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ISOMETRIES OF A GENERALIZED TRIDIAGONAL ALGEBRAS $\mathcal{A}_{2n}^{(m)}$''

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Abstract. Let $\mathcal{A}_{2n}^{(m)}$ be a generalization of a tridiagonal algebra which is defined in the introduction. In this paper it is proved that if $\varphi:\mathcal{A}_{2n}^{(m)}\rightarrow\mathcal{A}_{2n}^{(m)}$ is a surjective isometry, then there exists a unitary operator U such that $\varphi(A)=U^{*}AU$ for all A in $\mathcal{A}_{2n}^{(m)}$ or a unitary operator W such that $\varphi(A)=W{}^{t}AW^{*}$ for all A in $\mathcal{A}_{2n}^{(m)}$, where ${}^{t}A$ is the transpose matrix of A.

I. Introduction

In [3], Gilfeather and Larson discovered tridiagonal algebras and in [4], Jo characterized all linear isometric maps of a tridiagonal algebra onto itself. Let $\mathcal{H}$ be a complex Hilbert space with an orthonormal basis $\{f_1, f_2, \cdots, f_{2n}\}$. Then a member of the tridiagonal algebra on $\mathcal{H}$ has the form

$$\begin{pmatrix} * & * & & & & & * \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & \cdot & & \\ & & & & & \cdot & \\ & & & & & \cdot & * \\ & & & & & & * \end{pmatrix}$$

with respect to the basis $\{f_1, f_2, \cdots, f_{2n}\}$, where all non-starred entries are zero. If we write the given basis in the order $\{f_1, f_3, f_5, \cdots, f_{2n-1}, f_2, f_4, \cdots, f_{2n}\}$, then the above matrix looks like this

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$$\begin{pmatrix} * & & & & & * & & & & * \\ & * & & & & * & * & & & \\ & & \cdot & & & & * & & & \\ & & & \cdot & & & & \cdot & & \\ & & & & \cdot & & & & \cdot & \\ & & & & * & & & & * & * \\ & & & & & * & & & & \\ & & & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & * \end{pmatrix},$$

where all non-starred entries are zero. Let $\mathcal{H}$ be a complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$ and let

$$S_0=\begin{pmatrix} * & & & & & & & & * \\ * & * & & & & & & & \\ & * & * & & & & & & \\ & & * & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & * \end{pmatrix}$$

be an (n, n)-matrix, where all non-starred entries are zero.

Let S be an (n, n) matrix. Then $S_0 \leqq S$ means that if the (i, j)-component of S_0 is $*$, then the (i, j)-component of S is also $*$. Let $\mathcal{A}_{2n}^{(m)} = \left\{ \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix} : D_1\right.$ and D_2 are (n, n) diagonal matrices and S is an (n, n) matrix with m stars in each row and column and $\left. S_0 \leqq S \right\}$. Then $\mathcal{A}_{2n}^{(m)}$ is a generalization of a tridiagonal algebra. In this paper, we will prove the following.

THEOREM. *Let $\varphi: \mathcal{A}_{2n}^{(m)} \to \mathcal{A}_{2n}^{(m)}$ be a surjective isometry. Then there exists a unitary operator U such that $\varphi(A) = U^*AU$ for all A in $\mathcal{A}_{2n}^{(m)}$ or a unitary operator W such that $\varphi(A) = W\,{}^tAW^*$ for all A in $\mathcal{A}_{2n}^{(m)}$, where ${}^tA$ is the transposed matrix of A.*

From now, we will introduce the terminologies which are used in this paper. Let $\mathcal{H}$ be a complex Hilbert space. If x and y are two vectors in $\mathcal{H}$, then (x, y) means the inner product of the two vectors x and y. If S is a non-empty subset of $\mathcal{H}$, then $[S]$ means the closed subspace generated by the vectors of S. An operator is a continuous linear transformation on $\mathcal{H}$ and the set of all such is $\mathcal{B}(\mathcal{H})$. A projection on $\mathcal{H}$ is a self-adjoint idempotent operator in $\mathcal{B}(\mathcal{H})$. There is an obvious correspondence between projections and their ranges, which are always norm-closed subspaces of $\mathscr{H}$.

A lattice $\mathcal{L}$ of projections (or subspaces) is a collection of projections closed under the operations $\wedge$ and $\vee$, where $E\wedge F$ is the projection whose range is (range E)$\cap$(range F) and $E\vee F$ is the projection whose range is [(range E)$\cup$ (rauge F)]. An operator A leaves a projection E invariant in case $AE=EAE$, and we denote by $Alg\mathcal{L}$ the collection $\{A: AE=EAE$ for all $E\in\mathcal{L}\}$. $Alg\mathcal{L}$ is a weakly closed subalgebra of $\mathscr{B}(\mathscr{H})$.

Dually, if $\mathcal{A}$ is a subalgebra of $\mathscr{B}(\mathscr{H})$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $A=AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L}=LatAlg\mathcal{L}$. Let α be in C, then $\bar{\alpha}$ is the complex conjugate of α. Let i and j be non-zero natural numbers. Then E_{ij} is the matrix whose (i, j)-component is 1 and all other components are zero. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be subalgebras of $\mathscr{B}(\mathscr{H})$.

A linear map φ of $\mathcal{A}_1$ into $\mathcal{A}_2$ is isometry if it preserves norm.

2. Examples

EXAMPXE 1. Let $\mathscr{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$. Let $E_{1(n+i),n+i}, E_{2(n+i),n+i}, \cdots, E_{m(n+i),n+i}$ be in $\mathcal{A}_{2n}^{(m)}$ for all $i(1\leqq i\leqq n)$ and let $\mathcal{L}$ be the subspace lattice generated by $\{[e_1], [e_2], \cdots, [e_n], [e_{1(n+1)}, \cdots, e_{m(n+1)}, e_{n+1}], [e_{1(n+2)}, e_{2(n+2)}, \cdots, e_{m(n+2)}, e_{n+2}], \cdots, [e_{1(2n)}, \cdots, e_{2(2n)}, \cdots, e_{m(2n)}, e_{2n}]\}$. Then $\mathcal{A}_{2n}^{(m)}=Alg\mathcal{L}$ and $\mathcal{A}_{2n}^{(m)}$ is reflexive.

EXAMPLE 2. Let $\mathscr{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$ and let U be a $(2n, 2n)$ diagonal unitary matrix whose (i, i)-component is u_{ii} for all $i(1\leqq i\leqq 2n)$. Define $\varphi: \mathcal{A}_{2n}^{(m)}\rightarrow\mathcal{A}_{2n}^{(m)}$ by $\varphi(A)=U^*AU$ for all A in $\mathcal{A}_{2n}^{(m)}$. Then φ is an isometry such that $\varphi(E_{ii})=E_{ii}$ for all $i=1, 2, \cdots, 2n$. If E_{ij} is in $\mathcal{A}_{2n}^{(m)}$, then the (i, j)-component of $\varphi(A)$ is $\bar{u}_{ii}a_{ij}u_{jj}$ for $A=(a_{ij})$ in $\mathcal{A}_{2n}^{(m)}$ $(1\leqq i\leqq n$ and $n+1\leqq j\leqq 2n)$.

EXAMPXE 3. Let us consider $\mathcal{A}_{10}^{(3)}$ as the following algebra.

$$A=\begin{pmatrix} D_1 & S \\ \mathbf{0} & D_2 \end{pmatrix} \text{ is in } \mathcal{A}_{10}^{(3)} \text{ if and only if } S=\begin{pmatrix} * & 0 & * & 0 & * \\ * & * & 0 & 0 & * \\ 0 & * & * & * & 0 \\ 0 & * & * & * & 0 \\ * & 0 & 0 & * & * \end{pmatrix}.$$

Let V be a (10, 10) matrix whose (1, 2)-, (2, 1)-, (3, 3)-, (4, 4)-, (5, 5)-, (6, 10)-, (7, 8)-, (8, 7)-, (9, 9)-, and (10, 6)-component are 1 and all other components are zero. Define $\varphi: \mathcal{A}_{10}^{(3)}\rightarrow\mathcal{A}_{10}^{(3)}$ by $\varphi(A)=V^*AV$ for all A in $\mathcal{A}_{10}^{(3)}$. Then φ is an isometry such that $\varphi(I)=I$, $\varphi(E_{11})=E_{22}$, $\varphi(E_{22})=E_{11}$, $\varphi(E_{33})=E_{33}$, $\varphi(E_{44})=E_{44}$, $\varphi(E_{55})=E_{55}$, $\varphi(E_{66})=E_{10,10}$, $\varphi(E_{77})=E_{88}$, $\varphi(E_{88})=E_{77}$, $\varphi(E_{99})=E_{99}$, and $\varphi(E_{10,10})=E_{66}$.

EXAMPLE 4. Let us consider $\mathcal{A}_8^{(3)}$ as the following algebra.

$$A=\begin{pmatrix} D_1 & S \\ \mathbf{0} & D_2 \end{pmatrix} \text{ is in } \mathcal{A}_8^{(3)} \text{ if and only if } S=\begin{pmatrix} * & 0 & * & * \\ * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \end{pmatrix}.$$

Let U be the unitary matrix whose (1, 8)-, (2, 7)-, (3, 6)-(4, 5)-, (5, 4)-, (6, 3)-, (7, 2)-, and (8, 1)-component are 1 and all other components are 0. Define φ: $\mathcal{A}_8^{(3)}\to\mathcal{A}_8^{(3)}$ by $\varphi(A)=U{}^tAU$ for all A in $\mathcal{A}_8^{(3)}$, where ${}^tA$ is the transposed matrix of A. Then φ is an isometry such that $\varphi(I)=I$, $\varphi(E_{11})=E_{88}$, $\varphi(E_{22})=E_{77}$, $\varphi(E_{33})=E_{66}$, $\varphi(E_{44})=E_{55}$, $\varphi(E_{55})=E_{44}$, $\varphi(E_{66})=E_{33}$, $\varphi(E_{77})=E_{22}$, and $\varphi(E_{88})=E_{11}$.

EXAMPLE 5. Let us consider $\mathcal{A}_{10}^{(3)}$ as the following algebra.

$$A=\begin{pmatrix} D_1 & S \\ \mathbf{0} & D_2 \end{pmatrix} \text{ is in } \mathcal{A}_{10}^{(3)} \text{ if and only if } S=\begin{pmatrix} * & 0 & 0 & * & * \\ * & * & * & 0 & 0 \\ 0 & * & * & 0 & * \\ * & 0 & * & * & 0 \\ 0 & * & 0 & * & * \end{pmatrix}.$$

Let U be a (10, 10)-matrix whose (1, 8)-, (2, 9)-, (3, 10)-, (4, 6)-, (5, 7)-(6, 4)-, (7, 5)-, (8, 1)-, (9, 2)-, and (10, 3)-component are 1 and all other components are zero. Define φ: $\mathcal{A}_{10}^{(3)}\to\mathcal{A}_{10}^{(3)}$ by $\varphi(A)=U{}^tAU^*$ for all A in $\mathcal{A}_{10}^{(3)}$, where ${}^tA$ is the transposed matrix of A. Then φ is an isometry such that $\varphi(I)=I$, $\varphi(E_{11})=E_{88}$, $\varphi(E_{22})=E_{99}$, $\varphi(E_{33})=E_{10,10}$, $\varphi(E_{44})=E_{66}$, $\varphi(E_{55})=E_{77}$, $\varphi(E_{66})=E_{44}$, $\varphi(E_{77})=E_{55}$, $\varphi(E_{88})=E_{11}$, $\varphi(E_{99})=E_{22}$, $\varphi(E_{10,10})=E_{33}$.

3. Results

Through this section, $\mathcal{H}$ is a $2n$-dimensional complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, e_{2n}\}$. We see that there is a commutative subspace lattice $\mathcal{L}$ such that $\mathcal{A}_{2n}^{(m)}=Alg\mathcal{L}$. φ will denote an isometry from $\mathcal{A}_{2n}^{(m)}$ onto $\mathcal{A}_{2n}^{(m)}$. Let x and y be two non-zero vectors in $\mathcal{H}$. Then $x\otimes y$ is a rank one operator defined by $(x\otimes y)(h)=(h, x)y$ for every h in $\mathcal{H}$.

LEMMA 1 ([7]). *Let $\mathcal{L}$ be a subspace lattice and let x and y be two vectors. Then $x\otimes y$ is in $Alg\mathcal{L}$ if and only if there exists E in $\mathcal{L}$ such that y is in E and x is in $E_-^\perp$, where $E_-=\vee\{F: F\in\mathcal{L}$ and $F\ngeq E\}$ and $E_-^\perp=(E_-)^\perp$.*

LEMMA 2 ([8]). *Let $\mathcal{L}$ be a subspace lattice and let φ: $Alg\mathcal{L} \rightarrow Alg\mathcal{L}$ be a surjective isometry. If $\varphi(I)=A$ and if $x\otimes x$ is in $Alg\mathcal{L}$, then $\|Ax\|=\|x\|$, where I denetes the identity operator.*

THEOREM 3. *Let φ: $\mathcal{A}_{2n}^{(m)} \rightarrow \mathcal{A}_{2n}^{(m)}$ be an isometry. Then $\varphi(I)$ is a diagonal unitary operator.*

PROOF. Let $\varphi(I)=(b_{ij})$. Since $\|\varphi(I)e_i\|=\|e_i\|=1$ and $\varphi(I)e_i=b_{ii}e_i$, $|b_{ii}|=1$ for all $i=1, 2, \cdots, n$. Since $\|\varphi(I)\|=\|I\|=1$, $\varphi(I)$ is a diagonal unitary operator.

Let $\mathcal{D}=\{A: A$ is a diagonal operator in $\mathcal{A}_{2n}^{(m)}\}$. Then $\mathcal{D}$ is a maximal abelian subalgebra containing $\mathcal{L}$ and $\mathcal{D}=\mathcal{A}_{2n}^{(m)}\cap(\mathcal{A}_{2n}^{(m)})^*$, where $\mathcal{A}_{2n}^{(m)}=Alg\mathcal{L}$ and $(\mathcal{A}_{2n}^{(m)})^*=\{A^*: A$ is in $\mathcal{A}_{2n}^{(m)}\}$.

LEMMA 4 ([6]). *A linear map φ of one C^*-algebra into another which carries the identity into the identity and is isometric on normal elements preserves adjoint, i.e., $\varphi(A^*)=(\varphi(A))^*$.*

DEFINITION 5. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*-algebras. A Jordan isomorphism or C^*-isomorphism φ: $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a bijective linear map such that if $A=A^*$ in $\mathcal{A}_1$, then $\varphi(A)=(\varphi(A))^*$ and $\varphi(A^n)=(\varphi(A))^n$.

LEMMA 6 ([6]). a) *A linear bijection φ of one C^*-algebra $\mathcal{A}_1$ onto another $\mathcal{A}_2$ which is isometric is a C^*-isomorphism followed by left multiplication by a fixed unitary operator, viz, $\varphi(I)$.*

b) *A C^*-isomorphism φ of a C^*-algebra $\mathcal{A}_1$ onto a C^*-algebra $\mathcal{A}_2$ is isometric and preserves commutativity.*

Let φ: $\mathcal{A}_{2n}^{(m)} \rightarrow \mathcal{A}_{2n}^{(m)}$ be an isometry and let $\varphi(I)=U$. Then UA and U^*A are in $\mathcal{A}_{2n}^{(m)}$ for every A in $\mathcal{A}_{2n}^{(m)}$. Define $\hat{\varphi}$: $\mathcal{A}_{2n}^{(m)} \rightarrow \mathcal{A}_{2n}^{(m)}$ by $\hat{\varphi}(A)=U^*\varphi(A)$ for every A in $\mathcal{A}_{2n}^{(m)}$. Then $\hat{\varphi}$ is an isometry such that $\hat{\varphi}(I)=I$. Since $\mathcal{D}$ is a C^*-algebra, $\hat{\varphi}(I)=I$, and $\hat{\varphi}$ is an isometry, $\hat{\varphi}|\mathcal{D}$ preserves adjoint by Lemma 4. From this fact, we can prove the following lemma.

LEMMA 7. $\hat{\varphi}(\mathcal{D})=\mathcal{D}$.

Since $\hat{\varphi}$: $\mathcal{A}_{2n}^{(m)} \rightarrow \mathcal{A}_{2n}^{(m)}$ is a surjective isometry, just like φ, and since the main theorem would be true of φ if it were true of $\hat{\varphi}$, we now work exclusively with $\hat{\varphi}$ and drop the "$\wedge$". Equivalently we assume that $\varphi(I)=I$. Then we can get the following corollary.

COROLLARY 8. *If* $\varphi: \mathcal{A}_{2n}^{(m)} \to \mathcal{A}_{2n}^{(m)}$ *is an isometry such that* $\varphi(I)=I$, *then* $\varphi(\mathcal{D}) = \mathcal{D}$.

Let $\varphi: \mathcal{A}_{2n}^{(m)} \to \mathcal{A}_{2n}^{(m)}$ be an isometry such that $\varphi(I)=I$. Then since $\varphi|\mathcal{D}$ and $\varphi^{-1}|\mathcal{D}$ are Jordan isomorphisms, we can prove the following lemma.

LEMMA 9. *Let* $\varphi: \mathcal{A}_{2n}^{(m)} \to \mathcal{A}_{2n}^{(m)}$ *be an isometry such that* $\varphi(I)=I$. *Then* E *is a projection in* $\mathcal{D}$ *if and only if* $\varphi(E)$ *is a projection in* $\mathcal{D}$.

LEMMA 10 ([6]). *If* φ *is a Jordan isomorphism from a C*algebra* $\mathcal{A}_1$ *onto a C*-algebra* $\mathcal{A}_2$, *then* $\varphi(BAB)=\varphi(B)\varphi(A)\varphi(B)$ *with* A *and* B *in* $\mathcal{A}_1$.

Let E and F be orthogonal projections acting on a Hilbert space $\mathcal{H}$. Then a partial order relation $\leqq$ is described as follows: $E \leqq F$ if and only if $EF = FE = E$. From Lemmas 9 and 10, we can prove the following theorem.

THEOREM 11. *Let* $\varphi: \mathcal{A}_{2n}^{(m)} \to \mathcal{A}_{2n}^{(m)}$ *be an isometry such that* $\varphi(I)=I$. *Then* $\varphi([e_i])$ *is rank one for each* $i=1, 2, \cdots, 2n$.

LEMMA 12 ([8]). *Let* $\varphi: \mathcal{A}_{2n}^{(m)} = Alg\mathcal{L} \to \mathcal{A}_{2n}^{(m)} = Alg\mathcal{L}$ *be an isometry such that* $\varphi(I)=I$. *Let* E *be a projection in* $\mathcal{D}$ *and let* T *be in* $Alg\mathcal{L} = \mathcal{A}_{2n}^{(m)}$ *with* $T=ETE^{\perp}$. *Then we have* $\varphi(T)=\varphi(E)\varphi(T)\varphi(E)^{\perp}+\varphi(E)^{\perp}\varphi(T)\varphi(E)$.

From Lemma 12, we can get the following lemma.

LEMMA 13. *Let* $\varphi: \mathcal{A}_{2n}^{(m)} \to \mathcal{A}_{2n}^{(m)}$ *be an isometry such that* $\varphi(I)=I$. *Let* $E_{i,i(1)}, E_{i,i(2)}, \cdots, E_{i,i(m)}$ *be in* $\mathcal{A}_{2n}^{(m)} (n+1 \leqq i(1), \cdots, i(m) \leqq 2n$ *and* $1 \leqq i \leqq n)$. *Let* $\varphi(E_{ii}) = E_{ll}$ *and let* $\varphi(E_{i(j),i(j)}) = E_{x_j,x_j}$ *for all* $j=1, 2, \cdots, m$. *If* $1 \leqq l \leqq n$, *then* $x_j \geqq n+1$ *and there exists* α_{l,x_j} *in* C *such that* $|\alpha_{l,x_j}|=1$ *and* $\varphi(E_{i,i(j)}) = \alpha_{l,x_j} E_{l,x_j}$. *If* $n+1 \leqq l \leqq 2n$, *then* $1 \leqq x_j \leqq n$ *and there exists* $\alpha_{x_j,l}$ *in* C *such that* $|\alpha_{x_j,l}|=1$ *and* $\varphi(E_{i,i(j)}) = \alpha_{x_j,l} E_{x_j,l}$.

PROOF. Suppose that $1 \leqq l \leqq n$. Since $E_{i,i(j)} = E_{i(j),i(j)}^{\perp} E_{i,j(j)} E_{i(j),i(j)} = E_{ii} E_{i,i(j)} E_{ii}^{\perp}$, $\varphi(E_{i,i(j)}) = E_{x_j,x_j}^{\perp} \varphi(E_{i,i(j)}) E_{x_j,x_j} + E_{x_j,x_j} \varphi(E_{i,i(j)}) E_{x_j,x_j}^{\perp}$ and $\varphi(E_{i,i(j)}) = E_{ll} \varphi(E_{i,i(j)}) E_{ll}^{\perp} + E_{ll}^{\perp} \varphi(E_{i,i(j)}) E_{ll}$ by Lemma 12. So $x_j \geqq n+1$ and $\varphi(E_{i,i(j)}) = \alpha_{l,x_j} E_{l,x_j}$ for some α_{l,x_j} in C and $|\alpha_{l,x_j}|=1$. Similarly, we can prove the second part of lemma.

LEMMA 14. *Let* $\varphi: \mathcal{A}_{2n}^{(m)} \to \mathcal{A}_{2n}^{(m)}$ *be an isometry such that* $\varphi(I)=I$. *Let* $\varphi(E_{11}) = E_{kk}$. *If* $1 \leqq k \leqq n$ *and if* $\varphi(E_{ii}) = E_{ll} (1 \leqq i \leqq n)$, *then* $1 \leqq l \leqq n$. *If* $n+1 \leqq k \leqq 2n$

and if $\varphi(E_{ii})=E_{ll}(1\leqq i\leqq n)$, *then* $n+1\leqq l\leqq 2n$.

PROOF. Define a permutation σ on $\{1, 2, \cdots, 2n\}$ by $\sigma(a)=b$ if $\varphi(E_{aa})=E_{bb}$. Suppose $1\leqq k\leqq n$. Since $E_{1,n+1}$ is in $\mathcal{A}_{2n}^{(m)}$, $\sigma(n+1)\geqq n+1$ by Lemma 13. Since $E_{2,n+1}$ is in $\mathcal{A}_{2n}^{(m)}$, $\sigma(2)\leqq n$. Since $E_{2,n+2}$ is in $\mathcal{A}_{2n}^{(m)}$, $\sigma(n+2)\geqq n+1$. Since $E_{3,n+2}$ is in $\mathcal{A}_{2n}^{(m)}$, $\sigma(3)\leqq n$. Continue this way. Then $\sigma(i)\leqq n$ for all $i=1, 2, \cdots, n$. Similarly we can prove the second part of lemma.

LEMMA 15. *Let U be a unitary operator. Then* $\|I+U\|=2$ *if and only if* 1 *is in* $\sigma(U)$, *where I denotes the identity and* $\sigma(U)$ *is the spectrum of U.*

PROOF. Suppose that $\|I+U\|=2$. Since U is unitary, $I+U$ is a normal operator. So the norm of $I+U$ is equal to its spectral radius; that is, $2=\|I+U\|=\sup\{|1+\alpha| : \alpha\in\sigma(U)\}$. Hence 1 is in $\sigma(U)$ because $\sigma(U)$ is a compact subset of the unit circle in C. Suppose that 1 is in $\sigma(U)$. Since $I+U$ is a normal operator, $\|I+U\| = \sup\{|1+\alpha| : \alpha\in\sigma(U)\}$. But $\|I+U\| \leqq \|I\|+\|U\|=2$. Since 1 is in $\sigma(U)$, $\sup\{|1+\alpha| : \alpha\in\sigma(U)\}\geqq 2$. Hence $\|I+U\|=2$.

PROPOSITION 16. *Let A be an* (n, n) *matrix whose* (1, 1)-, (1, n)-, (2, 1)-, (2, 2)-, (3, 2)-, (3, 3)-, $\cdots$, (n, $n-1$)-, ((n, n)-*component are* 1 *and all other components are zero* ($n\geqq 2$). *Then* $\|A\|=2$.

PROPOSITION 17. *Let A be an* (n, n) *matrix whose* (1, 1)-, (1, 2)-, (2, 2)-, (2, 3)-, $\cdots$, ($n-1$, $n-1$)-, ($n-1$, n)-, (n, 1)-*component are* 1 *and all other components are zero* ($n\geqq 2$). *Then* $\|A\|=2$.

THEOREM 18. *Let* $\varphi: \mathcal{A}_{2n}^{(m)}\rightarrow\mathcal{A}_{2n}^{(m)}$ *be an isometry such that* $\varphi(E_{ii})=E_{ii}$ *for* $i=1, 2, \cdots, 2n$. *Then there exists a unitary operator U such that* $\varphi(A)=U^*AU$ *for every A in* $\mathcal{A}_{2n}^{(m)}$.

PROOF. Let $\varphi(E_{kj})=\alpha_{kj}E_{kj}$ for all E_{kj} in $\mathcal{A}_{2n}^{(m)}$, where $|\alpha_{kj}|=1$. Let $\alpha_{kj}=e^{i\theta_{kj}}$. Let $A=(a_{ij})$ be in $\mathcal{A}_{2n}^{(m)}$ and let $a_{k,k(i)}$ represent the $(k, k(i))$-component of A, where $1\leqq k\leqq n$, $1\leqq i\leqq m$ and $n+1\leqq k(i)\leqq 2n$. Let $U=(u_{ll})$ be a $(2n, 2n)$ unitary diagonal matrix and let $u_{ll}=e^{i\theta_l}$ $(l=1, 2, \cdots, 2n)$. Consider U^*AU. If the linear system (*): $\theta_{n+1}-\theta_1=\theta_{1,n+1}$, $\theta_{1(2)}-\theta_1=\theta_{1,1(2)}$, $\cdots$, $\theta_{1(m)}-\theta_1=\theta_{1,1(m)}$ $(1(m)=2n)$, $\theta_{2(1)}-\theta_{2,2(1)}$, $\theta_{2(2)}-\theta_2=\theta_{2,2(2)}$, $\cdots$, $\theta_{2(m)}-\theta_2 = \theta_{2,2(m)}$, $\cdots$, $\theta_{n(1)}-\theta_n=\theta_{n,n(1)}$, $\theta_{n(2)}-\theta_n = \theta_{n,n(2)}$, $\cdots$, $\theta_{n(m)}-\theta_n = \theta_{n,n(m)}$ has solutions, then $\varphi(A)=U^*AU$ for every A in $\mathcal{A}_{2n}^{(m)}$. Let K be the $(mn, 2n)$ matrix consisting of the coefficients of the linear system (*) the let $X=(\theta_{1,1(1)}, \theta_{1,1(2)}, \cdots, \theta_{n,n(m)})^t$. Then the linear system (*) has solutions if and only if rank K=rank (K, X). We know that rank $K=2n-1$. If $\theta_{k,n+l}-\theta_{l,n+l}+\theta_{l,n+l-1}-\cdots-\theta_{k,n+k}=0\ (l>k\geqq 1)$ and $\theta_{q,n+p}-\theta_{q,n+q-1}+\theta_{q-1,n+q-1}-\cdots+\theta_{p+1,n+p+1}-\theta_{p+1,n+p}=0\ (q-2\geqq p\geqq 1)$, then rank $(K, X)=2n-1$. Let

$$V=\begin{pmatrix} \alpha_{k,n+k} & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \alpha_{k,n+l} \\ \alpha_{k+1,n+k} & \alpha_{k+1,n+k+1} & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \alpha_{k+2,n+k+1} & \alpha_{k+2,n+k+2} & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \alpha_{l,n+l-1} & \alpha_{l,n+l} \end{pmatrix}.$$

Then we see that $\|V\|=\|\varphi(B)\|$ for some B in $\mathcal{A}_{2n}^{(m)}$. Since $\|B\|=2$ by Proposition 16, $\|V\|=2$. Since

$$V\begin{pmatrix} \bar{\alpha}_{k,n+k} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \bar{\alpha}_{k+1,n+k+1} & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 & \bar{\alpha}_{l-1,n+l-1} & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & \bar{\alpha}_{l,n+l} \end{pmatrix}=I+W,$$

where

$$W=\begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & a_{1,l-k+1} \\ a_{21} & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & a_{32} & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ 0 & \cdot & \cdot & \cdot & 0 & a_{l-k+1,l-k} & 0 \end{pmatrix},$$

where $a_{21}=\alpha_{k+1,n+k}\bar{\alpha}_{k,n+k}$, $a_{32}=\alpha_{k+2,n+k+1}\bar{\alpha}_{k+1,n+k+1}$, $\cdots$, $a_{l-k+1,l-k}=\alpha_{l,n+l-1}\bar{\alpha}_{l-1,n+l-1}$, and $a_{1,l-k+1}=\alpha_{k,n+l}\bar{\alpha}_{l,n+l}$. 1 is in $\sigma(W)$ by Lemma 15. So $\alpha_{k,n+l}\bar{\alpha}_{l,n+l}\alpha_{l,n+l-1}\bar{\alpha}_{l-1,n+l-1}\cdots\alpha_{k+1,n+k}\bar{\alpha}_{k,n+k}=1$ or equivalently $\theta_{k,n+l}-\theta_{l,n+l}+\theta_{l,n+l-1}-\cdots+\theta_{k+1,n+k}-\theta_{k,n+k}=0$. Let

$$V_1=\begin{pmatrix} \alpha_{p+1,n+p} & \alpha_{p+1,n+p+1} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \alpha_{p+2,n+p+1} & \alpha_{p+2,n+p+2} & 0 & \cdot & \cdot & 0 \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & \alpha_{q-1,n+q-2} & \alpha_{q-1,n+q-1} \\ \alpha_{q,n+p} & 0 & \cdot & \cdot & \cdot & 0 & \alpha_{q,n+q-1} \end{pmatrix}.$$

Then we see that $\|V_1\|=\|\varphi(B_1)\|$ for some B_1 in $\mathcal{A}_{2n}^{(m)}$. Since $\|B_1\|=2$ by Proposition 17, $\|V_1\|=2$. Since

$$V_1\begin{pmatrix} \bar{\alpha}_{p+1,n+p} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \bar{\alpha}_{p+2,n+p+1} & 0 & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & & \cdot & 0 & \bar{\alpha}_{q,n+q-1} \end{pmatrix}=I+W_1,$$

where

$$W_1=\begin{pmatrix} 0 & b_{12} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & b_{23} & 0 & \cdot & \cdot & 0 & 0 \\ \cdot & & & \cdot & & & & \cdot \\ \cdot & & & & \cdot & & & \cdot \\ \cdot & & & & & \cdot & & \cdot \\ \cdot & & & & & & \cdot & 0 \\ \cdot & & & & & & & b_{q-p-1,q-p} \\ b_{q-p,1} & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix},$$

where $b_{12}=\alpha_{p+1,n+p+1}\bar{\alpha}_{p+2,n+p+1}$, $b_{23}=\alpha_{p+2,n+p+2}\bar{\alpha}_{p+3,n+p+2}$, $\cdots$, $b_{q-p-1,q-p}=\alpha_{q-1,n+q-1}\bar{\alpha}_{q,n+q-1}$, $b_{q-p,1}=\alpha_{q,n+p}\bar{\alpha}_{p+1,n+p}$. 1 is in $\sigma(W_1)$ by Lemma 15. So $\alpha_{q,n+p}\bar{\alpha}_{p+1,n+p}\alpha_{p+1,n+p+1}\bar{\alpha}_{p+2,n+p+1}\cdots\alpha_{q-1,n+q-1}\bar{\alpha}_{q,n+q-1}=1$ or equivalently $\theta_{q,n+p}-\theta_{q,n+q-1}+\theta_{q-1,n+q-1}-\cdots+\theta_{p+1,n+p+1}-\theta_{p+1,n+p}=0$. Hence rank $(K, X)=2n-1$. Hence $\varphi(A)=U^*AU$ for all A in $\mathcal{A}_{2n}^{(m)}$.

LEMMA 19. *Let* $\varphi: \mathcal{A}_{2n}^{(m)}\to\mathcal{A}_{2n}^{(m)}$ *be an isometry such that* $\varphi(I)=I$ *and* $\varphi(E_{kk})=E_{i_k,i_k}$ *for* $k=1, 2, \cdots, 2n$. *If* $1\leqq i_1\leqq n$, *then there is a unitary operator* V *such that* $V\varphi(E_{kk})V^*=E_{kk}$ *for all* $k=1, 2, \cdots, 2n$ *and* $V\varphi(E_{k,k(l)})V^*=\alpha_{i_k,i_{k(l)}}E_{k,k(l)}$ *for* $l=1, 2, \cdots, m$, *and for some* $\alpha_{i_k,i_{k(l)}}$ *in* C.

PROOF. Let V be a $(2n, 2n)$ matrix whose (k, i_k)-component is 1 for all $k=1, 2, \cdots, 2n$ and all other entries are 0, where $\varphi(E_{kk})=E_{i_k,i_k}$ for all $k=1, 2, \cdots, 2n$. Then $V\varphi(E_{kk})V^*=E_{kk}$ and $V\varphi(E_{k,k(l)})V^*=\alpha_{i_k,i_{k(l)}}E_{i_k,i_{k(l)}}$ for all $k=1, 2, \cdots, 2n$ and $l=1, 2, \cdots, m$, and for some $\alpha_{i_k,i_{k(l)}}$ in C.

THEOREM 20. *Let* $\varphi: \mathcal{A}_{2n}^{(m)}\to\mathcal{A}_{2n}^{(m)}$ *be an isometry such that* $\varphi(I)=I$ *and* $\varphi(E_{kk})=E_{i_k,i_k}$. *If* $1\leqq i_1\leqq n$, *then there is a unitary operator* W *such that* $\varphi(A)=W^*AW$ *for all* A *in* $\mathcal{A}_{2n}^{(m)}$.

PROOF. By Lemma 19 there is a unitary operator V such that $V\varphi(E_{kk})V^*=E_{kk}$ for all $k=1, 2, \cdots, 2n$. Define $\varphi_1: \mathcal{A}_{2n}^{(m)}\to\mathcal{A}_{2n}^{(m)}$ by $\varphi_1(A)=V\varphi(A)V^*$ for all A in $\mathcal{A}_{2n}^{(m)}$. Then φ_1 is an isometry by Lemma 19 and $\varphi_1(E_{kk})=E_{kk}$ for all $k=1, 2, \cdots, 2n$. Then there is a unitary operator U such that $\varphi_1(A)=U^*AU$ for all A in $\mathcal{A}_{2n}^{(m)}$ by Theorem 18. Since $\varphi_1(A)=U^*AU=V\varphi(A)V^*$ for all A in $\mathcal{A}_{2n}^{(m)}$, $\varphi(A)=(V^*U^*)A(UV)$ for all A in $\mathcal{A}_{2n}^{(m)}$. Put $UV=W$. Then $\varphi(A)=W^*AW$ for all A in $\mathcal{A}_{2n}^{(m)}$.

LEMMA 21. *Let* $\varphi: \mathcal{A}_{2n}^{(m)} \to \mathcal{A}_{2n}^{(m)}$ *be an isometry such that* $\varphi(I)=I$ *and* $\varphi(E_{kk}) = E_{i_k, i_k}(k=1, 2, \cdots, 2n)$ *If* $n+1 \leqq i_1 \leqq 2n$, *theu there is a unitary operator* V *such that* $V^t\varphi(A)V^*$ *is in* $\mathcal{A}_{2n}^{(m)}$ *for all* A *in* $\mathcal{A}_{2n}^{(m)}$ *and* $V^t\varphi(E_{kk})V^*=E_{kk}$ *for all* $k=1, 2, \cdots, 2n$.

PROOF. Let V be a $(2n, n)$ matrix whose (k, i_k)-component is 1 for $k=1, 2, \cdots, 2n$ and all other components are 0. If $E_{k, k(l)}$ is in $\mathcal{A}_{2n}^{(m)}$ for $k=1, 2, \cdots, n$ and for $l=1, 2, \cdots, m$, then $\varphi(E_{k, k(l)})=\alpha_{i_{k(l)}, i_k} E_{i_{k(l)}, i_k}$. Since $V^t\varphi(E_{k, k(l)})V^* = \alpha_{i_{k(l)}, i_k} V E_{i_k, i_{k(l)}} V^* = \alpha_{i_{k(l)}, i_{k(l)}} E_{k, i_{k(l)}} V^* = \alpha_{i_{k(l)}, i_k} E_{k, k(l)}$ for all $k=1, 2, \cdots, n$ and all $l=1, 2, \cdots, m$, $V^t\varphi(A)V^*$ is in $\mathcal{A}_{2n}^{(m)}$ for all A in $\mathcal{A}_{2n}^{(m)}$ and $V^t\varphi(E_{kk})V^* = E_{kk}$ for all $k=1, 2, \cdots, 2n$.

THEOREM 22. *Let* $\varphi: \mathcal{A}_{2n}^{(m)} \to \mathcal{A}_{2n}^{(m)}$ *be an isometry such that* $\varphi(E_{kk})=E_{i_k, i_k}$ *for* $k=1, 2, \cdots, 2n$. *If* $n+1 \leqq i_1 \leqq 2n$, *then there is a unitary operator* W *such that* $\varphi(A)=W^tAW^*$ *for all* A *in* $\mathcal{A}_{2n}^{(m)}$.

PROOF. By Lemma 21 there is a unitary operator V such that $V^t\varphi(E_{kk})V^* = E_{kk}$ for all $k=1, 2, \cdots, 2n$ and $V^t\varphi(A)V^*$ is in $\mathcal{A}_{2n}^{(w)}$ for all A in $\mathcal{A}_{2n}^{(m)}$. So there is a unitary operator U such that $V^t\varphi(A)V^*=U^*AU$ for all A in $\mathcal{A}_{2n}^{(m)}$ by Theorem 18. Set ${}^t(UV)=W$. Then $\varphi(A)=W^tAW^*$ for all A in $\mathcal{A}_{2n}^{(m)}$.

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