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ON A SUFFICIENT CONDITIONS FOR MULTIVALENTLY STARLIKENESS

By

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Let $q\in N=\{1, 2, 3, \cdots\}$ and $A(q)$ denote the class of function

$$f(z)=z^{q}+\sum_{n=q+1}^{\infty}a_{n}z^{n}$$

which are analytic in the open disk $E=\{z: |z|<1\}$.

A function $f(z)\in A(q)$ is called q-valently starlike with respect to the origin if and only if

$$Re\frac{zf'(z)}{f(z)}>0 \quad \text{in } E.$$

There are many papers in which various sufficient conditions for multivalently starlikeness were obtained, but almost these results were got by using real part of some analytic functions.

Recently, Mocanu [3] obtained the following result by using the imaginary part of $zf''(z)/f'(z)$.

THEOREM A. *If* $f(z)\in A(1)$ *and*

$$\left|\mathrm{Im}\frac{zf''(z)}{f'(z)}\right|<\sqrt{3} \quad in\ E,$$

then $f(z)$ *is univalently starlike in* E.

We need the following lemma due to [1, 2].

LEMMA 1. *Let* $w(z)$ *be analytic in* E *and suppose that* $w(0)=0$. *If* $|w(z)|$ *attains its maximum value on the circle* $|z|=r<1$ *at a point* z_{0}, *then we can write*

$$z_{0}w'(z_{0})=kw(z_{0})$$

where k *is a real number and* $k\geqq 1$.

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Applying the same method as the proof of [4, Theorem 1], we can prove the following lemma:

LEMMA 2. *Let $p(z)$ be analytic in E, $p(0)=q$ and suppose that there exists a point $z_0\in E$ such that*

$$Re\ p(z)>0 \quad for \quad |z|<|z_0| \tag{1}$$

$Re\ p(z_0)=0$ and $p(z_0)=ia$ where a is a real number and not zero.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)}=ik$$

where

$$k\geqq\frac{1}{2}\left(\frac{q^2+a^2}{a}\right)\geqq q \quad \text{if} \quad a>0,$$

and

$$k\leqq\frac{-1}{2}\left(\frac{q^2+a^2}{a}\right)\leqq -q \quad \text{if} \quad a<0.$$

PROOF. Let us put

$$\phi(z)=\frac{q-p(z)}{q+p(z)}. \tag{2}$$

Then we have that $\phi(0)=0$, $|\phi(z)|<1$ for $|z|<|z_0|$ and $|\phi(z_0)|=1$. From (1), (2) and Lemma 1, we have

$$\frac{z_0\phi'(z_0)}{\phi(z_0)}=-\frac{2z_0p'(z_0)}{q^2-p(z_0)^2}=\frac{-2z_0p'(z_0)}{q^2+|p(z_0)|^2}\geqq 1.$$

This shows that

$$-z_0p'(z_0)\geqq\frac{1}{2}(q^2+|p(z_0)|^2)$$

and $z_0p'(z_0)$ is a negative real number.

Applying the same method as the proof of [4, Theorem 1], for $a>0$, we have

$$\operatorname{Im}\frac{z_0p'(z_0)}{p(z_0)}\geqq\frac{1}{2}\left(\frac{q^2+a^2}{a}\right)\geqq q$$

and for $a<0$, we have

$$\operatorname{Im}\frac{z_0p'(z_0)}{p(z_0)}\leqq-\frac{1}{2}\left(\frac{q^2+a^2}{|a|}\right)\leqq -q.$$

This completes our proof.

Applying Lemma 2, we will obtain a generalized result of Theorem A.

MAIN THEOREM. *Let $f(z)\in A(q)$ and suppose that*

$$1+\frac{zf''(z)}{f'(z)}\neq ik \quad \text{in} \quad E, \tag{3}$$

where k is a real number and $|k|\geqq\sqrt{3}q$.

Then $f(z)$ is q-valently starlike in E.

PROOF. Let us put

$$p(z)=\frac{zf'(z)}{f(z)}$$

where $p(0)=q$. From the assumption (3), we easily have

$$p(z)\neq 0 \quad \text{in} \quad E.$$

In fact, if $p(z)$ has a zero of order n at $z=\alpha\in E$, then we can put

$$p(z)=(z-\alpha)^n p_1(z), \qquad (n\in N)$$

where $p_1(z)$ is analytic in E and $p_1(\alpha)\neq 0$.

Then we have

$$1+\frac{zf''(z)}{f'(z)}=\frac{zp'(z)}{p(z)}+p(z) \tag{4}$$

$$=\frac{nz}{z-\alpha}+\frac{zp_1'(z)}{p_1(z)}+(z-\alpha)^n p_1(z).$$

But, the imaginary part of (4) can take any infinite values when z approaches α.

This contradicts (3). Hence we have

$$p(z)\neq 0 \quad \text{in} \quad E.$$

Therefore, if there exists a point $z_0\in E$ such that $Re\, p(z)>0$ for $|z|<|z_0|$,

$$Re\, p(z_0)=0 \quad \text{and} \quad p(z_0)=ia,$$

then we have

$$p(z_0)\neq 0 \quad \text{and} \quad a\neq 0.$$

From Lemma 2 and (4), for $a>0$, we have

$$1+\frac{z_0f''(z_0)}{f'(z_0)}=\frac{z_0p'(z_0)}{p(z_0)}+p(z_0)$$

$$=i\left(\text{Im}\frac{z_0p'(z_0)}{p(z_0)}+\text{Im}\, p(z_0)\right)$$

and

$$\mathrm{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}+p(z_0)\right) \geqq \frac{1}{2}\left(\frac{q^2+3a^2}{a}\right) \geqq \sqrt{3}\, q .$$

For $a<0$, we have

$$1+\frac{z_0 f''(z_0)}{f'(z_0)}=i\left(\mathrm{Im}\frac{z_0 p'(z_0)}{p(z_0)}+\mathrm{Im}\ p(z_0)\right)$$

and so

$$\mathrm{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}+p(z_0)\right) \leqq -\frac{1}{2}\left(\frac{q^2+3a^2}{|a|}\right) \leqq -\sqrt{3}\, q .$$

These contradict (3). Hence we have

$$Re\ p(z)>0 \quad \text{in} \quad E .$$

This shows that $f(z)$ is q-valently starlike in E.

This completes our proof.

From Main theorem, we easily have the following result.

COROLLARY. *Let $f(z) \in A(q)$ and suppose that there exists a real number R for which*

$$\left|\frac{z f''(z)}{f'(z)}-R\right|<\sqrt{(R+1)^2+3q^2} \qquad in \quad E .$$

Then $f(z)$ is q-valently starlike in E.

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