= \sum_{j(k) \in I_k(\sigma)} g(j(k)) \prod_{s=1}^{k} p_{j_s}(r h_{j_s})\, p_{j(k)}^{\perp}(0),$$

where $\sigma(j(k)) = \sum_{s=1}^{k} h_{j_s} \sigma_{j_s} = \sigma|_{H_{j(k)}}$ *and* $g(j(k)) = |\det Q_{j(k)}|$ *with a* $k \times k$-*matrix* $Q_{j(k)}$ *such that* $Q_{j(k)} \sigma_{j_s} = e_{j_s}$ *for every* $s=1, 2, \cdots, k$.

Note that the assumption of Theorem 3 implies that there is at least one $j(k) = \{j_1, \cdots, j_k\} \in I_k(\sigma)$ such that $p_{j(k)}^{\perp}(0) > 0$ and $p_{j_s}(r\sigma_{j_s}) \sim c(j_s) r^{-1-\alpha}$ as $r \to +\infty$ with a positive constant $c(j_s)$ for each $s=1, \cdots, k$.

REMARK 2. a) Note that $S(d) = \mathbf{Con\ Spt}\, \lambda$ and $T(1) = \mathbf{Spt}\, \lambda$.

b) In a similar way to the proof of Theorem 2 we can show that if μ is *rotation invariant*, that is, $\Psi(z) = -c|z|^{\alpha}(c>0)$, then

$$p(x) \approx \sum_{n=1}^{\infty} c_n |x|^{-d-n\alpha} \qquad \text{as} \quad |x| \to +\infty,$$

where

$$c_n=\pi^{-d/2-1}\alpha\frac{(-1)^{n-1}}{(n-1)!}2^{n\alpha-1}c^n\sin\frac{\pi n\alpha}{2}\Gamma\Big(\frac{n\alpha+d}{2}\Big)\Gamma\Big(\frac{n\alpha}{2}\Big).$$

This expansion means that

(1.2) $$p(x)=\sum_{n=1}^{N}c_n|x|^{-d-n\alpha}+O(|x|^{-d-(N+1)\alpha})\quad\text{as}\quad|x|\to+\infty\ \text{for all } N.$$

In particular, if $0<\alpha<1$, then $p(x)=\sum_{n=1}^{\infty}c_n|x|^{-d-n\alpha}$.

This result was shown by S. C. Port (A. 13 in [5]) by making use of a subordination technique.

2. Some Preliminary Results

For the proof of Theorem 2, we mention some results in the one-dimensional case which are well-known in [3].

a) $\alpha\neq 1$. In this case $p(x)$ is expressed with some constants $c_0>0$ and $|\beta_0|\leqq 1$ as follows:

(2.1) $$p(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\Big[-ixz-c_0|z|^\alpha\Big(1-i\beta_0\tan\frac{\pi\alpha}{2}\operatorname{sgn}z\Big)\Big]dz$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[-ixz-c|z|^\alpha e^{-i\theta}\operatorname{sgn}z]dz$$

(2.2) $$\approx\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}x^{-1-n\alpha}\Gamma(n\alpha+1)c^n\sin n\eta\quad\text{as}\quad 0<x\to+\infty,$$

where

(2.3) $$c=c_0\sec\theta,\quad \theta=\theta(\beta)=\pi L(\alpha)\beta/2\quad\text{and}\quad\eta=\eta(\theta)=\theta+\pi\alpha/2$$

$$=\pi(\alpha+L(\alpha)\beta)/2\quad\text{with}\quad L(\alpha)=\alpha(0<\alpha<1),\ =\alpha-2(1<\alpha<2)$$

$$\text{and}\quad\beta=2\pi^{-1}L(\alpha)^{-1}\arctan(\beta_0\tan\pi\alpha/2).$$

Note that $|\theta|<\pi/2$, $c>0$, $0\leqq\eta\leqq\pi$, $|\beta|\leqq 1$ and

(2.4) $\beta_0=\pm 1$ if and only if $\beta=\pm 1$ and then λ has mass at only $\{\pm 1\}$ respectively.

In particular if $\beta_0=-1$, then $\eta=0\,(0<\alpha<1)$, $=\pi\,(1<\alpha<2)$ and it holds that

(2.5) $$p(x)=0\qquad\text{for}\quad x\geqq 0\quad\text{if}\quad 0<\alpha<1,$$

$$\sim\frac{1}{\sqrt{2\pi(\alpha-1)}}(c_0\alpha)^{-1/(2\alpha-2)}x^{(2-\alpha)/(2\alpha-2)}\exp[-(\alpha-1)\alpha^{-\alpha/(\alpha-1)}c_0^{-1/(\alpha-1)}x^{\alpha/(\alpha-1)}]$$

$$\text{as}\quad 0<x\to+\infty\quad\text{if}\quad 1<\alpha<2.$$

b) $\alpha=1$.

$$(2.6)\qquad p(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\Big[-ixz-c\Big(|z|+i\frac{2\beta}{\pi}z\log|z|\Big)\Big]dz,\ c>0,\ |\beta|\leqq 1,$$

$$\approx\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{c^n}{n\,!}x^{-1-n}\int_0^{\infty}e^{-z}z^n\,\mathrm{Im}\Big[i(1+\beta)-\frac{2\beta}{\pi}\log\frac{z}{x}\Big]^n dz \text{ as } 0<x\to+\infty.$$

In this case (2.4) also holds. Moreover, if $\beta=-1$ (i.e., **Spt** $\lambda=\{-1\}$), then

$$(2.7)\qquad p(x)\sim\frac{1}{2\sqrt{ce}}\exp\Big[\frac{\pi}{4c}x-\frac{2}{\pi e}ce^{-\pi/(2c)}\Big] \quad\text{as}\quad 0<x\to+\infty.$$

c) The asymptotic behaviour of each derivative of $p(x)$ is obtained by differentiating the above formulae.

d) Moreover (cf. [9])

(2.8) $p(x)=0$ if and only if $0<\alpha<1$ and either $x\geqq 0$, $\beta=-1$ or $x\leqq 0$, $\beta=1$.
In particular if $\alpha\neq 1$ then

$$(2.9)\qquad p(0)=\pi^{-1}c^{-1/\alpha}\Gamma(\alpha^{-1}+1)\cos\Big(\frac{\pi}{2\alpha}L(\alpha)\beta\Big).$$

REMARK 3. In the case $0>x\to-\infty$, we obtain the same results by changing x, β_0 and β to $|x|$, $-\beta_0$ and $-\beta$ (thus, θ to $-\theta$) respectively. Because if we write $p(x;\alpha,\beta)=p(x)$ as $p(x)$ depends on (α,β), then $p(-x;\alpha,\beta)=p(x;\alpha,-\beta)$ holds.

3. Proof of Results

Before proceeding to the proof of Theorem 1, we present a general fact on multidimensional stable distributions, which is interesting in its own right. Let $p(x)$ be a density function of non-degenerate stable distribution μ of exponent $0<\alpha<2$. Recall that $b=0$ in $\Psi(z)$ and $S^0(d)$ is the smallest closed convex cone with vertex 0, which contains **Spt** λ. Note that **Int** $S^0(d)\neq\emptyset$ because of **Span Spt** $\lambda=\boldsymbol{R}^d$, where **Int** V denotes interior of a set V in $\boldsymbol{R}^d$.

LEMMA 1. *$p(x)=0$ if and only if $0<\alpha<1$ and $x\notin$* **Int** *$S^0(d)$.*

PROOF. Let (X_t, P) be a Lévy process on $\boldsymbol{R}^d$ corresponding to μ, then $P(X_t\in dx)=\mu^{t*}(dx)$. Of course for each $t>0$, $\mu^{t*}(dx)$ has a C^∞-density $p_t(x)$ with respect to the Lebesgue measure on $\boldsymbol{R}^d$, and $p_1=p$. We divide the proof into three cases: $\alpha=1$, $1<\alpha<2$ and $0<\alpha<1$, and use the Lévy-Ito decomposition of Lévy processes (see [4], [8]).

(1) $\alpha=1$. In this case $\Psi(z)$ is expressed by

$$\Psi(z)=-|z|\int_{S^{d-1}}|\langle\xi,\ \theta\rangle|\left[1+i\frac{2}{\pi}\mathrm{sgn}\langle\xi,\ \theta\rangle\log|\langle z,\ \theta\rangle|\right]\lambda(d\theta)$$

$$=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^\infty[e^{i\langle z,\,r\theta\rangle}-1-i\langle z,\ r\theta\rangle 1_{(0,1)}(r)]r^{-2}dr+\langle b_0,\ z\rangle,$$

where $\lambda_0=2\pi^{-1}\lambda$ and $b_0=-2\pi^{-1}c_0\int\theta\lambda(d\theta)$ with

$$c_0=\int_1^\infty r^{-2}\sin r dr+\int_0^1 r^{-2}(\sin r-r)dr.$$

Then by the Lévy-Ito decomposition we see that

$$X_t=\int_0^t\int_{0<|x|<1}x\tilde{N}(ds\ dx)+\int_0^t\int_{1\leqq|x|<\infty}xN(ds\ dx)+tb_0,$$

where $N(ds\ dx)=\#\{s\in ds: X_s-X_{s-}\in dx\}$ is a Poisson random measure corresponding to a Poisson point process with characteristic measure

$$n(dx)=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^\infty 1_{dx}(r\theta)r^{-2}dr \qquad \text{on} \quad \boldsymbol{R}^d\setminus\{0\}$$

and $\tilde{N}(ds\ dx)=N(ds\ dx)-ds\ n(dx)$. Now for each $0<\varepsilon<1$ we define

$$X_t^\varepsilon=\int_0^t\int_{\varepsilon\leqq|x|<1}x\tilde{N}(ds\ dx)+\int_0^t\int_{1\leqq|x|<\infty}xN(ds\ dx)+tb_0$$

$$=\int_0^t\int_{\varepsilon\leqq|x|<\infty}xN(ds\ dx)-tb^\varepsilon$$

with

$$b^\varepsilon=(-\log\varepsilon+c_0)\frac{2}{\pi}\int_{S^{d-1}}\theta\lambda(d\theta).$$

Then $X_t^\varepsilon+tb^\varepsilon$ is a compound Poisson Process with Lévy measure

$$n^\varepsilon(dx)=\int_{S^{d-1}}\lambda_0(d\theta)\int_\varepsilon^\infty 1_{dx}(r\theta)r^{-2}dr.$$

Thus, if we set $F_0^\varepsilon=\{0\}$, $F_1^\varepsilon=\mathbf{Spt}\ n^\varepsilon$, $F_{n+1}^\varepsilon=F_n^\varepsilon+F_1^\varepsilon\ (n\geqq 1)$, then it holds that $\mathbf{Spt}\ X_t^\varepsilon+tb^\varepsilon=\mathbf{CL}(\cup_{n=0}^\infty F_n^\varepsilon)$ for all $t>0$ and that $\uparrow\lim_{\varepsilon\downarrow 0}\mathbf{CL}(\cup_{n=0}^\infty F_n^\varepsilon)=S^0(d)$, where $\mathbf{Spt}\ X_t^\varepsilon$ denotes a support of a distribution of X_t^ε under P and $\mathbf{CL}\ V$ denotes closure of a set V in $\boldsymbol{R}^d$. From these results we can easily see that $\mathbf{Spt}\ \mu=\boldsymbol{R}^d$. In fact, if $\int\theta\lambda(d\theta)=0$ then $S^0(d)=\boldsymbol{R}^d$ because of $\mathbf{Span\ Spt}\ \lambda=\boldsymbol{R}^d$. Hence $\mathbf{Spt}\ X_t=\uparrow\lim_{\varepsilon\downarrow 0}\mathbf{Spt}\ X_t^\varepsilon=S^0(d)=\boldsymbol{R}^d$ for all $t>0$. Therefore $\mathbf{Spt}\ \mu=\mathbf{Spt}\ X_1=\boldsymbol{R}^d$. If $\int\theta\lambda(d\theta)\neq 0$ then $|b^\varepsilon|\to+\infty$ as $\varepsilon\to 0$ and $b^\varepsilon\in\mathbf{Int}\ S^0(d)$ for small ε because of $\int\theta\lambda(d\theta)\in\mathbf{Int}\ S^0(d)$. Thus for each $x\in\boldsymbol{R}^d$ we have $x+b^\varepsilon\in\mathbf{Int}\ S^0(d)$ if $0<\varepsilon<1$ is sufficiently small. Hence there is an $0<\varepsilon<1$ such that $x+b^\varepsilon\in\mathbf{CL}(\cup_{n=0}^\infty F_n^\varepsilon)$, that is, $x\in\mathbf{Spt}\ X_t^\varepsilon\subset\mathbf{Spt}\ X_t$ for all $t>0$. Therefore $\mathbf{Spt}\ \mu=\mathbf{Spt}\ X_1=\boldsymbol{R}^d$. Now if we assume that $p(x)=0$ for some $x\in\boldsymbol{R}^d$, then $L^*p(x)=(\partial/\partial_t)p_t(x)|_{t=1}=0$, where L^* is a Lévy generator of $-X_t$:

$$L^*p(x)=\int_{S^{d-1}}\lambda_0^*(d\theta)\int_0^\infty[p(x+r\theta)-p(x)-\langle r\theta, \nabla p(x)\rangle 1_{(0,1)}(r)]r^{-2}dr+\langle b_0, \nabla p(x)\rangle$$

with $\lambda_0^*(d\theta)=\lambda_0(-d\theta)$. Hence noting that $\nabla p(x)=0$, we have $p(x-r\theta)=0$ for a.e. $r\geqq 0$ and λ-a.e. $\theta\in\mathbf{Spt}\ \lambda$. By the continuity of p it holds that $p(x-r\theta)=0$ for all $r\geqq 0$, $\theta\in\mathbf{Spt}\ \lambda$. Furthermore we easily deduce that

$$p(x-r\theta)=0 \qquad \text{for all} \quad r\geqq 0,\ \theta\in\mathbf{Con\ Spt}\ \lambda.$$

This implies that $\mu(x-\mathbf{Int}\ S^0(d))=0$, but which is contrary to $\mathbf{Spt}\ \mu=\boldsymbol{R}^d$ and $\mathbf{Int}\ S^0(d)\neq\emptyset$. Therefore we get $p(x)>0$ for all $x\in\boldsymbol{R}^d$.

(2) $1<\alpha<2$. In this case $p>0$ on $\boldsymbol{R}^d$ has been already proved in [9] by using the scaling property of $p_t(x)$. We here give an alternative proof by the same way as in (1). In this case the previous arguments work replacing $\Psi(z)$, $n^\varepsilon(dx)$ and L^* by the following:

$$\Psi(z)=-|z|^\alpha\int_{S^{d-1}}|\langle\xi, \theta\rangle|^\alpha\left[1-i\tan\frac{\pi\alpha}{2}\mathrm{sgn}\langle\xi, \theta\rangle\right]\lambda(d\theta)$$
$$=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^\infty[e^{i\langle z, r\theta\rangle}-1-i\langle z, r\theta\rangle]r^{-1-\alpha}dr,$$

where $\lambda_0=c(\alpha)\lambda$ with $c(\alpha)=2\Gamma(\alpha+1)\sin(\pi\alpha/2)/\pi$.

$$X_t=\int_0^t\int_{0<|x|<\infty}x\tilde{N}(ds\ dx)$$

with Lévy measure

$$n(dx)=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^\infty 1_{dx}(r\theta)r^{-1-\alpha}dr \qquad \text{on} \quad \boldsymbol{R}^d\setminus\{0\}.$$

For each $0<\varepsilon<1$,

$$X_t^\varepsilon=\int_0^t\int_{\varepsilon\leq|x|<\infty}x\tilde{N}(ds\ dx)=\int_0^t\int_{\varepsilon\leq|x|<\infty}xN(ds\ dx)-tb^\varepsilon$$

where

$$b^\varepsilon=\varepsilon^{-\alpha}(\alpha-1)^{-1}\int_{S^{d-1}}\theta\lambda_0(d\theta),$$

and its Lévy measure is given by

$$n^\varepsilon(dx)=\int_{S^{d-1}}\lambda_0(d\theta)\int_\varepsilon^\infty 1_{dx}(r\theta)r^{-1-\alpha}dr.$$

The Lévy generator L^* of $-X_t$:

$$L^*p(x)=\int_{S^{d-1}}\lambda_0^*(d\theta)\int_0^\infty[p(x+r\theta)-p(x)-\langle r\theta, \nabla p(x)\rangle]r^{-1-\alpha}dr.$$

(3) $0<\alpha<1$. We show that $p(x)=0$ if and only if $x\notin \mathbf{Int}\, S^0(d)$. In this case $\Psi(z)$ and X_t are expressed by the following:

$$\Psi(z)=-|z|^{\alpha}\int_{S^{d-1}}|\langle\xi,\theta\rangle|^{\alpha}\Big[1-i\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle\xi,\theta\rangle\Big]\lambda(d\theta)$$
$$=\int_{S^{d-1}}\lambda_0(d\theta)\int_0^{\infty}[e^{i\langle z,r\theta\rangle}-1]r^{-1-\alpha}dr,$$

where λ_0 is the same as in (2), and

$$X_t=\int_0^t\int_{0<|x|<\infty}xN(ds\,dx).$$

Moreover for each $0<\varepsilon<1$ we define

$$X_t^{\varepsilon}=\int_0^t\int_{\varepsilon\le|x|<\infty}xN(ds\,dx),$$

then $\mathbf{Spt}\, X_t^{\varepsilon}=\mathbf{CL}(\cup_{n=0}^{\infty}F_n^{\varepsilon})$. Hence by limiting $\varepsilon\to 0$ we have $\mathbf{Spt}\, X_t=S^0(d)$, that is, $p(x)=0$ if $x\notin\mathbf{Int}\, S^0(d)$. Furthermore by a similar argument to (1) we can see that $p(x)>0$ if $x\in\mathbf{Int}\, S^0(d)$. In fact, if $p(x)=0$ for some $x\in\mathbf{Int}\, S^0(d)$, then $L^*p(x)=(\partial/\partial t)p_t(x)|_{t=1}=0$, where L^* is given by

$$L^*p(x)=\int_{S^{d-1}}\lambda_0^*(d\theta)\int_0^{\infty}[p(x+r\theta)-p(x)]r^{-1-\alpha}dr$$

with $\lambda_0^*(d\theta)=\lambda_0(-d\theta)$. Hence we have $\mu(x-\mathbf{Int}\, S^0(d))=0$, but this is contrary to $\mathbf{Spt}\,\mu=S^0(d)$. Therefore we get $p>0$ on $\mathbf{Int}\, S^0(d)$. **Q.E.D.**

We also mention the following result: To emphasize the dependence on λ we write $\Psi(z)=\Psi_{\lambda}(z)$ and $p(x)=p_{\lambda}(x)$. Let Q be a linear transformation on $\boldsymbol{R}^d$ and set $\lambda_Q(d\theta)=\lambda(Q^{-1}d\theta)$ on $Q(\boldsymbol{S}^{d-1})$. Then by the definition of $\Psi(z)$ we have $\Psi_{\lambda_Q}(z)=\Psi_{\lambda}({}^tQz)$, where ${}^tQ$ denotes a transposed matrix of Q. Moreover by using (1.1) we can easily deduce that if Q is invertible, then p_{λ_Q} is well-defined and

$$p_{\lambda}(x)=|\det Q|\,p_{\lambda_Q}(Qx) \tag{3.1}$$

holds.

PROOF OF THEOREM 1. First assume that $\lambda(\{\sigma_0\})>0$ for some $\sigma_0\in\boldsymbol{S}^{d-1}$, and also that $\sigma_0\in\mathbf{Int}\,(\mathbf{Con\ Spt}\,\lambda)$ if $0<\alpha<1$. For simplicity we write $\sigma_0=\sigma$. In (3.1) let Q be an orthogonal transformation, then $p_{\lambda}(x)=p_{\lambda_Q}(Qx)$. From this we may assume that $\sigma=(1,0,\cdots,0)$. Moreover it is easily deduced that $p(r\sigma)$ is expressed by

$$p(r\sigma)=cp_1(r)p_{d-1}(0,\cdots,0) \tag{3.2}$$

or

$$(3.3)\qquad p(r\sigma)=\int_{-\infty}^{\infty} p_1(r-y)p_d(y, 0, \cdots, 0)dy,$$

where p_j is a j-dimensional density ($j=1, d-1, d$) and $c>0$. In fact, we define λ^σ by $\lambda=\delta_{\{\sigma\}}+\lambda^\sigma$ and set $H=\mathbf{Span\ Spt}\ \lambda^\sigma$. Then $\dim H=d-1$ or d because of Span $\mathbf{Spt}\ \lambda=\boldsymbol{R}^d$. If $\dim H=d-1$, then by taking Q in (3.1) such that $Q\sigma=\sigma$ and $Q(H)=\{x_1=0\}$ we see that $p_{\lambda_Q}(r\sigma)=p_1(r)p_{d-1}(0, \cdots, 0)$, where p_1 (resp. p_{d-1}) is a one-dimensional density function (resp. $(d-1)$-dimensional density function) corresponing to $\delta_{\{\sigma\}}$ (resp. λ_Q^σ). Hence we get $p(r\sigma)=|\det Q|p_1(r)p_{d-1}(0, \cdots, 0)$. If $\dim H=d$, then we can define a d-dimensional density function p_d by λ^σ. Thus we have

$$\begin{aligned}(2\pi)^d p(x)&=\int_{\boldsymbol{R}^d}\exp[-i\langle x, z\rangle+\Psi_{\delta_{\{\sigma\}}}(z)+\Psi_{\lambda^\sigma}(z)]dz\\ &=\int_{-\infty}^{\infty}dy\ p_1(y)\int_{\boldsymbol{R}^d}\exp[-i\{(x_1-y)z_1+x_2z_2+\cdots+x_dz_d\}+\Psi_{\lambda^\sigma}(z)]dz\\ &=(2\pi)^d\int_{-\infty}^{\infty}p_1(y)p_d(x_1-y, x_2, \cdots, x_d)dy.\end{aligned}$$

Therefore (3.3) holds. Here in the second equation we use

$$\exp[\Psi_{\delta_{\{\sigma\}}}(z)]=\int_{-\infty}^{\infty}p_1(y)\exp[i\ yz_1]dy.$$

Now noting that (3.2) does not occur when $0<\alpha<1$ and $\mathbf{Con\ Spt}\ \lambda\neq \boldsymbol{S}^{d-1}$, we see that $p_{d-1}(0, \cdots, 0)>0$ and $p_d(y, 0, \cdots, 0)>0$ if at least $y>0$ by Lemma 1. Hence in the case of (3.2) our claim holds. In the case of (3.3) we have $p(r\sigma)\geqq cp_1(2r)$ for sufficiently large r with a positive constants c. In fact there are a compact set K in $(0, \infty)$ and a positive constant r_0 such that $\varepsilon\equiv\inf_{y\in K}p_d(y, 0, \cdots, 0)>0$ and $\inf_{y\in K}p_1(r-y)\geqq p_1(2r)$ for all $r\geqq r_0$. Thus $p(r\sigma)\geqq \varepsilon|K|\inf_{y\in K}p_1(r-y)\geqq\varepsilon|K|p_1(2r)$ for $r\geqq r_0$. Since $p_1(2r)\sim c'r^{-1-\alpha}$ as $r\to+\infty$, there is a constant $C_1>0$ such that $p(r\sigma)\geqq C_1r^{-1-\alpha}$ for all $r\geqq r_0$. **Q.E.D.**

PROOF OF THEOREM 2 AND THEOREM 3. Let $d=2, 3$ and let μ be a non-degenerate stable distribution on $\boldsymbol{R}^d$ with exponent $0<\alpha<2$. Recall that we are assuming that $\mathbf{Spt}\ \lambda$ is a finite set of $\boldsymbol{S}^{d-1}$, and we say that λ has mass at $(m+1)$-directions $\sigma_j\in\boldsymbol{S}^{d-1}$, $j=0, 1, 2, \cdots, m$, if λ has mass at σ_j and/or $-\sigma_j$ for each $j=0, 1, 2, \cdots, m$ (of course $\sigma_j\neq\pm\sigma_k$ if $j\neq k$).

Now we begin with the case $d=2$. The proof is divided into three cases.

CASE 1. λ has mass at only two directions σ_0, σ_1 ($\sigma_0\neq\pm\sigma_1$). By (3.1) we may assume that $\sigma_0=(1, 0)$, $\sigma_1=(a, b)$ and with $a\neq 1$, $b>0$ such that $a^2+b^2=1$. Then

$$p(r\sigma)=b^{-1}p_0(rh_0)p_1(rh_1)$$

where, h_j are defined by the decomposition $\sigma=h_0\sigma_0+h_1\sigma_1$, and $p_j(y)$, $y\in\boldsymbol{R}$ are defined by (2.1) with some constants $(c_{j,0}, \beta_{j,0})$ instead of (c_0, β_0), $j=0, 1$. Here one can easily check that $b^{-1}=g(\{0, 1\})$; which is defined in Theorem 3, and that $p_0^{\frac{1}{0}}(0)=b^{-1}p_1(0)$ and $p_1^{\frac{1}{1}}(0)=b^{-1}p_0(0)$. Hence our claim immediately follows by using the facts (2.2), (2.4), (2.8) and (2.9). In particular if $1<\alpha<2$ and $\sigma\notin$ **Con Spt** λ, then by (2.5) and (2.7),

$$p(r\sigma)\sim K_1r^{K_2}\exp[-K_3r^{K_4}] \quad \text{as} \quad r\to+\infty \text{ if } 1<\alpha<2, \tag{3.4}$$

$$p(r\sigma)\sim \tilde{K}_1\exp[\tilde{K}_2r-\tilde{K}_3e^{\tilde{K}_4r}] \quad \text{as} \quad r\to+\infty \text{ if } \alpha=1, \tag{3.5}$$

where K_j, $\tilde{K}_j$ are positive constants which are independent of r. For instance, when **Spt** $\lambda=\{\pm\sigma_0, \sigma_1\}$ with $\sigma_0=(1, 0)$ and $\sigma_1=(0, 1)$, let $\sigma=(s, t)$,

if $\sigma\in T(2)$, i.e., $t>0$ and $\sigma\neq\sigma_1$, then $p(r\sigma)\sim cr^{-2(1+\alpha)}$ as $r\to+\infty$;

if $\sigma\in T(1)\cap$**Int** $S(2)$, i.e., $\sigma=\sigma_1$, then $p(r\sigma)\sim cr^{-(1+\alpha)}$ as $r\to+\infty$;

if $\sigma\in T(1)\cap\partial S(2)$, i.e., $\sigma=\pm\sigma_0$, then $p(r\sigma)=0$ $(0<\alpha<1)$, $p(r\sigma)\sim cr^{-(1+\alpha)}(1\leqq\alpha<2)$ as $r\to+\infty$;

if $\sigma\notin S(2)$, i.e., $t<0$, then $p(r\sigma)=0$ for all $r\geqq 0$ $(0<\alpha<1)$ (3.4) $(1<\alpha<2)$ and (3.5) $(\alpha=1)$ hold.

CASE 2. $\alpha\neq 1$ and λ has mass at only $(m+1)$-directions σ_j, $j=0, 1, 2, \cdots, m(m\geqq 2)$. Then $\Psi(z)$, $z=(z_1, z_2)$, is expressed by

$$\begin{aligned}\Psi(z)&=-\sum_{j=0}^{m}c_{j,0}|\langle\sigma_j, z\rangle|^{\alpha}\left[1-i\beta_{j,0}\tan\frac{\pi\alpha}{2}\operatorname{sgn}\langle\sigma_j, z\rangle\right]\\&=-\sum_{j=0}^{m}c_j|\langle\sigma_j, z\rangle|^{\alpha}\exp[-i\theta_j\operatorname{sgn}\langle\sigma_j, z\rangle],\end{aligned}$$

where $c_{j,0}>0$, $|\beta_{j,0}|\leqq 1$ and c_j, θ_j are defined by (2.3).

In order to prove Theorem 2 and Theorem 3 in Case 2 we first consider the special case, however we show that the general case is reduced to this special one (see Second step).

First step. Set $\sigma=\sigma_0=(1, 0)$ and let $\sigma_j=(s_j, t_j)$, $j=0, 1, 2, \cdots, m$, where $s_j=\cos\varphi_j$ and $t_j=\sin\varphi_j$ with $0=\varphi_0<\varphi_1<\cdots<\varphi_m=\pi/2$. Note that if λ has no mass at $\sigma=(1, 0)$, then λ has mass at $-\sigma=(-1, 0)$ by our definition of directions, and $\beta_{0,0}=-1$.

We define the following α-stable densities:

(i) For $y, z\in\boldsymbol{R}$, $p_0(y)$ (resp. $p_0^{\frac{1}{0}}(y)$) is a one-dimensional density with a log-characteristic function $\Psi_0(z)=-c_0|z|^\alpha \exp[-i\theta_0 \operatorname{sgn} z]$ (resp. $\Psi_0^\perp(z)=\Psi(0, z)$)

(ii) For $x, z\in \boldsymbol{R}^2$ and $j\neq k$, $p_{j,k}(x)$ is a two-dimensional density with a log-characteristic function $\Psi_{j,k}(z)=-\sum_{r=j,k} c_r |\langle\sigma_r, z\rangle|^\alpha \exp[-i\theta \operatorname{sgn}\langle\sigma_r, z\rangle]$.

PROPOSITION. *Let* $r\geqq 0$.

a) *If* $\sigma\in$ **Spt** λ *and* $p_0^\perp(0)>0$, *then*

$$p(r\sigma)\sim p_0(r)p_0^\perp(0) \qquad as \quad r\to+\infty; \tag{3.6}$$

b) *If* $\sigma\notin$ **Spt** λ *and* $\sigma\in$ **Con Spt** λ, *then*

$$p(r\sigma)\sim \sum_{1\leq j<k\leq m} p_{j,k}(r\sigma) \qquad as \quad r\to+\infty; \tag{3.7}$$

c) *If* $1\leqq\alpha<2$ *and* $\sigma\notin$ **Con Spt** λ, *then* $p(r\sigma)$ *is rapidly decreasing as* $r\to+\infty$;

d) *If* $0<\alpha<1$ *and* $\sigma\notin$ **Int** (**Con Spt** λ), *then* $p(r\sigma)=0$.

Note that (b), (c) and (d) also hold in the case that λ has no mass at $\{\pm\sigma\}$ (in this case $c_{0,0}=c_0=0$ in $\Psi(z)$) and that, by (2.9)

$$\operatorname{Re}\int_0^\infty \exp\Psi(0, z_2)dz_2=\pi p_0^\perp(0)=\bar{c}^{-1/\alpha}\Gamma(\alpha^{-1}+1)\cos\left(\frac{\pi}{2\alpha}L(\alpha)\bar{\beta}\right),$$

where $(\bar{c}, \bar{\beta})$ is (c, β) in (2.3) which is given by using $(\bar{c}_0, \bar{\beta}_0)=(\sum_{j=1}^m c_{j,0}t_j^\alpha, \sum_{j=1}^m c_{j,0}\beta_{j,0}t_j^\alpha/\bar{c}_0)$ instead of (c_0, β_0) in (2.3). Hence by (2.4) and (2.8) $p_0^\perp(0)=0$ if and only if $0<\alpha<1$ and $\beta_{1,0}=\beta_{2,0}=\cdots=\beta_{m,0}=\pm 1$ (i.e., $\sigma\notin$ **Int** (**Con Spt** λ)).

From this proposition we can easily deduce Theorem 2 and Theorem 3 in Case 2 by using the one-dimensional results.

To prove Proposition we need some lemmas. The following lemma is obtained by elementary analysis.

LEMMA 2. *Set* $a_j=t_j/s_j=\tan\phi_j (a_0=0,\ a_m=\infty)$. *Then*

$$\begin{aligned} p(r\sigma)&=(2\pi)^{-2}\int_{\boldsymbol{R}^2}\exp[-irz_1+\Psi(z)]dz \\ &\approx r^{-1}\pi^{-2}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n!} r^{-n\alpha}c_0^n \sin n\eta_0 \int_0^\infty du\, e^{-u}u^{n\alpha}\operatorname{Re}\int_{u/(ra_1)}^\infty \exp\Psi\left(-i\frac{u}{r}, \nu\right)d\nu \\ &\quad + r^{-2}\pi^{-2}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n!} r^{-n\alpha}c_0^n \sin n\eta_0 \int_0^\infty du\, e^{-u}u^{n\alpha+1} \\ &\qquad \int_0^{\pi/2} d\phi e^{i\phi}\sum_{n=0}^\infty \frac{(-1)^n}{n!} r^{-n\alpha}\operatorname{Im}\left[\sum_{j=1}^m c_j u^\alpha (s_j+ie^{i\phi}t_j/a_1)^\alpha e^{-i\eta_j}\right]^n \\ &\quad + r^{-2}\pi^{-2}\sum_{j=1}^{m-1}\int_0^\infty d\nu \int_{a_j\nu}^{a_{j+1}\nu} du e^{-u} \end{aligned} \tag{3.8}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}r^{-n\alpha}\,\mathrm{Im}[c_0u^{\alpha}e^{-i\eta_0}+c_1s_1^{\alpha}(u-a_1\nu)^{\alpha}e^{-i\eta_1}+\cdots$$
$$+c_js_j^{\alpha}(u-a_j\nu)^{\alpha}e^{-i\eta_j}]^n$$
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}r^{-n\alpha}\,\mathrm{Im}[c_{j+1}s_{j+1}^{\alpha}(a_{j+1}\nu-u)^{\alpha}e^{-i\hat{\eta}_{j+1}}+\cdots+c_m\nu^{\alpha}e^{-i\hat{\eta}_m}]^n$$

as $r\to+\infty$, where $\eta_j=\eta(\theta_j)$, $\hat{\eta}_j=\eta(-\theta_j)$ are defined by (2.3). This expansion holds in equal provided $0<\alpha<1$, and if $1<\alpha<2$, then it holds in the sense of (1.2).

PROOF. For simplicity we only prove the case that $m=3$, $\sigma=\sigma_0=(1, 0)$, $\sigma_1=(s_1, t_1)$, $\sigma_2=(s_2, t_2)$ and $\sigma_3=(0, 1)$. That is, for $\tilde{c}_j=c_js_j^{\alpha}(j=1, 2)$,

$$\Psi(z)=-c_0|z_1|^{\alpha}\exp[-i\theta_0\,\mathrm{sgn}\,z_1]-\tilde{c}_1|z_1+a_1z_2|^{\alpha}\exp[-i\theta_1\,\mathrm{sgn}(z_1+a_1z_2)]$$
$$-\tilde{c}_2|z_1+a_2z_2|^{\alpha}\exp[-i\theta_2\,\mathrm{sgn}(z_1+a_2z_2)]-c_3|z_2|^{\alpha}\exp[-i\theta_3\,\mathrm{sgn}\,z_2].$$

Then

$$p(r\sigma)=\frac{\mathrm{Re}}{2\pi^2}\int_0^{\infty}dz_2\int_0^{\infty}dz_1\exp[-irz_1-c_0z_1^{\alpha}e^{-i\theta_0}]$$
$$(\exp[-\tilde{c}_1(z_1+a_1z_2)^{\alpha}e^{-i\theta_1}-\tilde{c}_2(z_1+a_2z_2)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{-i\theta_3}]$$
$$+\exp[-\tilde{c}_1|z_1-a_1z_2|^{\alpha}e^{-i\theta_1\,\mathrm{sgn}(z_1-a_1z_2)}$$
$$-\tilde{c}_2|z_1-a_2z_2|^{\alpha}e^{-i\theta_2\,\mathrm{sgn}(z_1-a_2z_2)}-c_3z_2^{\alpha}e^{i\theta_3}]).$$

By changing variable rz_1 to u we have

$$2\pi^2rp(r\sigma)=\mathrm{Re}\int_0^{\infty}dz_2\left\{\int_0^{ra_1z_2}du\,\exp[-iu-c_0r^{-\alpha}u^{\alpha}e^{-i\theta_0}]\right.$$
$$\left(\exp\left[-\tilde{c}_1\left(a_1z_2+\frac{u}{r}\right)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\left(a_2z_2+\frac{u}{r}\right)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{-i\theta_3}\right]\right.$$
$$\left.+\exp\left[-\tilde{c}_1\left(a_1z_2-\frac{u}{r}\right)^{\alpha}e^{i\theta_1}-\tilde{c}_2\left(a_2z_2-\frac{u}{r}\right)^{\alpha}e^{i\theta_2}-c_3z_2^{\alpha}e^{i\theta_3}\right]\right)$$
$$+\int_{ra_1z_2}^{ra_2z_2}du\,\exp[-iu-c_0r^{-\alpha}u^{\alpha}e^{-i\theta_0}]$$
$$\left(\exp\left[-\tilde{c}_1\left(\frac{u}{r}+a_1z_2\right)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\left(a_2z_2+\frac{u}{r}\right)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{-i\theta_3}\right]\right.$$
$$\left.+\exp\left[-\tilde{c}_1\left(\frac{u}{r}-a_1z_2\right)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\left(a_2z_2-\frac{u}{r}\right)^{\alpha}e^{i\theta_2}-c_3z_2^{\alpha}e^{i\theta_3}\right]\right)$$
$$+\int_{ra_2z_2}^{\infty}du\,\exp[-iu-c_0r^{-\alpha}u^{\alpha}e^{-i\theta_0}]$$

$$\Big(\exp\Big[-\tilde{c}_1\Big(\frac{u}{r}+a_1z_2\Big)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\Big(\frac{u}{r}+a_2z_2\Big)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{-i\theta}\Big]$$

$$+\exp\Big[-\tilde{c}_1\Big(\frac{u}{r}a_1z_2\Big)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\Big(\frac{u}{r}-a_2z_2\Big)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{i\theta_3}\Big]\Big)\Big\}.$$

First assume $0<\alpha<1$. Rotate the contour of integration with respect to du through an angle $-\pi/2$. Then

$$(3.9)\quad 2\pi^2 rp(r\sigma)=\mathrm{Re}\Big[-i\int_0^{\infty}dz_2\Big\{\int_0^{ra_1z_2}du\ e^{-u}\exp[-c_0r^{-\alpha}u^{\alpha}e^{-i\eta_0}]$$

$$\Big(\exp\Big[-\tilde{c}_1\Big(a_1z_2-i\frac{u}{r}\Big)^{\alpha}e^{-i\theta_1}-\tilde{c}_2\Big(a_2z_2-i\frac{u}{r}\Big)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{-i\theta_3}\Big]$$

$$+\exp\Big[-\tilde{c}_1\Big(a_1z_2+i\frac{u}{r}\Big)^{\alpha}e^{i\theta_1}-\tilde{c}_2\Big(a_2z_2+i\frac{u}{r}\Big)^{\alpha}e^{i\theta_2}-c_3z_2^{\alpha}e^{i\theta_3}\Big]\Big)$$

$$+\int_{r_1az_2}^{ra_2z_2}du\ \exp[-u-c_0r^{-\alpha}u^{\alpha}e^{-i\eta_0}]$$

$$\Big(\exp\Big[-\tilde{c}_1\Big(\frac{u}{r}+ia_1z_2\Big)^{\alpha}e^{-i\eta_1}-\tilde{c}_2\Big(a_2z_2-i\frac{u}{r}\Big)^{\alpha}e^{-i\theta_2}-c_3z_2^{\alpha}e^{-i\theta_3}\Big]$$

$$+\exp\Big[-\tilde{c}_1\Big(\frac{u}{r}-ia_1z_2\Big)^{\alpha}e^{-i\eta_1}-\tilde{c}_2\Big(a_2z_2+i\frac{u}{r}\Big)^{\alpha}e^{i\theta_2}-c_3z_2^{\alpha}e^{i\theta_3}\Big]\Big)$$

$$+\int_{ra_2z_2}^{\infty}du\ \exp[-u-c_0r^{-\alpha}u^{\alpha}e^{-i\eta_0}]$$

$$\Big(\exp\Big[-\tilde{c}_1\Big(\frac{u}{r}+ia_1z_2\Big)^{\alpha}e^{-i\eta_1}-\tilde{c}_2\Big(\frac{u}{r}+ia_2z_2\Big)^{\alpha}e^{-i\eta_2}-c_3z_2^{\alpha}e^{-i\theta_3}\Big]$$

$$+\exp\Big[-\tilde{c}_1\Big(\frac{u}{r}-ia_1z_2\Big)^{\alpha}e^{-i\eta_1}-\tilde{c}_2\Big(\frac{u}{r}-ia_2z_2\Big)^{\alpha}e^{-i\eta_2}-c_3z_2^{\alpha}e^{i\theta_3}\Big]\Big)\Big\}$$

$$+i\int_0^{\infty}dz_2ra_1z_2\int_0^{\pi/2}d\varphi e^{-i\varphi}$$

$$\exp[-ra_1z_2e^{i(\pi/2-\varphi)}-c_0a_1^{\alpha}z_2^{\alpha}e^{-i(\theta_0+\alpha\varphi)}-\tilde{c}_2z_2^{\alpha}(a_2-a_1e^{-i\varphi})^{\alpha}e^{i\theta_2}-c_3z_2^{\alpha}e^{i\theta_3}]$$

$$\{\exp[-\tilde{c}_1a_1^{\alpha}z_2^{\alpha}(1-e^{-i\varphi})^{\alpha}e^{i\theta_1}]-\exp[-\tilde{c}_1a_1^{\alpha}z_2^{\alpha}(e^{-i\varphi}-1)^{\alpha}e^{-i\theta_1}]\}$$

$$+i\int_0^{\infty}dz_2ra_2z_2\int_0^{\pi/2}d\varphi e^{-i\varphi}$$

$$\exp[-ra_2z_2e^{i(\pi/2-\varphi)}-c_0a_2^{\alpha}z_2^{\alpha}e^{-i(\theta_0+\alpha\varphi)}-\tilde{c}_1z_2^{\alpha}(a_2e^{-i\varphi}-a_1)^{\alpha}e^{-i\theta_1}-c_3z_2^{\alpha}e^{i\theta_3}]$$

$$\{\exp[-\tilde{c}_2a_2^{\alpha}z_2^{\alpha}(1-e^{-i\varphi})^{\alpha}e^{i\theta_2}]-\exp[-\tilde{c}_2a_2^{\alpha}z_2^{\alpha}(e^{-i\varphi}-1)^{\alpha}e^{-i\theta_2}]\}\Big].$$

In the last two terms change ra_1z_2 and ra_2z_2 to u, $\pi/2-\varphi$ to ϕ respectively and rotate the contour of the integration with respect to du through an angle $-\phi$. Moreover, in the second and third terms exchange the order of integration with respect to dz_2 and du and change rz_2 to ν. Then $2\pi^2 rp(r\sigma)$ is equal to

$$\int_0^\infty du\int_{u/(ra_1)}^\infty dz e^{-u}\,\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^\alpha e^{-i\eta_0}]\}$$

$$\mathrm{Re}\Big\{\exp\Big[-\tilde{c}_1\Big(a_1z_2-i\frac{u}{r}\Big)^\alpha e^{-i\theta_1}-\tilde{c}_2\Big(a_2z_2-i\frac{u}{r}\Big)^\alpha e^{-i\theta_2}-c_3z_2^\alpha e^{-i\theta_3}\Big]\Big\}$$

$$+\frac{\mathrm{Im}}{r}\int_0^\infty du\ e^{-u}\exp[-c_0r^{-\alpha}u^\alpha e^{-i\eta_0}]$$

$$\Big\{\int_{u/a_2}^{u/a_1}d\nu(\exp[-\tilde{c}_1r^{-\alpha}(u+ia_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(a_2\nu-iu)^\alpha e^{-i\theta_2}-c_3r^{-\alpha}\nu^\alpha e^{-i\theta_3}]$$

$$+\exp[-\tilde{c}_1r^{-\alpha}(u-ia_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(a_2\nu+iu)^\alpha e^{i\theta_2}-c_3r^{-\alpha}\nu^\alpha e^{i\theta_3}])$$

$$+\int_0^{u/a_2}d\nu(\exp[-\tilde{c}_1r^{-\alpha}(u+ia_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(u+ia_2\nu)^\alpha e^{-i\eta_2}-c_3r^{-\alpha}\nu^\alpha e^{-i\theta_3}]$$

$$+\exp[-\tilde{c}_1r^{-\alpha}(u-ia_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(u-ia_2\nu)^\alpha e^{-i\eta_2}-c_3r^{-\alpha}\nu^\alpha e^{i\theta_3}])\Big\}$$

$$+\frac{\mathrm{Re}}{r}\int_0^\infty du\ e^{-u}\exp[-c_0r^{-\alpha}u^\alpha e^{-i\eta_0}]\frac{u}{a_1}\int_0^{\pi/2}d\phi$$

$$\exp[-i\phi-\tilde{c}_2r^{-\alpha}u^\alpha(1-ie^{-i\phi}a_2/a_1)^\alpha e^{i\eta_2}-c_3r^{-\alpha}a_1^{-\alpha}u^\alpha e^{i(\theta_3-\alpha\phi)}]$$

$$\{\exp[-\tilde{c}_1r^{-\alpha}u^\alpha(1-ie^{-i\phi})^\alpha e^{i\eta_1}]-\exp[-\tilde{c}_1r^{-\alpha}u^\alpha(1-ie^{-i\phi})^\alpha e^{-i\eta_1}]\}$$

$$+\frac{\mathrm{Re}}{r}\int_0^\infty du\ e^{-u}\exp[-c_0r^{-\alpha}u^\alpha e^{-i\eta_0}]\frac{u}{a_2}\int_0^{\pi/2}d\phi$$

$$\exp[-i\phi-\tilde{c}_1r^{-\alpha}u^\alpha(1-ie^{-i\phi}a_1/a_2)^\alpha e^{i\eta_1}-c_3r^{-\alpha}a_2^{-\alpha}u^\alpha e^{i(\theta_3-\alpha\phi)}]$$

$$\{\exp[-\tilde{c}_2r^{-\alpha}u^\alpha(1-ie^{-i\phi})^\alpha e^{i\eta_2}]-\exp[-\tilde{c}_2r^{-\alpha}u^\alpha(1-ie^{-i\phi})^\alpha e^{-i\eta_2}]\}.$$

Moreover in the second term we see that

$$\int_{u/a_2}^{u/a_1}d\nu(\exp[\tilde{c}_1r^{-\alpha}(u+ia_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(a_2\nu-iu)^\alpha e^{-i\theta_2}-c_3r^{-\alpha}\nu^\alpha e^{-i\theta_3}]$$

$$+\exp[-\tilde{c}_1r^{-\alpha}(u-ia_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(a_2\nu+iu)^\alpha e^{i\theta_2}-c_3r^{-\alpha}\nu^\alpha e^{i\theta_3}])$$

$$+\int_0^{u/a_2}d\nu(\exp[-\tilde{c}_1r^{-\alpha}(u+ia_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(u+ia_2\nu)^\alpha e^{-i\eta_2}-c_3r^{-\alpha}\nu^\alpha e^{-i\theta_3}]$$

$$+\exp[-\tilde{c}_1r^{-\alpha}(u-ia_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(u-ia_2\nu)^\alpha e^{-i\eta_2}-c_3r^{-\alpha}\nu^\alpha e^{i\theta_3}])$$

$$=i\int_{u/a_2}^{u/a_1}d\nu(\exp[-\tilde{c}_1r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(a_2\nu-u)^\alpha e^{i\hat{\eta}_2}-c_3r^{-\alpha}\nu^\alpha e^{i\hat{\eta}_3}]$$

$$-\exp[-\tilde{c}_1r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(a_2\nu-u)^\alpha e^{-i\hat{\eta}_2}-c_3r^{-\alpha}\nu^\alpha e^{-i\hat{\eta}_3}])$$

$$+i\int_0^{u/a_2}d\nu(\exp[-\tilde{c}_1r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2r^{-\alpha}(u-a_2\nu)^\alpha e^{-i\eta_2}-c_3r^{-\alpha}\nu^\alpha e^{i\hat{\eta}_3}]$$

$$-\exp[-\tilde{c}_1 r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\eta_1}-\tilde{c}_2 r^{-\alpha}(u-a_2\nu)^\alpha e^{-i\eta_2}-c_3 r^{-\alpha}\nu^\alpha e^{-i\hat{\eta}_3}])$$

$$-i\frac{u}{a_1}\int_0^{\pi/2} d\phi(\exp[i\phi-\tilde{c}_1 r^{-\alpha}u^\alpha(1+ie^{i\phi})e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^\alpha(1+ie^{i\phi}a_2/a_1)^\alpha e^{-i\eta_2}-c_3 r^{-\alpha}a_1^{-\alpha}u^\alpha e^{-i(\theta_3-\alpha\phi)}]$$

$$-\exp[-i\phi-\tilde{c}_1 r^{-\alpha}u^\alpha(1-ie^{-i\phi})^\alpha e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^\alpha(1-ie^{i\phi}a_2/a_1)^\alpha e^{i\eta_2}-c_3 r^{-\alpha}a_1^{-\alpha}u^\alpha e^{i(\theta_3-\alpha\phi)}])$$

$$+i\frac{u}{a_2}\int_0^{\pi/2} d\phi(\exp[i\phi-\tilde{c}_1 r^{-\alpha}u^\alpha(1+ie^{i\phi}a_1/a_2)^\alpha e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^\alpha(1+ie^{i\phi})^\alpha e^{-i\eta_2}-c_3 r^{-\alpha}a_2^{-\alpha}u^\alpha e^{-i(\theta_3-\alpha\phi)}]$$

$$-\exp[-i\phi-\tilde{c}_1 r^{-\alpha}u^\alpha(1-ie^{i\phi}a_1/a_2)^\alpha e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^\alpha(1-ie^{i\phi})^\alpha e^{i\eta_2}-c_3 r^{-\alpha}a_2^{-\alpha}u^\alpha e^{i(\theta_3-\alpha\phi)}])$$

$$-i\frac{u}{a_2}\int_0^{\pi/2} d\phi(\exp[i\phi-\tilde{c}_1 r^{-\alpha}u^\alpha(1+ie^{i\phi}a_1/a_2)^\alpha e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^\alpha(1+ie^{i\phi})^\alpha e^{-i\eta_2}-c_3 r^{-\alpha}a_2^{-\alpha}u^\alpha e^{-i(\theta_3-\alpha\phi)}]$$

$$-\exp[-i\phi-\tilde{c}_1 r^{-\alpha}u^\alpha(1-ie^{i\phi}a_1/a_2)^\alpha e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^\alpha(1-ie^{i\phi})^\alpha e^{-i\eta_2}-c_3 r^{-\alpha}a_2^{-\alpha}u^\alpha e^{i(\theta_3-\alpha\phi)}]),$$

where we rotate the contours through angles $\pm\pi/2$. Substitute this equation for the above one, then we get

$$p(r\sigma)=\frac{1}{\pi^2 r}\int_0^\infty du\ e^{-u}\,\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^\alpha e^{-i\eta_0}]\}$$

$$\mathrm{Re}\int_{u/(ra_1)}^\infty dz_2\left\{\exp\left[-\tilde{c}_1\left(a_1 z_2-i\frac{u}{r}\right)^\alpha e^{-i\theta_1}-\tilde{c}_2\left(a_2 z_2-i\frac{u}{r}\right)^\alpha e^{-i\theta_2}-c_3 z_2^\alpha e^{-i\theta_3}\right]\right\}$$

$$+\frac{1}{\pi^2 r^2}\int_0^\infty du\left\{\int_{u/a_2}^{u/a_1} d\nu\ e^{-u}\,\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^\alpha e^{-i\eta_0}-\tilde{c}_1 r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\eta_1}]\}\right.$$

$$\mathrm{Im}\{\exp[-\tilde{c}_2 r^{-\alpha}(a_2\nu-u)^\alpha e^{-i\hat{\eta}_2}-c_3 r^{-\alpha}\nu^\alpha e^{-i\hat{\eta}_3}]\}$$

$$+\int_0^{u/a_2} d\nu e^{-u}\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^\alpha e^{-i\eta_0}-\tilde{c}_1 r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\eta_1}$$

$$\left.-\tilde{c}_2 r^{-\alpha}(u-a_2\nu)^\alpha e^{-i\eta_2}]\}\,\mathrm{Im}\{\exp[-c_3 r^{-\alpha}\nu^\alpha e^{-i\hat{\eta}_3}]\}\right\}$$

$$+\frac{1}{\pi^2 r^2}\int_0^\infty du\ e^{-u}\,\mathrm{Im}\{\exp[-c_0 r^{-\alpha}u^\alpha e^{-i\eta_0}]\}\frac{u}{a_1}$$

$$\int_0^{\pi/2} d\phi\ \exp[i\phi-\tilde{c}_1 r^{-\alpha}u^\alpha(1+ie^{i\phi})^\alpha e^{-i\eta_1}$$

$$-\tilde{c}_2 r^{-\alpha}u^\alpha(1+ie^{i\phi}a_2/a_1)^\alpha e^{-i\eta_2}-c_3 r^{-\alpha}a_1^{-\alpha}u^\alpha e^{-i(\theta_3-\alpha\phi)}].$$

This implies (3.8). Next let $1<\alpha<2$. In this case it is impossible to proceed in the same way as above, because the integral in (3.9) may diverge. However in a similar way to the one-dimensional case (cf. [3] Th. 2.4.2), if we choose suitable angles in the rotation of the contours of integration and use Taylor's formula: For $x>0$, $y\in\boldsymbol{R}$

$$\exp[-x+iy]=\sum_{n=0}^{N}\frac{(-x+iy)^n}{n\,!}+\varepsilon\frac{(-x+iy)^{N+1}}{(N+1)\,!},\quad \varepsilon\in\boldsymbol{C},\ |\varepsilon|\leqq 1,$$

then we will obtain the same asymptotic expansion (3.8). In fact, first we see that

$$2\pi^2 rp(r\sigma)=\mathrm{Re}\int_0^\infty dz_2\int_0^{ra_1z_2}du\exp[-iu-c_0r^{-\alpha}u^\alpha e^{-\theta_0}]^n/n\,!$$

$$\left(\exp\left[-\tilde{c}_1\left(a_1z_2+\frac{u}{r}\right)^\alpha e^{-i\theta_1}-\tilde{c}_2\left(a_2z_2+\frac{u}{r}\right)^\alpha e^{-i\theta_2}-c_3z_2^\alpha e^{-i\theta_3}\right]\right.$$

$$\left.+\exp\left[-\tilde{c}_1\left(a_1z_2-\frac{u}{r}\right)^\alpha e^{i\theta_1}-\tilde{c}_2\left(a_2z_2-\frac{u}{r}\right)^\alpha e^{i\theta_2}-c_3z_2^\alpha e^{i\theta_3}\right]\right)$$

$$+\frac{\mathrm{Re}}{r}\int_0^\infty d\nu\left\{\int_{a_1\nu}^{a_2\nu}du\exp[-iu-c_0r^{-\alpha}u^\alpha e^{-i\theta_0}]\right.$$

$$\left(\sum_{n=0}^{N}[-\tilde{c}_1r^{-\alpha}(u+a_1\nu)^\alpha e^{-i\theta_1}]^n/n\,!\sum_{n=0}^{N}[-\tilde{c}_2r^{-\alpha}(a_2\nu+u)^\alpha e^{-i\theta_2}]^n/n\,!\right.$$

$$\sum_{n=0}^{N}[-c_3r^{-\alpha}\nu^\alpha e^{-i\theta_3}]^n/n\,!+\sum_{n=0}^{N}[-\tilde{c}_1r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\theta_1}]^n/n\,!$$

$$\sum_{n=0}^{N}[-\tilde{c}_2r^{-\alpha}(a_2\nu-u)^\alpha e^{i\theta_2}]^n/n\,!$$

$$\left.\sum_{n=0}^{N}[-c_3r^{-\alpha}\nu^\alpha e^{i\theta_3}]^n/n\,!\right)+\int_{a_2\nu}^{\infty}du\exp[-iu-c_0r^{-\alpha}u^\alpha e^{-i\theta_0}]$$

$$\left(\sum_{n=0}^{N}[-\tilde{c}_1r^{-\alpha}(u+a_1\nu)^\alpha e^{-i\theta_1}]^n/n\,!\sum_{n=0}^{N}[-\tilde{c}_2r^{\alpha}(u+a_2\nu)^\alpha e^{-i\theta_2}]^n/n\,!\right.$$

$$\sum_{n=0}^{N}[-c_3r^{-\alpha}\nu^\alpha e^{-i\theta_3}]^n/n\,!+\sum_{n=0}^{N}[-\tilde{c}_1r^{-\alpha}(u-a_1\nu)^\alpha e^{-i\theta_1}]^n/n\,!$$

$$\left.\left.\sum_{n=0}^{N}[-\tilde{c}_2r^{-\alpha}(u-a_2\nu)^\alpha e^{-i\theta_2}]^n/n\,!\sum_{n=0}^{N}[-c_3r^{-\alpha}\nu^\alpha e^{i\theta_3}]^n/n\,!\right)\right\}$$

$$+O(r^{-1-(N+1)\alpha}).$$

In each term we rotate the contour of integration with respect to du through an angle $\gamma=\pi[(\alpha-2)\beta_0-1]/(2\alpha)$, then $\exp[-iu]$ is to $\exp[-ue^{i(\pi/2+\gamma)}]$ and $\exp[-c_0r^{-\alpha}u^\alpha e^{-i\theta_0}]$ is to $\exp[ic_0r^{-\alpha}u^\alpha]=\sum_{n=0}^{N}[ic_0r^{-\alpha}u^\alpha]^n/n\,!+\varepsilon[ic_0r^{-\alpha}u^\alpha]^{N+1}/(N+1)\,!$ with $\varepsilon\in\boldsymbol{C}$, $|\varepsilon|\leqq 2$. Note that $-\pi<\gamma<0$ and $|\pi/2+\gamma|<\pi/2$. Moreover we rotate the contour through an angle $-\pi/2-\gamma$. Then we have the expansion which is similar to (3.9). Then by the same way to the case of $0<\alpha<1$ we can easily obtain (3.8). Q.E.D.

Thus if $\sigma\in\mathbf{Spt}\,\lambda$ and $p_0^{\perp}(0)>0$, then

$$\mathrm{Re}\int_{u/r}^{\infty}\exp\Psi\left(-i\frac{u}{r},\ \nu\right)d\nu\longrightarrow\pi p_0^{\perp}(0)\qquad\text{as}\quad r\to+\infty,$$

and

$$p(r\sigma)\sim r^{-1-\alpha}\pi^{-1}c_0\sin\eta_0\Gamma(\alpha+1)p_0^{\perp}(0)\qquad\text{as}\quad r\to+\infty.$$

Therefore we have (a) in Proposition:

If $\sigma\notin\mathbf{Spt}\,\lambda$ then $\beta_{0,0}=-1$, i.e., $\eta_0=0$ or π (see §2), thus the first and second terms of (3.8) vanish. Hence by change of variables $u-a_1\nu$ to u' we have the following:

LEMMA 3. *Set* $b_j=a_j-a_1(b_1=0,\ b_m=\infty)$. *Then for* $\sigma\notin\mathbf{Spt}\,\lambda$,

$$\begin{aligned}(3.10)\quad p(r\sigma)\approx r^{-2}\pi^{-2}\sum_{j=1}^{m-1}\int_0^{\infty}d\nu\int_{b_j\nu}^{b_{j+1}\nu}du\ e^{-u-a_1\nu}\\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,!}r^{-n\alpha}\,\mathrm{Im}\,[c_1s_1^{\alpha}u^{\alpha}e^{-i\eta_1}+c_2s_2^{\alpha}(u-b_2\nu)^{\alpha}e^{-i\eta_2}+\cdots\\ +c_js_j^{\alpha}(u-b_j\nu)^{\alpha}e^{-i\eta_j}]^n\\ \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,!}r^{-n\alpha}\,\mathrm{Im}\,[c_{j+1}s_{j+1}^{\alpha}(b_{j+1}\nu-u)^{\alpha}e^{-i\hat{\eta}_{j+1}}+\cdots+c_m\nu^{\alpha}e^{-i\hat{\eta}_m}]^n.\end{aligned}$$

as $r\to+\infty$.

This lemma also holds in the case that λ has mass at neither σ nor $-\sigma$, because $c_0=0$ in (3.8).

Thus if $\sigma\notin\mathbf{Spt}\,\lambda$ and $\sigma\in\mathbf{Int}\,S(2)$, then

$$\begin{aligned}p(r\sigma)&\sim r^{-2(1+\alpha)}\pi^{-2}\Gamma(\alpha+1)^2\sum_{1\leq j<k\leq m}g_{j,k}c_j|h_{j,k}|^{-1-\alpha}\sin\eta_jc_k|h_{k,j}|^{-1-\alpha}\sin\hat{\eta}_k\\ &\sim\sum_{1\leq j<k\leq m}g_{j,k}p_j(rh_{j,k})p_k(rh_{k,j})\qquad\text{as}\quad r\to+\infty\\ &=\sum_{1\leq j<k\leq m}p_{j,k}(r\sigma),\end{aligned}$$

where $g_{j,k}=(s_jt_k-s_kt_j)^{-1}>0$ for $j<k$, $h_{j,k}$ and $h_{k,j}$ are defined by $\sigma=h_{j,k}\sigma_j+h_{k,j}\sigma_k$ (i.e., $h_{j,k}=t_k/(s_jt_k-s_kt_j)$). Thus, we get (b) in Proposition.

Moreover if $1<\alpha<2$ and $\sigma\notin S(2)$, then $\beta_{1,0}=\cdots=\beta_{m,0}=\pm1$ (i.e., $\hat{\eta}_1=\cdots=\hat{\eta}_m=\pi$ or $\eta_1=\cdots=\eta_m=\pi$). Hence every term of (3.10) vanish. We have (c) in Proposition.

Finally (d) is followed by Lemma 2.

Second step. Suppose that λ has mass at only $(m+1)$-directions σ_j, $j=0,1,2,\cdots,m$. We may assume that $\sigma=(1,0)$ and $0\leqq\varphi_0<\varphi_1<\varphi_2<\cdots<\varphi_{m-1}<\varphi_m<$

$\pi(\varphi_j=\arg\sigma_j)$. If λ has no mass at $\{\pm\sigma\}$, then by taking $c_0=0$ in $\Psi(z)$ and seting $\sigma_0=\sigma$ we may include σ as a member of directions σ_j, $j=0, 1, \cdots, m$. Moreover in (3.1) let Q be a linear transformation such that $Q\sigma_0=\sigma_0$ and $Q\sigma_m=(0, 1)$, then $0=\tilde{\varphi}_0<\tilde{\varphi}_1<\cdots<\tilde{\varphi}_m=\pi/2$ where $\tilde{\varphi}_j=\arg Q\sigma_j$. Thus by $Qr\sigma=r\sigma$ we have $p_\lambda(r\sigma)=|\det Q|p_{\lambda_Q}(r\sigma)$ and λ_Q has mass at only $(m+1)$-directions $Q\sigma_j$, $j=0, 1, \cdots, m$. Therefore the general case is reduced to the special case of First step.

The proof of Theorem 2 and Theorem 3 in Case 2 is complete.

CASE 3. $\alpha=1$ and λ has mass at $(m+1)$-directions $\sigma_0, \sigma_1, \cdots, \sigma_m$ $(m\geqq 2)$. We may also take $\{\sigma_j, j=0, 1, 2, \cdots, m\}$ as in First step of Case 2. Then for $z=(z_1, z_2)\in\boldsymbol{R}^2$

$$\Psi(z)=-\sum_{j=0}^{m} c_j\left\{|\langle\sigma_j, z\rangle|+i\frac{2}{\pi}\beta_j\langle\sigma_j, z\rangle\log|\langle\sigma_j, z\rangle|\right\},$$

where $c_j>0$, $|\beta_j|\leqq 1$, $j=0, 1, 2, \cdots, m$ are constants.

The following lemma is corresponding to Lemma 2 and Lemma 3.

LEMMA 4. *Let* $r\geqq 0$ *and* $\sigma=\sigma_0=(1, 0)$.

(i) *Then for* $a_j=\tan\phi_j$

$$p(r\sigma)=(2\pi)^{-2}\int_{\boldsymbol{R}^2}\exp[-irz_1+\Psi(z)]\,dz$$

$$\approx r^{-1}\pi^{-2}\sum_{n=1}^{\infty}\frac{r^{-n}}{n!}c_0^n\int_0^\infty e^{-u}u^n\,\mathrm{Im}\left[i(1+\beta_0)-\frac{2\beta_0}{\pi}\log\frac{u}{r}\right]^n du$$

$$\mathrm{Re}\int_{u/r}^{\infty}\exp\Psi\left(-i\frac{u}{r}, \nu\right)d\nu$$

$$+r^2\pi^{-2}\sum_{n=1}^{\infty}\frac{r^{-n}}{n!}c_0^n\int_0^\infty e^{-u}u^{n+1}\,\mathrm{Im}\left[i(1+\beta_0)-\frac{2}{\pi}\beta_0\log\frac{u}{r}\right]^n du$$

$$+r^{-2}\pi^{-2}\sum_{j=1}^{m-1}\int_0^\infty d\nu\int_{a_j\nu}^{a_{j+1}\nu}du\; e^{-u}\sum_{n=1}^{\infty}\frac{r^{-n}}{n!}\mathrm{Im}\left[c_0u\left\{i(1+\beta_0)-\frac{2}{\pi}\beta_0\log\frac{u}{r}\right\}\right.$$

$$+c_1s_1(u-a_1\nu)\left\{i(1+\beta_1)-\frac{2}{\pi}\beta_1\log[s_1(u-a_1\nu)/r]\right\}+\cdots$$

$$\left.+c_js_j(u-a_j\nu)\left\{i(1+\beta_j)-\frac{2}{\pi}\beta_j\log[s_j(u-a_j\nu)/r]\right\}\right]^n$$

$$\sum_{n=1}^{\infty}\frac{r^{-n}}{n!}\mathrm{Im}\left[c_{j+1}s_{j+1}(a_{j+1}\nu-u)\left\{i(1-\beta_{j+1})+\frac{2}{\pi}\beta_{j+1}\log[s_{j+1}(a_{j+1}\nu-u)/r]\right\}\right.$$

$$+\cdots+c_m\nu\Big\{i(1-\beta_m)+\frac{2}{\pi}\beta_m\log\frac{\nu}{r}\Big\}\Big]^n$$

as $r\to+\infty$.

(ii) *If* $\sigma\notin\mathbf{Spt}\,\lambda$, *set* $b_j=a_j-a_1$, *then*

$$p(r\sigma)\approx r^{-2}\pi^{-2}\sum_{j=1}^{m-1}\int_0^\infty d\nu\int_{b_j\nu}^{b_{j+1}\nu}du\ e^{-u-a_1\nu}$$

$$\sum_{n=1}^\infty\frac{r^{-n}}{n!}\mathrm{Im}\Big[c_1s_1u\Big\{i(1+\beta_1)-\frac{2}{\pi}\beta_1\log[s_1(u-b_1)/r]\Big\}+\cdots$$

$$+c_js_j(u-b_j\nu)\Big\{i(1+\beta_j)-\frac{2}{\pi}\beta_j\log[s_j(u-b_j\nu)/r]\Big\}\Big]^n$$

$$\sum_{n=1}^\infty\frac{r^{-n}}{n!}\mathrm{Im}\Big[c_{j+1}s_{j+1}(b_{j+1}\nu-u)\Big\{i(1-\beta_{j+1})+\frac{2}{\pi}\beta_{j+1}\log[s_{j+1}(b_{j+1}\nu-u)/r]\Big\}$$

$$+\cdots+c_m\nu\Big\{i(1-\beta_m)+\frac{2}{\pi}\beta_m\log\frac{\nu}{r}\Big\}\Big]^n$$

as $r\to+\infty$.

From this lemma we obtain Theorem 2 and Theorem 3 by the same way as in case of $1<\alpha<2$.

Next we proceed the proof of Theorem 2 in case of $d=3$.

(1) First we see that

$$(3.11)\quad (2\pi)^3p(x)=\int_{R^3}\exp[-i\langle x,z\rangle+\Psi(z)]dz$$

$$=2\,\mathrm{Re}\int_{R^3_+}dz\,\{\exp[-i(x_1z_1+x_2z_2+x_3z_3)+\Psi(z_1,z_2,z_3)]$$

$$+\exp[-i(x_1z_1-x_2z_2+x_3z_3)+\Psi(z^1,-z_2,z_3)]$$

$$+\exp[-i(x_1z_1+x_2z_2-x_3z_3)+\Psi(z_1,z_2,-z_3)]$$

$$+\exp[-i(x_1z_1-x_2z_2-x_3z_3)+\Psi(z_1,-z_2,-z_3)]\}.$$

(2) We divide the integral domain in order to omit the notation "sgn" in $\Psi(z)$.

(3) We change variables z_1, z_2, z_3 appropriately according to σ.

Then we deduce that Theorem 2 and Theorem 3 hold. We will describe the outline of the proof in some details. Here we only consider the case that λ has mass at $(m+1)$-directions $\sigma_0,\sigma_1,\cdots\sigma_m(m\geqq3)$ but that $0<\alpha<1$ and $\sigma\notin$ **Int** $S(3)$, because it is evident in the others.

a) If $\sigma\in T(1)$, i.e., $\lambda(\{\sigma\})>0$, we may take $\sigma=\sigma_0=(1,0,0)$ and change z_1 to $-iu/r$, then we have $p(r\sigma)\sim p_0(r)p_0^{\perp}(0)\sim cr^{-1-\alpha}(c>0)$ as $r\to+\infty$.

EXAMPLE 1. Let $m=5$, $\sigma=\sigma_0=(1, 0, 0)$, $\sigma_1=(0, 1, 0)$, $\sigma_2=(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $\sigma_3=(0, 1/\sqrt{2}, 1/\sqrt{2})$, $\sigma_4=(1/\sqrt{2}, 0, 1/\sqrt{2})$ and $\sigma_5=(0, 0, 1)$. In (3.11) we divide the integral domain as follows:

$$\begin{aligned}(3.12)\qquad \int_{R_+^3} dz=\int_0^\infty dz_3\Big\{&\int_0^{z_3/2} dz_2\Big(\int_0^{z_2} dz_2\int_{z_2}^{z_3-z_2}+\int_{z_3-z_2}^{z_3}+\int_{z_3}^{z_3+z_2}+\int_{z_3+z_2}^{\infty} dz_1\Big)\\ &+\int_{z_3/2}^{z_3} dz_2\Big(\int_0^{z_3-z_2}+\int_{z_3-z_2}^{z_2}+\int_{z_2}^{z_3}+\int_{z_2}^{z_3+z_2}+\int_{z_3+z_2}^{\infty} dz_1\Big)\\ &+\int_{z_3}^{2z_3} dz_2\Big(\int_0^{z_2-z_3}+\int_{z_2-z_3}^{z_3}+\int_{z_3}^{z_2}+\int_{z_2}^{z_2+z_3}+\int_{z_2+z_3}^{\infty} dz_1\Big)\\ &+\int_{2z_3}^{\infty} dz_2\Big(\int_0^{z_3}+\int_{z_3}^{z_2-z_3}+\int_{z_2-z_3}^{z_2}+\int_{z_2}^{z_2+z_3}+\int_{z_2+z_3}^{\infty} dz_1\Big)\Big\}\end{aligned}$$

and change z_1 to $-iu/r$, then we can see that the sum of terms in (3.11) corresponding to the first integral with respect to dz_1 of each term in (3.12) decreases like $p_0(r)p_0^{\perp}(0)\sim cr^{-1-\alpha}(c>0)$ as $r\to+\infty$. Moreover, the remaining terms are $o(r^{-1-\alpha})$ as $r\to+\infty$.

b) If $\sigma\in T(2)$, then the following two cases arise.

(i) There exists only one plane H which is spanned by some elements $\sigma_0, \sigma_1, \cdots, \sigma_k (k\geqq 1)$ of **Spt** λ and contains σ. In this case we may assume that H is x_1x_2-plane, $\sigma=(1/\sqrt{2}, 1/\sqrt{2}, 0)$, $\sigma_0=(1, 0, 0)$, $\sigma_1=(0, 1, 0)$ and $\sigma_2, \cdots, \sigma_k\in\{\theta_3=0\}\setminus\{\theta_1\geqq 0, \theta_2\geqq 0, \theta_3=0\}$ in S^2. Set $r'=r/\sqrt{2}$. We divide the integral domain as mentioned in (2) and change $(z_1, \pm z_2)$ to $-i(u_1/r', \pm u_2/r')$ in order to $\exp[-ir'(z_1\pm z_2)]$ become $\exp[-u_1-u_2]$ in (3.11). Then we have an asymptotic behaviour $p(r\sigma)\sim r^{-2(1+\alpha)}$ as $r\to+\infty$.

EXAMPLE 2. Let $m=3$, $k=1$, $\sigma_0=(1, 0, 0)$, $\sigma_1=(0, 1, 0)$, $\sigma_2=(0, 1/\sqrt{5}, 2/\sqrt{5})$, $\sigma_3=(0, 0, 1)$ and $\sigma=(1/\sqrt{2}, 1/\sqrt{2}, 0)\in$ **Con** $\{\sigma_0, \sigma_1\}$. In (3.11) we divide the integral as follows:

$$\int_{R_+^3} dz=\int_0^\infty dz_3\int_0^\infty dz_1\Big\{\int_0^{2z_3}+\int_{2z_3}^{\infty} dz_2\Big\}.$$

Change variables z_1 and z_2. Then from the term in (3.11) corresponding to the first integral in the above we have an asymptotic $p_{0,1}(r\sigma(0, 1))p_{0,1}^{\perp}(0)(\sim cr^{-2(1+\alpha)}, c>0)$ as $r\to+\infty$, where $\sigma(0, 1)$ is a restriction of σ to **Span** $\{\sigma_0, \sigma_1\}$. Moreover, from the other we have $o(r^{-2(1+\alpha)})$ as $r\to+\infty$.

(ii) There exist at least two planes H_1, H_2 which are spanned by some elements of **Spt** λ and $H_1\cap H_2$ is a line containing σ. In this case we take $\sigma=(1, 0, 0)$. We change z_1 to $-iu_1/r$ and also z_2 appropriately as seen in the following example. Then we have $p(r\sigma)\sim r^{-2(1+\alpha)}$ as $r\to+\infty$.

EXAMPLE 3. The setting is the same as in Example 1 except $\sigma_0=(1/\sqrt{2}, 1/\sqrt{2}, 0)\neq\sigma=(1, 0, 0)$, and also divide the integral domain as in it. First in each integral we change z_1 to $-iu_1/r$ then in (3.11) terms vanish which correspond to the first integrals with respect to z_1 in (3.12). In the integral $\int_0^\infty dz_3\int_0^{z_3/2}dz_2\int_{rz_2}^{r(z_3-z_2)}du_1$ we change z_2 to $+iu_2/r$, $-iu_2/r$, $+iu_2/r$ and $-iu_2/r$ according to each term of (3.11). Then we have the asymptotic $p_{0,1}(r\sigma(0, 1))\,p_{0,1}^{\perp}(0)$ as $r\rightarrow+\infty$. Moreover by the same change of variables we have $o(r^{-2(1+\alpha)})$ as $r\rightarrow+\infty$ from the integrals of

$$\int_0^\infty dz_3\Big\{\int_0^{z_3/2}dz_2\Big(\int_{r(z_3+z_2)}^{rz_3}+\int_{rz_3}^{r(z_3+z_2)}+\int_{r(z_3+z_2)}^{\infty}du_1\Big)$$
$$+\int_{z_3/2}^{z_3}dz_2\Big(\int_{rz_2}^{rz_3}+\int_{rz_3}^{r(z_3+z_2)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)\Big\}.$$

Similarly, in the integral $\int_0^\infty dz_3\int_{2z_3}^\infty dz_2\int_{rz_3}^{r(z_2-z_3)}du_1$ we change z_3 to $+iu_3/r$, $+iu_3/r$, $-iu_3/r$ and $-iu_3/r$ according to each term of (3.11). Then we have the asymptotic $p_{4,5}(r\sigma(4, 5))\,p_{4,5}^{\perp}(0)$ as $r\rightarrow+\infty$, and by the same change of variables we have $o(r^{-2(1+\alpha)})$ as $r\rightarrow+\infty$ from the integrals of

$$\int_0^\infty dz_3\Big\{\int_{z_3}^{2z_3}dz_2\Big(\int_{rz_3}^{rz_2}+\int_{rz_2}^{r(z_2+z_3)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)$$
$$+\int_{2z_3}^{\infty}dz_2\Big(\int_{r(z_2-z_3)}^{rz_2}+\int_{rz_2}^{r(z_2+z_3)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)\Big\}$$
$$=\int_0^\infty dz_2\Big\{\int_{z_2/2}^{z_2}dz_2\Big(\int_{rz_3}^{rz_2}+\int_{rz_2}^{r(z_2+z_3)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)$$
$$+\int_0^{z_2/2}dz_3\Big(\int_{r(z_2-z_3)}^{rz_2}+\int_{rz_2}^{r(z_2+z_3)}+\int_{r(z_2+z_3)}^{\infty}du_1\Big)\Big\}.$$

Finally we have the asymptotic $p_{2,3}(r\sigma(2, 3))\,p_{2,3}^{\perp}(0)$ as $r\rightarrow+\infty$ from the remaining terms. In fact, in the integral $\int_0^\infty dz_3\int_{z_3/2}^{z_3}dz_2\int_{rz_3}^{rz_2}du_1$ we change z_2 (resp. z_3) to $-iu_2/r$, $+iu_2/r$, $-iu_2/r$ and $+iu_2/r$ (resp. $+iu_3/r$, $+iu_3/r$, $-iu_3/r$ and $-iu_3/r$) according to each term of (3.11). Moreover change variables (u_1, u_2, u_3) to $(\nu_1+\nu_3, \nu_2, \nu_2+\nu_3)$. Then the sum of the first and 4-th terms vanish and we change ν_2 to $-i\nu_2$ (resp. $+i\nu_2$) in the second term (resp. third term). Similarly in $\int_0^\infty dz_3\int_{z_2}^{2z_3}dz_2\int_{r(z_2-z_3)}^{rz_3}du_1$ change z_2 (resp. z_3) to $+iu_2/r$, $-iu_2/r$, $+iu_2/r$ and $-iu_2/r$ (resp. $-iu_3/r$, $-iu_3/r$, $+iu_3/r$ and $+iu_3/r$) according to each terms of (3.11), and (u_1, u_2, u_3) to $(\nu_1+\nu_2, \nu_2+\nu_3, \nu_3)$. Then the sum of the first and 4-th terms vanish. Hence, we change ν_3 to $+i\nu_3$ (resp. $-i\nu_3$) in the second term (resp. third term). By this way we have $p_{2,3}(r\sigma(2, 3))p_{2,3}^{\perp}(0)$ as $r\rightarrow+\infty$. Therefore we see that $p(r\sigma)\sim p_{0,1}(r\sigma(0, 1))p_{0,1}^{\perp}(0)+p_{2,3}(r\sigma(2, 3))p_{2,3}^{\perp}(0)+p_{4,5}(r\sigma(4, 5))$ $p_{4,5}^{\perp}(0)$ $(\sim cr^{-2(1+\alpha)})$ as $r\rightarrow+\infty$.

c) If $\sigma\in T(3)$, it is sufficient to consider the case that $\sigma=(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $\sigma_0=(1, 0, 0)$, $\sigma_1=(0, 1, 0)$, $\sigma_2=(0, 0, 1)$ and $\sigma_3, \cdots, \sigma_m\subset S^2\backslash\{\theta_1\geqq 0, \theta_2\geqq 0, \theta_3\geqq 0\}$. Set $r'=r/\sqrt{3}$. We divide the integral domain as mentioned in (2) and change (z_1, z_2, z_3) to $-i(u_1/r', \pm u_2/r', \pm u_3/r')$ in order to $\exp[-ir'(z_1\pm z_2\pm z_3)]$ be to $\exp[-u_1-u_2-u_3]$ in (3.11). For instance, for $\exp[-ir'(z_1-z_2+z_3])$ we change (z_1, z_2, z_3) to $-i(u_1/r', -u_2/r', u_3/r')$. Then we have an asymptoic $p(r\sigma)$ $\sim r^{-3(1+\alpha)}$ as $r\rightarrow+\infty$.

EXAMPLE 4. Let $m=3$, $\sigma_0=(1, 0, 0)$, $\sigma_1=(0, 1, 0)$, $\sigma_2=(0, 0, 1)$, $\sigma_3=(0, -1/\sqrt{2}, 1/\sqrt{2})$ and $\sigma=(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. In (3.11) we divide the integral

$$\int_{R_{+}^{3}} dz=\int_0^{\infty}dz_3\int_0^{\infty}dz_1\left\{\int_0^{z_3}+\int_{z_3}^{\infty}dz_2\right\}.$$

Change variables z_1, z_2 and z_3 as above. Then we can easily deduce that $p(r\sigma)\sim p_{0,1,2}(r\sigma)p_{0,1,2}^{\perp}(0)+p_{0,1,3}(r\sigma)p_{0,1,3}^{\perp}(0)+p_{0,2,3}(r\sigma)p_{0,2,3}^{\perp}(0)$ $(\sim cr^{-3(1+\alpha)}, c>0)$ as $r\rightarrow+\infty$.

d) If $\sigma\in S(3)$ and $1\leqq\alpha<2$, then by the same way as in (c) we can see that $p(r\sigma)$ is rapidly decreasing as $r\rightarrow+\infty$.

All of the above change of variables are informal, however we can justify the computations by a similar way to the case of $d=2$.

Then we conclude Theorem 2 and Theorem 3.

REMARK 4. As mentioned in §1, in higher dimensions $(d\geqq 4)$ we belive that our method should work, although the calculations may be more tedious and complicated.

Acknowledgements

The author would like to thank Professor T. Shiga for his valuable suggestions and teachings. He also wishes to thank the refree for his useful suggestions and comments.

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