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YANG-MILLS CONNECTIONS IN ORTHONORMAL FRAME BUNDLE OVER $SU(2)$

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Abstract. The main result in this note is that the connection form with respect to frames in the orthonormal frame bundle for a left invariant Riemannian metric g on $SU(2)$ becomes a Yang-Mills connection if and only if g is bi-invariant.

1. Introduction and statement of results

To find Yang-Mills connections in a principal fibre bundle is important. In this paper, we get necessary and sufficient conditions for the connection form in the orthonormal frame bundle on $SU(2)$ with respect to an arbitrary given left invariant Riemannian metric to be a Yang-Mills connection. We get the following main Theorem and Corollary.

THEOREM. *Let g be left invariant Rimannian metric on $SU(2)$. Then the connection form in the orthonormal frame bundle defined by the Levi-Civita connection of g becomes a Yang-Mills connection if and only if g is bi-invariant.*

COROLLARY. *The connection in the above Theorem is a Yang-Mills connection if and only if the Lie group $SU(2)$ with a left invariant Riemmannian metric g is a space of constant curvature.*

2. The proof of the main theorem

Let G denote the Lie group $SU(2)$. Let the Lie algebra of all left invariant vector fields on $SU(2)$ denote $\mathfrak{g}$. The Killing form B of $\mathfrak{g}$ satisfies $B(X, Y)= 4\,Trace(XY)$, $(X, Y\in\mathfrak{g})$. We define an inner product $\langle\ ,\ \rangle_0$ on $\mathfrak{g}$ by

$$(1) \qquad \langle\ ,\ \rangle_0 := -B(X, Y), \qquad (X, Y\in\mathfrak{g}).$$

The following Lemma is known (cf. [4, p. 154]).

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LEMMA 1. *Let g be an arbitrary given left invariant Riemannian metric on G. Let $\langle\,,\,\rangle$ be an inner product on $\mathfrak{g}$ defined by $\langle X, Y\rangle := g_e(X_e, Y_e)$, where $X, Y \in \mathfrak{g}$ and e is the identity matrix of G. Then there exists an orthonormal basis (X_1, X_2, X_3) of $\mathfrak{g}$ with respect to $\langle\,,\,\rangle_0$ such that*

$$(2)\qquad \begin{aligned} &[X_1, X_2]=(1/\sqrt{2})X_3, &&[X_2, X_3]=(1/\sqrt{2})X_1,\\ &[X_3, X_1]=(1/\sqrt{2})X_2, &&\langle X_i, X_j\rangle=\delta_{ij}a_i^2, \end{aligned}$$

where a_i, $(i=1, 2, 3)$, are positive constant real numbers determined by the given left invariant Riemannian metric g of G.

The connection function α on $\mathfrak{g}\times\mathfrak{g}$ corresponding to the left invariant Riemanian connection of (G, g) is given as follows (cf. [3, p. 52]):

$$(3)\qquad \begin{cases} \alpha(X, Y)=1/2[X, Y]+U(X, Y), \qquad (X, Y\in\mathfrak{g}),\\ \langle X, Y\rangle := g_e(X_e, Y_e), \end{cases}$$

where $U(X, Y)$ is determined by

$$(4)\qquad 2\langle U(X, Y), Z\rangle=\langle [Z, X], Y\rangle+\langle X, [Z, Y]\rangle, \ (X, Y, Z\in\mathfrak{g}).$$

Now, putting $Y_1 := 2\sqrt{2}X_1$, $Y_2 := 2\sqrt{2}X_2$, and $Y_3 := 2\sqrt{2}X_3$ for the orthonormal basis (X_1, X_2, X_3) with respect to $\langle\,,\,\rangle_0$ in Lemma 1, we have the following:

$$(5)\qquad \begin{cases} [Y_1, Y_2]=2Y_3, \quad [Y_2, Y_3]=2Y_1, \quad [Y_3, Y_1]=2Y_2,\\ (V_1, V_2, V_3)\text{: orthonomal frame fields on } (G, g), \end{cases}$$

where $V_i := X_i/a_i$, $(i=1, 2, 3)$.

Using (4) and (5), we get

LEMMA 2.

$$(6)\qquad \begin{cases} U(Y_1, Y_1)=U(Y_2, Y_2)=U(Y_3, Y_3)=0,\\ U(Y_1, Y_2)=(a_2^2-a_1^2)a_3^{-2}Y_3,\\ U(Y_2, Y_3)=(a_3^2-a_2^2)a_1^{-2}Y_1,\\ U(Y_3, Y_1)=(a_1^2-a_3^2)a_2^{-2}Y_2. \end{cases}$$

The connection function α on $\mathfrak{g}\times\mathfrak{g}$ which is corresponding to a given left invariant Riemannian connection of (G, g) is uniquely expressed as

$$(7)\qquad \alpha(V_i, V_j)=\textstyle\sum_k \Gamma_{ij}^k V_k, \qquad (i, j=1, 2, 3).$$

Let $(\theta^1, \theta^2, \theta^3)$ be the dual 1-forms to the orthonormal frame basis (V_1, V_2, V_3). Then, the connection form ω and the curvature form Ω with respect to frames in the orthonormal frame bundle of (G, g) are defined as follows:

$$(8)\qquad \omega_j^i=\sum_{k=1}^3 \Gamma_{kj}^i\theta^k,$$

$$(9)\qquad \Omega_j^i=\sum_{k,l}\theta^i(\alpha(V_k, \alpha(V_l, V_j))-\alpha(V_l, \alpha(V_k, V_j))-\alpha([V_k, V_l], V_j))\theta^k\wedge\theta^l.$$

From (3), (4), (7) and Lemma 2, the non-zero terms of Γ_{jk}^i are

$$(10)\qquad \begin{cases} \Gamma_{23}^1=-\Gamma_{21}^3=(a_3^2+a_1^2-a_2^2)/2\sqrt{2}\,a_1a_2a_3, \\ \Gamma_{31}^2=-\Gamma_{32}^1=(a_1^2+a_2^2-a_3^2)/2\sqrt{2}\,a_1a_2a_3, \\ \Gamma_{12}^3=-\Gamma_{13}^2=(a_3^2+a_2^2-a_1^2)/2\sqrt{2}\,a_1a_2a_3. \end{cases}$$

Using (7)-(10), we get

$$(11)\qquad \begin{cases} \omega_1^1=\omega_2^2=\omega_3^3=0, & \omega_2^1=-\omega_1^2=\dfrac{(a_3^2-a_1^2-a_2^2)}{2\sqrt{2}\,a_1a_2a_3}\theta^3, \\ \omega_3^1=-\omega_1^3=\dfrac{(a_3^2+a_1^2-a_2^2)}{2\sqrt{2}\,a_1a_2a_3}\theta^2, & \omega_3^2=-\omega_2^3=\dfrac{(a_1^2-a_2^2-a_3^2)}{2\sqrt{2}\,a_1a_2a_3}\theta^1, \end{cases}$$

$$(12)\qquad \begin{cases} \Omega_2^1=-\Omega_1^2=\dfrac{-3a_3^4+2(a_1^2+a_2^2)a_3^2+(a_1^2-a_2^2)^2}{8(a_1a_2a_3)^2}\theta^1\wedge\theta^2, \\ \Omega_3^1=-\Omega_1^3=\dfrac{-3a_2^4+2(a_1^2+a_3^2)a_2^2+(a_1^2-a_3^2)^2}{8(a_1a_2a_3)^2}\theta^1\wedge\theta^3, \\ \Omega_3^2=-\Omega_2^3=\dfrac{-3a_1^4+2(a_2^2+a_3^2)a_1^2+(a_2^2-a_3^2)^2}{8(a_1a_2a_3)^2}\theta^2\wedge\theta^3, \\ \Omega_1^1=\Omega_2^2=\Omega_3^3=0. \end{cases}$$

We denote $(\nabla_{V_k}\Omega)(V_j, V_i)$, $\omega(V_j)$ and $\Omega(V_j, V_i)$ by $\nabla_k\Omega_{ji}$, ω_j and Ω_{ji}, respectively. The connection in (8) is a Yang-Mills connection (cf. [1, p. 107]) if and only if

$$(13)\qquad (\delta_\omega\Omega)(V_i)=-\sum_j(\nabla_j\Omega_{ji}+[\omega_j, \Omega_{ji}])=0, \qquad (i=1, 2, 3).$$

We put

$$(14)\qquad \begin{cases} (\delta_\omega\Omega)(V_1)=:(b_{(1)}{}_j^i), & (\delta_\omega\Omega)(V_2)=:(b_{(2)}{}_j^i), \quad \text{and} \\ (\delta_\omega\Omega)(V_3)=:(b_{(3)}{}_i^j), & (i, j=1, 2, 3). \end{cases}$$

From (7) and (10)-(14), the non-zero terms of $(b_{(1)}{}_j^i)$, $(b_{(2)}{}_j^i)$ and $(b_{(3)}{}_j^i)$ are

$$(15)\qquad b_{(1)}{}_3^2=-b_{(1)}{}_2^3=c^{-1}(2a_1^6-a_1^4a_2^2-a_1^4a_3^2-a_2^6+a_2^4a_3^2+a_2^2a_3^4-a_3^6),$$

$$(16)\qquad b_{(2)}{}_1^3=-b_{(2)}{}_3^1=c^{-1}(2a_2^6-a_2^4a_3^2-a_1^2a_2^4-a_3^6+a_1^2a_3^4+a_1^4a_3^2-a_1^6),$$

$$(17)\qquad b_{(3)}{}_2^1=-b_{(3)}{}_1^2=c^{-1}(2a_3^6-a_1^2a_3^4-a_2^2a_3^4-a_1^6+a_1^4a_2^2+a_1^2a_2^4-a_2^6),$$

where $c=4\sqrt{2}(a_1a_2a_3)^2$.

Hence, positive numbers a_1, a_2 and a_3 satisfy (15)=(16)=(17)=0 if and only if ω in (8) is a Yang-Mills connection. Moreover, positive numbers a_1, a_2 and a_3 satisfy (15)=(16)=(17)=0 if and only if $a_1=a_2=a_3$. Thus, the proof of the main theorem is completed.

On the other hand, using (12) and the main Theorem, we get the Corollary.

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