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# ON THE HIGHER WAHL MAPS

Dedicated to Prof. S. Koizumi on his 70-th birthday

By

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## 0. Introduction

Let $C$ be a complete non-singular curve defined over rn algebraically closed field $k$ and let $\mathcal{L}$ be an invertible sheaf of positive degree on $C$. J. Wahl defines a natural map

$$\Phi_{\mathcal{L}}: \wedge^2\Gamma(C, \mathcal{L}) \longrightarrow \Gamma(C, \omega_C\otimes\mathcal{L}^{\otimes 2})$$

given by $\Phi_{\mathcal{L}}(s\wedge t)=s(dt)-t(ds)$ where $s, t\in\Gamma(C, \mathcal{L})$ (see [18]). This notation is locally defined and well-defined on $C$. We often call this map a Wahl map. The original study of a Wahl map is the study of $\Phi_{\omega_C}$ where $\omega_C$ is the canonical sheaf on $C$. This map is very much useful. For example it gives a property which must be satisfied in order that a curve sits on a K3 surface. Precisely if $C$ lies on a K3 surface, then $\Phi_{\omega_C}$ is not surjective (see [17]). In [3], we have that if $C$ is a general curve of genus 10 or $\geqq 12$, then $\Phi_{\omega_C}$ is surjective. This result gives an answer of Mukai conjecture. Let $\mathcal{L}$ and $\mathcal{M}$ be two invertible sheaves on $C$ and let

$$\mathcal{R}(\mathcal{L}, \mathcal{M})=\ker[\Gamma(C, \mathcal{L})\otimes\Gamma(C, \mathcal{M}) \xrightarrow{\text{cup product}} \Gamma(C, \mathcal{L}\otimes\mathcal{M})].$$

In [18], Wahl constructs another Wahl map

$$\Phi_{\mathcal{L},\mathcal{M}}: \mathcal{R}(\mathcal{L}, \mathcal{M}) \longrightarrow \Gamma(C, \omega_C\otimes\mathcal{L}\otimes\mathcal{M}),$$

(if $\mathcal{L}=\mathcal{M}$, then $\wedge^2\Gamma(C, \mathcal{L})\hookrightarrow\mathcal{R}(\mathcal{L}\otimes\mathcal{L})$ and $\Phi_{\mathcal{L},\mathcal{L}}=\Phi_{\mathcal{L}}$). And he proves that if $\deg(\mathcal{L})\geqq 5g+2$ and $\deg(\mathcal{M})\geqq 2g+2$, then $\Phi_{\mathcal{L},\mathcal{M}}$ is surjective, and if $C$ is a non-hyperelliptic curve and $\deg(\mathcal{L})\geqq 5g+2$, then $\Phi_{\omega_C,\mathcal{L}}$ is surjective. These results give several informations about first ordered deformations of a cone over the $C\hookrightarrow \boldsymbol{P}(\Gamma(C, \mathcal{L}))$ when $\mathcal{L}$ is a normally generated (very ample) invertible sheaf on $C$ (if $\deg(\mathcal{L})\geqq 5g+2$ then $\mathcal{L}$ is clearly normally generated (and very ample)). For example we havə that if $\deg(\mathcal{L})\geqq 5g+2$ then a cone over a non-hyperelliptic curve embedded by $\mathcal{L}$ has only one canonical deformation

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from the above results (see [17]). $\Phi_{\mathcal{L}}$ also has a geometric meaning. The geometric aspect of a Wahl map is the following. Let $\mathcal{L}$ be a very ample invertible sheaf and $C \hookrightarrow \boldsymbol{P}^m$ is an embedding defined by $\mathcal{L}$. Then we can consider a Gaussian map

$$g : C \longrightarrow Grass(\boldsymbol{P}^1, \boldsymbol{P}^m)$$

which is given by $g(p)$=the tangent line of $C$ in $\boldsymbol{P}^m$ at $p$ where $Grass(\boldsymbol{P}^1, \boldsymbol{P}^m)$ is a Grassmannian variety of all projective lines in $\boldsymbol{P}^m$. Let

$$\iota : Grass(\boldsymbol{P}^1, \boldsymbol{P}^m) \hookrightarrow \boldsymbol{P}^M$$

be a Plücker embedding. Then the restriction map

$$g^*\iota^* : \Gamma(\boldsymbol{P}^M, \mathcal{O}_{\boldsymbol{P}^M}(1)) \longrightarrow \Gamma(\boldsymbol{P}^M, g^*\iota^*\mathcal{O}_{\boldsymbol{P}^M}(1))$$

gives the above $\Phi_{\mathcal{L}}$. Sometimes we call the image $\iota g(C)$ a dual curvə of $C$. If $\Phi_{\mathcal{L}}$ is surjective, then the dual curve $\iota g(C)$ is linearly normal and if $\Phi_{\mathcal{L}}$ is injective, then the dual curve $\iota g(C)$ is non-degenerate. Therefore the above dual curve $\iota g(C)$ is linearly normal if $\deg(\mathcal{L})$ is sufficiently large. In this paper, we want to generalize a Wahl map from the viewpoint of projective geometry. The notion of dual curve is generalized as follows. Let $C \rightarrow \boldsymbol{P}^m$ be a birational morphism to its image, let

$$g_n : C \cdots \longrightarrow Grass(\boldsymbol{P}^n, \boldsymbol{P}^m)$$

be a Gaussian map defined by $g_n(p)$=the osculating tangent $n$-th plaine at $p$ and let

$$\iota_n : Grass(\boldsymbol{P}^1, \boldsymbol{P}^m) \hookrightarrow \boldsymbol{P}^{M_n}$$

be a Plücker embedding. Then the image $\iota_n g_n(C)$ is also called a dual curve, and in projective geometry, whether $\iota_{m-1}g_{m-1}(C)$ is linearly normal or not and whether $\iota_{m-1}g_{m-1}(C)$ is non-degenerate or not are very big problems. These conditions are equivalent to surjectivity or injectivity of $g^*_{m-1}\iota^*_{m-1}$. Let $\mathcal{L}$ be a very ample invertible sheaf on $C$. In section 1, we define a generalized Wahl map

$$\Phi^{(n)}_{\mathcal{L}} : \wedge^n \Gamma(C, \mathcal{L}) \longrightarrow \Gamma(C, \omega_C^{\otimes n(n-1)/2} \otimes \mathcal{L}^{\otimes n})$$

which is equal to $g^*_n\iota^*_n$ if $\deg(\mathcal{L})$ is sufficiently large. Unfortunately we can not give the sufficient conditions for surjectivity or injectivity of $g^*_{m-1}\iota^*_{m-1}$. But in the sections 2 and 3, we have the following main theorem:

THEOREM. *Let $C$ be a non-singular curve of genus $g$ defined over an algebraically closed field $k$ and let $\mathcal{L}$ be an invertible sheaf on $C$. We assume that $char(k)=0$ or $char(k)>deg(\mathcal{L})$. If $deg(\mathcal{L})>(g-1)(2n^2-2n+3)+2(n^2-1)$, then*

$\Phi_{\mathcal{L}}^{(n)}$ *is surjective.*

I would like to express my sincerely gratitude to Professor Masaaki Homma for his help and continuous support.

NOTATIONS

char$(k)$: The characteristic of a field $k$

$\mathcal{O}_C$: The structure sheaf of a variety $C$

$\omega_C$: The canonical invertible sheaf on a non-singular variety $C$

$f^*$: The pull back defined by a morphism $f$

$\deg(\mathcal{L})$: The degree of an invertible sheaf $\mathcal{L}$

$\mathcal{O}_C(D)$: The invertible sheaf associated with a divisor $D$

$\Gamma(C, \mathcal{F})$: The global sections of a sheaf $\mathcal{F}$

$H^i(C, \mathcal{F})$: The $i$-th cohomology group of a sheaf $\mathcal{F}$

$\mathfrak{S}_n$: The symmetric group of degree $n$

$\wedge^n V$: The exterior product of a vector space $V$

$V^*$: The dual space of a vector space $V$

$\sum_{\lambda \in \Lambda} V_\lambda$: The direct sum of vector spaces $V_\lambda$ $(\lambda \in \Lambda)$

$V \oplus W$: The direct sum of vector spaces $V$ and $W$

$\boldsymbol{P}(V)$: The projective space of all 1-dimensional subspaces of $V$

$Grass(\boldsymbol{P}^n, \boldsymbol{P}^m)$: The Gramann variety of all $n$-plaines in $\boldsymbol{P}^m$.

## 1. The definition of a higher Wahl map and its basic property

Let $C$ be a complete non-singular algebraic curve of genus $g$ defined over an algebraically closed field $k$ and let $\mathcal{L}$ be an invertible sheaf on $C$. Throughout of this paper, we assume that char$(k)=0$ or char$(k)<deg(\mathcal{L})$.

DEFINITION 1. Let $V$ be a vector subspace of $\Gamma(C, \mathcal{L})$. We define the $n$-Wahl map

$$\Phi_V^{(n)}: \wedge^n V \longrightarrow \Gamma(C, \omega_C^{\otimes n(n-1)/2} \otimes \mathcal{L}^{\otimes n})$$

by

$$\Phi_V^{(n)}(s_1 \wedge \cdots \wedge s_n) = \begin{vmatrix} s_1 & \cdots & d^{n-1}s_1 \\ \vdots & & \vdots \\ s_n & \cdots & d^{n-1}s_n \end{vmatrix}.$$

If $V=\Gamma(C, \mathcal{L})$, we dəfine $\Phi_{\mathcal{L}}^{(n)}$ to be $\Phi_V^{(n)}$.

This definition is well-defined. Because if $n=1$, then the proof is found in

Wahl [18] (see p. 77) and if $n>1$, then the proof is given by the same argument.

DEFINITION 2. Let $V$ be a finite dimensional vector space over $k$. Then for each $x_1\wedge\cdots\wedge x_n$, $y_1\wedge\cdots\wedge y_n\in\wedge^n V$, we define

$$
\begin{aligned}
&[x_1\wedge\cdots\wedge x_n]\wedge[y_1\wedge\cdots\wedge y_n]\\
&=\frac{1}{2}(-1)^{n+1}\Big(x_1\wedge\cdots\wedge x_n\sum_{i=1}^{n}(-1)^{i-1}y_i\otimes y_1\wedge\cdots\wedge\hat{y}_i\wedge\cdots\wedge y_n\\
&\qquad -y_1\wedge\cdots\wedge y_n\wedge\sum_{i=1}^{n}(-1)^{i-1}x_i\otimes x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge x_n\Big)
\end{aligned}
$$

where $\hat{x}$ means that the term $x$ is omitted.

Clearly this definition is also well-defined. In the above definition,

$$x_1\wedge\cdots\wedge x_n]\wedge[y_1\wedge\cdots\wedge y_n]$$

is contained in $\wedge^{n+1}V\otimes\wedge^{n-1}V$. According to Definition 1 and Definition 2, we have the following lemma:

LEMMA 1. *For every* $x_1,\cdots,x_{n-1}$, $y_1,\cdots,y_{n-1}\in\Gamma(C,\mathcal{L})$, *we have*

$$
\begin{aligned}
&\Phi_{\mathcal{L}}^{(n)}\cap\Phi_{\mathcal{L}}^{(n-2)}([x_1\wedge\cdots\wedge x_{n-1}]\wedge[y_1\wedge\cdots\wedge y_{n-1}])\\
&\quad=\Phi^{(n)}_{\omega_C^{\otimes(n-1)(n-2)/2}\otimes\mathcal{L}^{\otimes n-1}}(\Phi_{\mathcal{L}}^{(n-1)}(x_1\wedge\cdots\wedge x_{n-1})\wedge\Phi_{\mathcal{L}}^{(n-1)}(y_1\wedge\cdots\wedge y_{n-1}))
\end{aligned}
$$

PROOF. By the definition,

$$
\begin{aligned}
&\Phi_{\mathcal{L}}^{(n)}\cap\Phi_{\mathcal{L}}^{(n-2)}([x_1\wedge\cdots\wedge x_{n-1}]\wedge[y_1\wedge\cdots\wedge y_{n-1}])\\
&=\frac{1}{2}(-1)^{n}\Phi_{\mathcal{L}}^{(n)}\cap\Phi_{\mathcal{L}}^{(n-2)}\Big(x_1\wedge\cdots\wedge x_{n-1}\wedge\sum_{i=1}^{n}(-1)^{i-1}y_i\otimes y_1\wedge\cdots\wedge\hat{y}_i\wedge\cdots\wedge y_{n-1}\\
&\quad -x_1\wedge\cdots\wedge x_{n-1}\wedge\sum_{i=1}^{n}(-1)^{i-1}x_i\otimes x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge x_{n-1}\Big)\\
&=\frac{1}{2}(-1)^{n}\Bigg(\sum_{i=1}^{n-1}(-1)^{i-1}\begin{vmatrix} x_1 & \cdots & d^{n-1}x\\ \vdots & & \vdots\\ x_{n-1} & \cdots & d^{n-1}x_{n-1}\\ y_i & \cdots & d^{n-1}y_i\end{vmatrix}\begin{vmatrix} y_1 & \cdots & d^{n-3}y_1\\ \vdots & & \vdots\\ >y_i & \cdots & d^{n-3}y_i<\\ \vdots & & \vdots\\ y_{n-1} & \cdots & d^{n-3}y_{n-1}\end{vmatrix}\\
&\quad -\sum_{i=1}^{n-1}(-1)^{i-1}\begin{vmatrix} y_1 & \cdots & d^{n-1}y_1\\ \vdots & & \vdots\\ y_{n-1} & \cdots & d^{n-1}y_{n-1}\\ x_i & \cdots & d^{n-1}x_i\end{vmatrix}\begin{vmatrix} x_1 & \cdots & d^{n-3}x_1\\ \vdots & & \vdots\\ >x_i & \cdots & d^{n-3}x_i<\\ \vdots & & \vdots\\ x_{n-1} & \cdots & d^{n-3}x_{n-1}\end{vmatrix}\Bigg)
\end{aligned}
$$

where $><$ means that this low (or column) is omitted. First we calculate the first term of the right-hand side of this equation.

$$(-1)^n \sum_{i=1}^{n-1} (-1)^{i-1} \begin{vmatrix} x_1 & \cdots & d^{n-1}x_1 \\ \vdots & & \vdots \\ x_{n-1} & \cdots & d^{n-1}x_{n-1} \\ y_i & \cdots & d^{n-1}y_i \end{vmatrix} \begin{vmatrix} y_1 & \cdots & d^{n-3}y_1 \\ \vdots & & \vdots \\ >y_i & \cdots & d^{n-3}y_i< \\ \vdots & & \vdots \\ y_{n-1} & \cdots & d^{n-1}y_{n-1} \end{vmatrix}$$

$$=(-1)^{2n} \sum_{i=1}^{n-1} \sum_{j=1}^{n} (-1)^{i+j-1}(d^{j-1}y_i)$$

$$\begin{vmatrix} x_1 & \cdots & \overset{\vee}{d^{j-1}x_1} & \cdots & d^{n-1}x_1 \\ \vdots & & \vdots & & \\ x_{n-1} & \cdots & \underset{\wedge}{d^{j-1}x_{n-1}} & \cdots & d^{n-1}x_{n-1} \end{vmatrix} \begin{vmatrix} y_1 & \cdots & d^{n-3}y_1 \\ \vdots & & \vdots \\ >y_i & \cdots & d^{n-3}y_i< \\ \vdots & & \vdots \\ y_{n-1} & \cdots & d^{n-3}y_{n-1} \end{vmatrix}$$

$$=-\sum_{j=1}^{n} (-1)^{j-1} \left( \sum_{i=1}^{n-1} (-1)^{i-1}(d^{j-1}y_i) \begin{vmatrix} y_1 & \cdots & d^{n-3}y_1 \\ \vdots & & \vdots \\ >y_i & \cdots & d^{n-3}y_i< \\ \vdots & & \vdots \\ y_{n-1} & \cdots & d^{n-3}y_{n-1} \end{vmatrix} \right.$$

$$\left. \begin{vmatrix} x_1 & \cdots & \overset{\vee}{d^{j-1}x_1} & \cdots & d^{n-1}x_1 \\ \vdots & & \vdots & & \vdots \\ x_{n-1} & \cdots & \underset{\wedge}{d^{j-1}x_{n-1}} & \cdots & d^{n-1}x_{n-1} \end{vmatrix} \right).$$

$$=-\sum_{j=1}^{n} (-1)^{j-1} \left( \sum_{i=1}^{n-1} \begin{vmatrix} 0 & y_1 & \cdots & d^{n-3}y_1 \\ \vdots & \vdots & & \vdots \\ d^{j-1}y_i & y_i & \cdots & d^{n-3}y_i \\ \vdots & \vdots & & \vdots \\ 0 & y_{n-1} & \cdots & d^{n-3}y_{n-1} \end{vmatrix} \right.$$

$$\left. \begin{vmatrix} x_i & \cdots & \overset{\vee}{d^{j-1}x_1} & \cdots & d^{n-1}x_1 \\ \vdots & & \vdots & & \vdots \\ x_{n-1} & \cdots & \underset{\wedge}{d^{j-1}x_{n-1}} & \cdots & d^{n-1}x_{n-1} \end{vmatrix} \right)$$

$$
=-\sum_{j=1}^{n}(-1)^{j-1}\begin{vmatrix} d^{j-1}y_1 & y_1 & \cdots & d^{n-3}y_1 \\ \vdots & \vdots & & \vdots \\ d^{j-1}y_i & y_i & \cdots & d^{n-3}y_i \\ \vdots & \vdots & & \vdots \\ d^{j-1}y_{n-1} & y_{n-1} & \cdots & d^{n-3}y_{n-1} \end{vmatrix}
$$

$$
\left.\begin{vmatrix} x_1 & \cdots & \overset{\vee}{d^{j-1}x_1} & \cdots & d^{n-1}x_1 \\ \vdots & & \vdots & & \vdots \\ x_{n-1} & \cdots & \underset{\wedge}{d^{j-1}x_{n-1}} & \cdots & d^{n-1}x_{n-1} \end{vmatrix}\right)
$$

$$
=\begin{vmatrix} y_1 & \cdots & d^{n-3}y_1 & d^{n-1}y_1 \\ \vdots & & \vdots & \vdots \\ y_i & \cdots & d^{n-3}y_i & d^{n-1}y_i \\ \vdots & & \vdots & \vdots \\ y_{n-1} & \cdots & d^{n-3}y_{n-1} & d^{n-1}y_{n-1} \end{vmatrix}\begin{vmatrix} x_1 & \cdots & d^{n-2}x_1 \\ \vdots & & \vdots \\ x_{n-1} & \cdots & d^{n-2}x_{n-1} \end{vmatrix}
$$

$$
-\begin{vmatrix} y_1 & \cdots & d^{n-2}y_1 \\ \vdots & & \vdots \\ y_{n-1} & \cdots & d^{n-2}y_{n-1} \end{vmatrix}\begin{vmatrix} x_1 & \cdots & d^{n-3}x_1 & d^{n-1}x_x \\ \vdots & & \vdots & \vdots \\ x_i & \cdots & d^{n-3}x_i & d^{n-1}x_i \\ \vdots & & \vdots & \vdots \\ x_{n-1} & \cdots & d^{n-3}x_{n-1} & d^{n-1}x_{n-1} \end{vmatrix}
$$

$$
=\begin{vmatrix} x_1 & \cdots & d^{n-2}x_1 \\ \vdots & & \vdots \\ x_{n-1} & \cdots & d^{n-2}x_{n-1} \end{vmatrix} d \begin{vmatrix} y_1 & \cdots & d^{n-2}y_1 \\ \vdots & & \vdots \\ y_{n-1} & \cdots & d^{n-2}y_{n-1} \end{vmatrix}
$$

$$
-\begin{vmatrix} y_1 & \cdots & d^{n-2}y_1 \\ \vdots & & \vdots \\ y_{n-1} & \cdots & d^{n-2}y_{n-1} \end{vmatrix} d \begin{vmatrix} x_1 & \cdots & d^{n-2}x_1 \\ \vdots & & \vdots \\ x_{n-1} & \cdots & d^{n-2}x_{n-1} \end{vmatrix}
$$

$$
=(\Phi_{\mathcal{L}}^{(n-1)}(x_1\wedge\cdots\wedge x_{n-1}))d(\Phi_{\mathcal{L}}^{(n-1)}(y_1\wedge\cdots\wedge y_{n-1}))
$$

$$
-(\Phi_{\mathcal{L}}^{(n-1)}(y_1\wedge\cdots\wedge y_{n-1}))d(\Phi_{\mathcal{L}}^{(n-1)}(x_1\wedge\cdots\wedge x_{n-1}))
$$

$$
=\Phi^{(2)}_{\omega_C^{\otimes(n-1)(n-2)/2}\otimes\mathcal{L}^{\otimes n-1}}(\Phi_{\mathcal{L}}^{(n-1)}([x_1\wedge\cdots\wedge x_{n-1}])
$$

$$
\wedge\Phi_{\mathcal{L}}^{(n-1)}([y_1\wedge\cdots\wedge y_{n-1}]))\,.
$$

We can calculate the second term in the first equation in the same way, and we are done. Q. E. D.

LEMMA 2. *If $V$ is an $(n+1)$-dimensional vector space, then we have an isomorphism*

$$
g:\wedge^2(\wedge^{n-1}V)\oplus\wedge^{n-3}V\longrightarrow\wedge^n V\otimes\wedge^{n-2}V\,.
$$

*The isomorphism g is given by*

$$g(x_1\wedge\cdots\wedge x_{n-1}\wedge y_1\wedge\cdots\wedge y_{n-1},\ \sigma)$$
$$=[x_1\wedge\cdots\wedge x_{n-1}]\wedge[y_1\wedge\cdots\wedge y_{n-1}]+\partial\wedge\sigma\,,$$

*where* $\partial=\sum_{i=1}^{n}(-1)^{n-i}e_0\wedge\cdots\wedge\hat{e}_i\wedge\cdots\wedge e_n\otimes e_i$ *and* $e_0,\cdots,e_n$ *make a basis of* $V$.

PROOF. If $e_0,\cdots,e_n\in V$ and $0\leqq i_0\leqq n,\cdots,0\leqq i_n\leqq n$, then

$$[e_{i_0}\wedge\cdots\wedge e_{i_{n-2}}]\wedge[e_{i_{n-1}}\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-3}}]$$
$$=(-1)^n\frac{1}{2}\{e_{i_0}\wedge\cdots\wedge e_{i_{n-2}}\wedge e_{i_{n-1}})\otimes(e_{i_0}\wedge\cdots\wedge e_{i_{n-3}}$$
$$-(e_{i_{n-1}}\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-3}}\wedge(-1)^{n-3}e_{i_{n-2}})\otimes(e_{i_0}\wedge\cdots\wedge e_{i_{n-3}})\{$$
$$=(-1)^n(e_{i_0}\wedge\cdots\wedge e_{i_{n-1}})\otimes(e_{i_0}\wedge\cdots\wedge e_{i_{n-3}}),$$

and

$$[e_i\wedge\cdots\wedge e_{i_{n-2}}]\wedge[e_{i_{n-1}}\wedge e_{i_n}\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}]$$
$$=(-1)^n\frac{1}{2}(e_{i_0}\wedge\cdots\wedge e_{i_{n-1}})\otimes(e_{i_n}\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-4}})$$
$$-(e_{i_0}\wedge\cdots\wedge e_{i_{n-2}}\wedge e_{i_n})\otimes(e_{i_{n-1}}\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-4}})$$
$$-(e_{i_{n-1}}\wedge e_{i_n}\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}\wedge(-1)^{n-}{}_{8}e_{i_{n-3}})\otimes(e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}\wedge e_{i_{n-3}})$$
$$-(e_{i_{n-1}}\wedge e_{i_n}\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}\wedge(-1)^{n-2}e_{i_{n-2}})\otimes(e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}\wedge e_{i_{n-3}})$$
$$=(-1)^n\frac{1}{2}(e_{i_0}\wedge\cdots\wedge e_{i_{n-1}}\otimes e_{i_n}-e_{i_0}\wedge\cdots\wedge e_{i_{n-2}}\wedge e_{i_n}\otimes e_{i_{n-1}}$$
$$-e_{i_0}\wedge\cdots\wedge e_{i_{n-3}}\wedge e_{i_{n-1}}\wedge e_{i_n}\otimes e_{i_{n-2}}$$
$$+e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}\wedge e_{i_{n-2}}\wedge e_{i_{n-1}}\wedge e_{i_n}\otimes e_{i_{n-3}})\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}$$
$$=(-1)^n\frac{1}{2}(e^*_{i_n}\otimes e_{i_n}+e^*_{i_{n-1}}\otimes e_{i_{n-1}}-e^*_{i_{n-2}}\otimes e_{i_{n-2}}-e^*_{i_{n-3}}\otimes e_{i_{n-3}})e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}$$

where $e^*_0,\cdots,e^*_n\in\wedge^nV\cong V^*$ is the dual basis of $e_0,\cdots,e_n$. Therefore

$$[e_{i_0}\wedge\cdots\wedge e_{i_{n-2}}]\wedge[e_{i_{n-1}}\wedge e_{i_n}\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}]$$
$$=(-1)^n\frac{1}{2}(e^*_{i_n}\otimes e_{i_n}+e^*_{i_{n-1}}\otimes e_{i_{n-1}}-e^*_{i_n{}^-{}_2}\otimes e_{i_{n-2}}-e^*_{i_{n3}}\otimes\partial_{i_{n-3}})\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}$$
$$=(-1)^n\Big(e^*_{i_n}\otimes e_{i_n}+e^*_{i_{n-1}}\otimes e_{i_{n-1}}-\frac{1}{2}\partial\Big)\wedge e_{i_0}\wedge\cdots\wedge e_{i_{n-4}}\,.$$

Therefore a basis of $\wedge^nV\otimes\wedge^{n-2}V$ is contained in $g(\wedge^{n-1}V)\oplus\wedge^{n-3}V)$. Hence $g$ is surjective. Moreover

$$\dim_k(\wedge^2(\wedge^{n-1}V)\oplus V)=(n+1)\frac{(n+1)n(n-1)}{3\cdot 2\cdot 1}\, dim_k(\wedge^n V\otimes\wedge^{n-2}V).$$

Hence $g$ is an isomorphism. Q.E.D.

LEMMA 3. *If* $\partial=\sum_{i=0}^{n}(-1)^{n-i}e_0\wedge\cdots\wedge\hat{e}_i\wedge\cdots\wedge e_n\in\wedge^n V\otimes V$ *where* $V$ *is an* $(n+1)$*-dimensional vector subspace of* $\Gamma(C, \mathcal{L})$ *and* $e_0, \cdots, e_n$ *make a basis of* $V$, *then*

$$\Phi_{\mathcal{L}}^{(n)}\cap id_V(\partial)=0.$$

PROOF. By the definition,

$$\begin{aligned}
\Phi_{\mathcal{L}}^{(n)}\cap id_V(\partial) &= \sum_{i=0}^{n}(-1)^{n-i}\Phi_{\mathcal{L}}^{(n)}(e_0\wedge\cdots\wedge\hat{e}_i\wedge\cdots\wedge e_n)e_i \\
&= \sum_{i=0}^{n}(-1)^{n-i}\begin{vmatrix} e_1 & \cdots & d^{n-1}e_1 \\ \vdots & & \vdots \\ >e_i & \cdots & d^{n-1}e_i< \\ \vdots & & \vdots \\ e_n & \cdots & d^{n-1}e_n \end{vmatrix} e_i \\
&= (-1)^n\sum_{i=0}^{n}\begin{vmatrix} 0 & e_1 & \cdots & d^{n-1}e_1 \\ \vdots & \vdots & & \vdots \\ e_i & e_i & \cdots & d^{n-1}e_i \\ \vdots & \vdots & & \vdots \\ 0 & e_n & \cdots & d^{n-1}e_n \end{vmatrix} \\
&= (-1)^n\begin{vmatrix} e_1 & e_1 & \cdots & d^{n-1}e_1 \\ \vdots & \vdots & & \vdots \\ e_i & e_i & \cdots & d^{n-1}e_i \\ \vdots & \vdots & & \vdots \\ e_n & e_n & \cdots & d^{n-1}e_n \end{vmatrix} \\
&= 0.
\end{aligned}$$

Therefore we have the result. Q.E.D.

## 2. Generalized Castelnuovo's lemma

In this section, we will give a generalization of Castelnuovo's lemmas in Wahl [18] (p. 86 Theorem 2.6.)

DEFINITION 3. Let $V$ be a vector subspace of $\Gamma(C, \mathcal{L})$ and let $s_1, \cdots, s_N \in V$ be a basis. For any $p\in C$, if

$$\operatorname{rank}\begin{pmatrix} s_1(p) & \cdots & d^n s_1(p) \\ \vdots & & \vdots \\ s_N(p) & \cdots & d^n s_N(p) \end{pmatrix} = n+1,$$

then we say that $V$ is $n$-immersive. In particular if $n=1$, then we say that $V$ is immersive.

DEFINITION 4. Let $V$ be a vector subspace of $\Gamma(C, \mathcal{L})$. If $\dim_k(V)=n$, then we call that $V$ is an $n$-net.

LEMMA 4. *Let $\mathcal{L}$ be an invertible sheaf on a curve $C$ of genus $g$. If* $\deg(\mathcal{L}) \geqq 2g+n$, *then a general $(n+2)$-dimensional subspace of $\Gamma(C, \mathcal{L})$ is an $n$-immersive $(n+2)$-net.*

PROOF. By the same argument of Hartshorne (see Hartshorne [9] p. 310 Porposition 3.5.), we have

$$\dim\Big(\bigcup_{p\in C}(\text{osculating tangent } n\text{-plaine at } p\in C\subset \boldsymbol{P}(\Gamma(C, \mathcal{L})^*))\Big) \leqq n+1\,.$$

This completes the proof. Q. E. D.

LEMMA 5. *Let $\mathcal{L}$ be an invertible sheaf on $C$, and let $V$ be a vector subspace of $\Gamma(C, \mathcal{L})$. If $V$ is an $(n-1)$-immersive, then a sheaf homomorphism*

$$\wedge^n V\otimes \mathcal{O}_C \longrightarrow \omega_C^{\otimes n(n-1)/2}\otimes \mathcal{L}^{\otimes n}$$

*induced by $\Phi_V^{(n)}$ is surjective.*

PROOF. Let $s_1, \cdots, s_n \in V$. Then we have

$$\Phi_V^{(n)}(s_1\wedge \cdots \wedge s_n)(p) = \begin{vmatrix} s_1(p) & \cdots & d^{n-1}s_1(p) \\ \vdots & & \vdots \\ s_n(p) & \cdots & d^{n-1}s_n(p) \end{vmatrix}$$

for every $p\in C$. As $V$ is $(n-1)$-immersive, therefore this completes the proof. Q. E. D.

The following definition and lemma are famous (see Hizeburch [11]).

DEFINITION 5. Let $\mathcal{W}$ and $\mathcal{W}'$ be locally free sheaves and let $\mathcal{F}$ be an invertible sheaf. Suppose that a sequence $0\to\mathcal{W}' \xrightarrow{\varphi} \mathcal{W} \xrightarrow{\phi} \mathcal{F}\to 0$ is exact. The the homomorphism $\wedge^p\phi\colon \wedge^p\mathcal{W}\to\mathcal{F}\otimes\wedge^{p-1}\mathcal{W}'$ is defined by

$$\wedge^p\psi(w_1\wedge\cdots\wedge w_p)=\sum_{i=0}^{p}(-1)^i\psi(w_i)w_1\wedge\cdots\wedge\hat{w}_i\wedge\cdots\wedge w_p\,.$$

LEMMA 6. *The above* $\wedge^p\psi$ *is surjective and induces an exact sequence*

$$0\longrightarrow\wedge^p\mathcal{W}'\xrightarrow{\wedge^p\varphi}\wedge^p\mathcal{W}\xrightarrow{\wedge^p\psi}\mathcal{F}\otimes\wedge^{p-1}\mathcal{W}\longrightarrow 0\,.$$

PROOF. See Hirzeburch [11] p. 55 Theorem 4.1.3. Q. E. D.

Let $V$ be a finite dimensional vector space and $V^*$ be a dual vector space of $V$. Let $\partial\in V\otimes V^*\cong\mathrm{Hom}_k(V, V)$ be an element corresponding to the identity id. We consider the Koszul complex

$$\mathcal{K}:0\longrightarrow\mathcal{O}_P\xrightarrow{\partial}V^*\otimes\mathcal{O}_P(1)\xrightarrow{\partial}\wedge^2V^*\otimes\mathcal{O}_P(2)\xrightarrow{\partial}\cdots$$

defined by $\partial(f)=f\wedge\partial$ for $f\in\wedge^iV^*\otimes\mathcal{O}_P(i)$ $(i=1, 2, \cdots)$ where $\boldsymbol{P}=\boldsymbol{P}(V^*)$.

LEMMA 7. *The above Koszul complex K is exact and the image sheaf*

$$\mathrm{im}(\partial)=\mathrm{im}(\wedge^pV^*\otimes\mathcal{O}_P(p)\xrightarrow{\partial}\wedge^{p+1}V^*\otimes\mathcal{O}_P(p+1))$$

*is isomorphic to* $\wedge^pT_P$, *where* $T_P$ *is a tangent sheaf on* $\boldsymbol{P}$.

PROOF. If $p=1$, this is obvious (for example, see Hartshorne [9] p. 176). If $p$ is arbitary, then this follows directly from Lemma 6. Q. E. D.

DEFINITION 6. Let $V$ be an $(n-1)$-immersive $(n+1)$-net. A locally free sheaf $Q_n$ is given by

$$Q_n=\ker(\wedge^nV\otimes\mathcal{O}_C\to\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n})\,.$$

REMARK 1. As $\Phi_{\mathcal{L}}^{(n)}:\wedge^nV\otimes\mathcal{O}_C\to\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n}$ is surjective, $Q_n$ is a locally free sheaf of rank $n$.

Let $V$ be an $(n-1)$-immersive $(n+1)$-net. As $V^*\cong\wedge^nV$, $\partial\in V^*\otimes V$ is given by

$$\partial=\sum_{i=0}^{n}(-1)^{n-i}e_0\wedge\cdots\wedge\hat{e}_i\wedge\cdots\wedge e_n\otimes e_i\,.$$

Therefore $\partial$ is contained in $\Gamma(C, Q_n)$ by Lemma 3 in §1. As $V$ is $(n-1)$-immersive, $V$ is base point free, so $V$ defines a morphism $C\to\boldsymbol{P}(V^*)$. We restrict the above Koszul complex $\mathcal{K}$ to $C\to\boldsymbol{P}(V^*)$ and we have the following exact sequence:

$$(\mathrm{B}): 0 \longrightarrow \mathcal{O}_C \stackrel{\partial}{\longrightarrow} V^*\otimes\mathcal{L} \stackrel{\partial}{\longrightarrow} \wedge^2 V^*\otimes\mathcal{L} \stackrel{\partial}{\longrightarrow} \cdots.$$

As $\partial\in\Gamma(C, Q_n)$, $\partial$ defines the following complex:

$$(\mathrm{A}): 0 \longrightarrow \mathcal{O}_C \stackrel{\partial}{\longrightarrow} Q_n\otimes\mathcal{L} \stackrel{\partial}{\longrightarrow} \wedge^2 Q_n\otimes\mathcal{L} \stackrel{\partial}{\longrightarrow} \cdots.$$

where the map $\partial$ is defined qy $\partial(f)=f\wedge\partial$. Moreover we consider a complex $(C)=(A)\otimes\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n}$:

$$0 \longrightarrow \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n} \stackrel{\partial}{\longrightarrow} Q_n\otimes\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n+1}$$
$$\stackrel{\partial}{\longrightarrow} \wedge^2 Q_n\otimes\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n+2} \cdots$$

By Definition 6, we have the following short exact sequence:

$$(1): 0 \longrightarrow Q_n\otimes\mathcal{L} \stackrel{\psi}{\longrightarrow} V^*\otimes\mathcal{L} \longrightarrow \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n+1} \longrightarrow 0.$$

Therefore we have the following short exact sequences by Lemma 6:

$$(\mathrm{i}): 0 \to \wedge^i Q_n\otimes\mathcal{L}^{\otimes i} \stackrel{\wedge^i\varphi}{\longrightarrow} \wedge^i V^*\otimes\mathcal{L}^{\otimes i} \stackrel{\wedge^i\psi}{\longrightarrow} \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n+i}\otimes\wedge^{i-1}Q_n \to 0$$

where $i=1, 2, \cdots$. Hence we have the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \stackrel{ia}{\longrightarrow} & \mathcal{O}_C & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & (I_1) & \downarrow & (II_1) & \downarrow & & \\
0 & \longrightarrow & Q_n\otimes\mathcal{L} & \stackrel{\varphi}{\longrightarrow} & V^*\otimes\mathcal{L} & \stackrel{\psi}{\longrightarrow} & \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n+1} & \longrightarrow & 0 \\
 & & \downarrow & (I_2) & \downarrow & (II_2) & \downarrow & & \\
0 & \longrightarrow & \wedge^2 Q_n\otimes\mathcal{L}^{\otimes 2} & \stackrel{\wedge^2\varphi}{\longrightarrow} & \wedge^2 V^*\mathcal{L}^{\otimes 2} & \stackrel{\wedge^2\psi}{\longrightarrow} & \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n+2}\otimes Q_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & (I_n) & \downarrow & (II_n) & \downarrow & & \\
0 & \longrightarrow & \wedge^n Q_n\otimes\mathcal{L}^{\otimes n} & \stackrel{\wedge^n\varphi}{\longrightarrow} & \wedge^n V^*\otimes\mathcal{L}^{\otimes n} & \stackrel{\wedge^n\psi}{\longrightarrow} & \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes 2n}\otimes\wedge^{n-1}Q_n & \longrightarrow & 0 \\
 & & \downarrow & (I_{n+1}) & \downarrow & (II_{n+1}) & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \wedge^{n+1}V^*\otimes\mathcal{L}^{\otimes n+1} & \stackrel{\sim}{\longrightarrow} & \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes 2n+1}\otimes\wedge^n Q_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 
\end{array}$$

where $\psi=\Phi_{\mathcal{L}}^{(n)}$.

LEMMA 8. *The above diagram is commutative.*

PROOF. It is easy that $(I_i)$ part is commutative for $i=1, 2, \cdots$. Because

$$)(\wedge^{i-1}\varphi)(x_i\wedge\cdots\wedge x_{i-1}))\wedge\partial=(\wedge^i\varphi)(x_1\wedge\cdots\wedge x_{i-1}\wedge\partial)$$

by the definition of (A) and (B). We now show that $(II_i)$ part is commutative for $i=1, z, \cdots$. This is equivalent to

$$((\wedge^{i-1}\psi)(x_i\wedge\cdots\wedge x_{i-1}))\wedge\partial=(\wedge^i\psi)(x_1\wedge\cdots\wedge x_{i-1}\wedge\partial)$$

for $i=1, 2, \cdots$. As

$$(\wedge^{i-1}\psi)(x_i\wedge\cdots\wedge x_{i-1})\wedge\partial=\sum_{j=1}^{i-1}(-1)^j\psi(x_j)x_1\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_{i=1}\wedge\partial,$$

and

$$\begin{aligned}&(\wedge^i\psi)(x_1\wedge\cdots\wedge x_{i-1}\wedge\partial)\\&=\sum_{j=1}^{i-1}(-1)^j\psi(x_j)x_1\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_{i-1}\wedge\partial+(-1)^i\psi(\partial)x_1\wedge\cdots\vee x_{i-1}\\&=\sum_{j=1}^{i-1}(-1)^j\psi(x_j)x_1\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_{i-1}\wedge\partial,\end{aligned}$$

we have the commutativity of $(II_i)$ by Lemma 3. Q. E. D.

LEMMA 9. *The complexes* (A) *and* (B) *are exact.*

PROOF. As the exact sequence (1) splits locally, there is a local section $\iota_1$ of $\varphi$. We put $\iota_i=\wedge^i\iota$ $(i=0, 1, \cdots)$ and put $\varphi_i=\wedge^i\varphi$ $(i=0, 1, \cdots)$. By the definition of $\iota_1$, it is clear that $\partial\iota_0=\iota_1\partial$. Therefore we have $\iota_{m+1}\partial=\partial\iota_m$ and $\iota_m\varphi_m=id$. Hence the exact sequence of complexes

$$0\longrightarrow(\mathrm{A})\longrightarrow(\mathrm{B})\longrightarrow(\mathrm{C})\longrightarrow 0$$

splits locally. As (B) is exact, therefore (A) and (C) are exact complexes. Q. E. D.

LEMMA 10. *Let $V$ be a subspace of $\Gamma(C, \mathcal{L})$ with* $\dim_k V\geqq n+1$. *If $V$ is $(n+1)$-immersive, then the subspace*

$$\Phi_{\mathcal{L}}^{(n-1)}(\wedge^{n-1}V)\subset\Gamma(C, \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n-1})$$

*is immersive.*

PROOF. Let $x_1, \cdots, x_{n-1}\in V$. Then

$$\Phi^{(2)}_{\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{n-1}}(\Phi^{(n-1)}_{\mathcal{L}}(x_1\wedge\cdots\wedge x_{n-1})\wedge\Phi^{(n-1)}_{\mathcal{L}}(x_n\wedge x_1\wedge\cdots\wedge x_{n-2}))$$

$$=\begin{vmatrix} x_1 & \cdots & d^{n-1}x_1 \\ \vdots & & \vdots \\ x_n & \cdots & d^{n-1}x_n \end{vmatrix}\begin{vmatrix} x_1 & \cdots & d^{n-3}x_1 \\ \vdots & & \vdots \\ x_{n-2} & \cdots & d^{n-3}x_{n-2} \end{vmatrix}.$$

Hence for every $p\in C$ there are some $v$, $v'\in\wedge^{n-1}V$ such that

$$\Phi^{(2)}_{\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n-1}}(\Phi^{(n-1)}_{\mathcal{L}}(v)\wedge\Phi^{(n-1)}_{\mathcal{L}}(v'))(p)\neq 0,$$

because $V$ is $(n-1)$-immersive. Therefore we have the result. Q.E.D.

LEMMA 11. *Let $\mathcal{E}$ be a locally free sheaf on $C$, let $V$ be a subspace of $\Gamma(C, \mathcal{E})$ and let $K$ be a function field of $C$. If the canonical map $V\otimes_k K\to \mathcal{E}\otimes_{\mathcal{O}_C}K$ is injective, then $V\otimes_k\mathcal{O}_C$ is subsheaf of $\mathcal{E}$.*

PRFOO. This is obvious. Q.E.D.

LEMMA 12. *Let $V$ be a subspace of $\Gamma(C, \mathcal{L})$ with $\dim_k V=n+1$. If $W_0$ is a general 3-dimensional linear subspace of $\wedge^{n-1}V$, then a composition of two canonical maps*

$$\wedge^2 W_0\otimes\mathcal{O}_C \longrightarrow \wedge^n V\otimes\wedge^{n-2}V\otimes\mathcal{O}_C \longrightarrow T_{P^n|C}\otimes\omega_C^{\otimes(n-2)(n-3)/2}\otimes\mathcal{L}^{\otimes n-3}$$

*is injective.*

PROOF. This condition is an open condition. Therefore we construct an example of $W_0$ which satisfies the property of this lemma. Let $x_1, \cdots, x_{n+1}$ be a basis of $V$ and let $W_0=[x_1\wedge\cdots\wedge x_{n-1}, x_n\wedge x_1\wedge\cdots< x_{n-2}, x_{n+1}\wedge x_1\wedge\cdots\wedge x_{n-2}]$. By Lemma 2, a basis of $\Phi^{(n)}_{\mathcal{L}}\cap\Phi^{(n-2)}_{\mathcal{L}}(\wedge^2 W_0)$ is $(x_1\wedge\cdots\wedge x_n)\otimes(x_1\wedge\cdots\wedge x_{n-2})$, $(x_1\wedge\cdots\wedge x_{n-1}\wedge x_{n+1})\otimes(x_1\wedge\cdots\wedge x_{n-2})$, $(x_1\wedge\cdots\wedge x_{n-2}\wedge x_n\wedge x_{n+1})\otimes(x_1\wedge\cdots\wedge x_{n-2})$. Therefore we can construct an example of $W_0$. Q.E.D.

LEMMA 13. *Let $V$ be $(n-1)$-immersive and $\dim_k(V)\geqq n+1$. If $W_0$ is a general 3-dimensional linear subspace of $\wedge^{n-1}V$, then $\Phi^{(n-1)}_{\mathcal{L}}(W_0)$ is an immersive net.*

PROOF. This condition is also an open condition. Therefore this lemma follows from Lemma 10. Q.E.D.

LEMMA 14. *If $V\subset\wedge^{n-1}V$ is an $(n-1)$-immersive $(n+1)$-net, then there is an $n$-dimensional subspace $W\subset\wedge^2(\wedge^{n-1}V)$ such that*

(1) *there is $W_0\subset\wedge^{n-1}V$ such that $\dim_k W_0=3$, $\wedge^2 W_0\subset W$ and $\Phi^{(2)}_{\mathcal{L}}(W_0)$ is im-*

*mersive net,*

(2) $\wedge^2 W_0 \otimes \mathcal{O}_C \to \wedge^n V \otimes \wedge^{n-2} V \otimes \mathcal{O}_C \leftarrow T_{P^n|C} \otimes \omega_C^{\otimes (n-2)(n-3)/2} \otimes \mathcal{L}^{\otimes n-3}$ *is injective.*

PROOF. By Lemma 12 and Lemma 13, there is a 3-dimensional subspace $W_0 \subset \wedge^{n-1} V$ such that

(1) there is $W_0 \subset \wedge^{n-1} V$ such that $\dim_k W_0 = 3$ and $\Phi_{\mathcal{L}}^{(2)}(W_0)$ is immersive net,

(2) $\wedge^2 W_0 \otimes \mathcal{O}_C \to \wedge^n V \otimes \wedge^{n-2} V \otimes \mathcal{O}_C \to T_{P^n|C} \otimes \omega_{C(n-2)(n-3)/2}^{\otimes} \otimes \mathcal{L}^{\otimes n-3}$ is injective.

As $0 \to \mathcal{O}_C \to \wedge^n V \otimes \mathcal{L} \to T_{P^n|C} \to 0$ is some $\overline{W} \subset \wedge^n V \otimes \wedge^{n-2} V$ such that $(\wedge^2 W_0 \oplus \overline{W}) \otimes K = T_{P^n|C} \otimes_{\mathcal{O}_C} \otimes K$. We put $W = \wedge^2 W_0 \oplus \overline{W}$. By Lemma 11, we get the result. Q. E. D.

DEFINITION 7. Let $V \subset \Gamma(C, \mathcal{L})$ be an $(n-1)$-immersive $(n+1)$-net. By the following exact commutative diagrams, we define a locally free sheaf $\mathcal{E}_n$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_C & \xrightarrow{id} & \mathcal{O}_C & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Q_n \otimes \mathcal{L} & \xrightarrow{\varphi} & V^* \otimes \mathcal{L} & \xrightarrow{\phi} & \omega_C^{\otimes n(n-1)/2} \otimes \mathcal{L}^{\otimes n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow id & & \\
0 & \longrightarrow & \mathcal{E}_n & \longrightarrow & T_{P^n|C} & \longrightarrow & \omega_C^{\otimes n(n-1)/2} \otimes \mathcal{L}^{\otimes n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Let $W_0 \subset \wedge^{n-1} V$ and $W \subset \wedge^2(\wedge^{n-1} V)$ be vector subspaces in Lemma 14. By the following exact sequences, we define locally free shəaves $Q_{W_0}$ and $Q_0$:

$$0 \longrightarrow Q_0 \longrightarrow W \otimes \mathcal{O}_C \xrightarrow{\Phi^{(2)}_{\omega_C^{\otimes (n-1)(n-2)/2} \otimes \mathcal{L}^{\otimes n+1}}} \omega_C^{\otimes n^2-3n+3} \otimes \mathcal{L}^{\otimes 2n-2} \longrightarrow 0$$

and

$$0 \longrightarrow Q_{W_0} \longrightarrow \wedge^2 W_0 \otimes \mathcal{O}_C \xrightarrow{\Phi^{(2)}_{\omega_C^{\otimes (n-1)(n-2)/2} \otimes \mathcal{L}^{\otimes n+1}}} \omega_C^{\otimes n^2-3n+3} \otimes \mathcal{L}^{\otimes 2n-2} \longrightarrow 0.$$

PROPOSITION 1. *We have the following exact sequences:*

(A): $0 \longrightarrow \mathcal{O}_C \longrightarrow Q_n \otimes \mathcal{L} \longrightarrow \mathcal{E}_n\, 0$

(B): $0 \longrightarrow Q_0 \longrightarrow \mathcal{E}_n \otimes \omega_C^{\otimes (n-2)(n-3)/2} \otimes \mathcal{L}^{\otimes n-3} \longrightarrow (\textit{torsion sheaf}) \longrightarrow 0$

$$(\mathrm{C}):\ 0 \longrightarrow Q_{W_0} \longrightarrow Q_0 \longrightarrow \overline{W}\otimes\mathcal{O}_C \longrightarrow 0$$

$$(\mathrm{D}):\ 0 \longrightarrow (\omega_C^{\otimes(n-1)(n-2)/2}\otimes\mathcal{L}^{\otimes n-1})^{\otimes -1} \longrightarrow Q_{W_0}$$
$$\longrightarrow (\omega_C^{\otimes 1+((n-1)(n-2))/2}\otimes\mathcal{L}^{\otimes u-1})^{\otimes -1} \longrightarrow 0$$

*where* $\overline{W}=W/(\wedge^2 W_0)$.

PROOF. The sequence (A) is given by the definition of $\mathcal{E}_n$. Now we consider the sequence (B). By Lemma 14, we have the following commutative exact diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{E}_n\otimes\omega_C^{\otimes(n-2)(n-3)/2}\otimes\mathcal{L}^{\otimes n-3} & \longrightarrow & T_{P^n|C}\otimes\omega_C^{\otimes(n-2)(n-3)/2}\otimes\mathcal{L}^{\otimes n-3} & \longrightarrow & \omega_C^{\otimes n^2-3n+3}\otimes\mathcal{L}^{\otimes 2n-2} & \longrightarrow 0 \\
 & & & \uparrow \mathit{inclusion} & & \uparrow \mathit{id} & \\
0 \longrightarrow & Q_0 & \longrightarrow & W\otimes\mathcal{O}_C & \xrightarrow{\Phi^{(2)}_{\omega_C^{\otimes(n-1)(n-2)/2}\otimes\mathcal{L}^{\otimes n+1}}} & \omega_C^{\otimes n^2-3n+3}\otimes\mathcal{L}^{\otimes 2n-2} & \longrightarrow 0
\end{array}$$

Therefore there is an injective morphism:

$$Q_0 \longrightarrow \mathcal{E}_n\otimes\omega_C^{\otimes(n-2)(n-3)/2}\otimes\mathcal{L}_{\otimes}^{n-3}.$$

As $Q_0$ and $\mathcal{E}_n\otimes\omega_C^{\otimes(n-2)(n-3)/2}\otimes\mathcal{L}^{\otimes n-3}$ are both locally free invertible sheaves of rank $n-1$, hence we have (B). By the exact sequence

$$0 \longrightarrow Q_0 \longrightarrow W\otimes\mathcal{O}_C \longrightarrow \omega_C^{\otimes n^2-3n+3}\otimes\mathcal{L}^{\otimes 2n-2} \longrightarrow 0,$$

the following exact commutative diagram is obtained:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_{W_0} & \longrightarrow & \wedge^2 W_0\otimes\mathcal{O}_C & \longrightarrow & \omega_C^{\otimes n^2-3n+3}\otimes\mathcal{L}^{\otimes 2n-2} & \longrightarrow & 0 \\
 & & & & \downarrow & & \mathit{id}\downarrow & & \\
0 & \longrightarrow & Q_0 & \longrightarrow & W\otimes\mathcal{O}_C & \longrightarrow & \omega_C^{\otimes n^2-3n+3}\otimes\mathcal{L}^{\otimes 2n-2} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{W}\otimes\mathcal{O}_C & \xrightarrow[\mathit{id}]{} & \overline{W}\otimes\mathcal{O}_C & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Therefore we have the exact sequence (C) by the snake lemma. As $W_0$ is an immersive net, the exact sequence (D) is obtained by the definition of $Q_2$ (see

Wahl [18] p. 85 Lemma 2.4.). Q. E. D.

By these lemmas and proposition, we have the following theorem which is a generalization of Castelnuovo's lemma in [18].

THEOREM 1. *If* $V \subset \Gamma(C, \mathcal{L})$ *is an* $(n-1)$*-immersive* $(n+1)$*-net and if* $\mathcal{F}$ *is a coherent* $\mathcal{O}_C$*-module on* $C$ *such that*

$$H^1(C, \mathcal{L}^{\otimes -2} \otimes \mathcal{F}) = 0,$$

$$H^1(C, (\omega_C^{\otimes (n-2)(n-3)/2} \otimes \mathcal{L}^{\otimes n-1})^{\otimes -1} \otimes \mathcal{F}) = 0,$$

$$H^1(C, (\omega_C^{\otimes n^2-4n+4} \otimes \mathcal{L}^{\otimes 2n-2})^{\otimes -1} \otimes \mathcal{F}) = 0,$$

$$H^1(C, (\omega_C^{\otimes n^2-4n+5} \otimes \mathcal{L}^{\otimes 2n-2})^{\otimes -1} \otimes \mathcal{F}) = 0,$$

*then a canonical map*

$$\wedge^n V \otimes \Gamma(C, \mathcal{L}^{\otimes -1} \otimes \mathcal{F}) \longrightarrow \Gamma(C, \omega_C^{\otimes n(n-1)/2} \otimes \mathcal{L}^{\otimes n-1} \otimes \mathcal{F})$$

*which is induced by an* $n$*-Wahl map* $\Phi_{\mathcal{L}}^{(n)}$ *is surjective.*

PROOF. We put $(A)_1 = (A) \otimes \mathcal{L}^{\otimes -2} \otimes \mathcal{F}$, $(B)_1 = (B) \otimes (\omega_C^{\otimes (n-2)(n-3)/2} \otimes \mathcal{L}^{\otimes n-1})^{\otimes -1} \otimes \mathcal{F}$, $(C)_1 = (C) \otimes (\omega_C^{\otimes (n-2)(n-3)/2} \otimes \mathcal{L}^{\otimes n-1})^{\otimes -1} \otimes \mathcal{F}$, $(D)_1 = (D) \otimes (\omega_C^{\otimes (n-2)(n-3)/2} \otimes \mathcal{L}^{\otimes n-1})^{\otimes -1} \otimes \mathcal{F}$. By $(B)_1$, we have the following two exact sequences:

$$0 \longrightarrow (\mathit{kernel}) \longrightarrow Q_0 \otimes (\omega_C^{\otimes (n-2)(n-3)/2} \otimes \mathcal{L}^{\otimes n-3})^{\otimes -1} \mathcal{F} \longrightarrow (\mathit{image}) \longrightarrow 0$$

$$0 \longrightarrow (\mathit{image}) \longrightarrow \mathcal{E}_n \otimes \mathcal{L}^{\otimes -2} \otimes \mathcal{F} \longrightarrow (\mathit{torsion\ sheaf}) \longrightarrow 0.$$

By the exact sequence $(D)_1$ and an assumption, we have that

$$H^1(C, Q_{W_0} \otimes (\omega_C^{\otimes (n-2)(n-3)/2} \otimes \mathcal{L}^{\otimes n-1} \otimes \mathcal{F}) = 0.$$

Therefore we have

$$H^1(C, Q_0 \otimes (\omega_C^{\otimes (n-2)(n-3)} \otimes \mathcal{L}^{\otimes n-1})^{\otimes -1} \otimes \mathcal{F}) = 0$$

by the exact sequence $(C)_1$. Hence

$$H^1(C, \mathcal{E}_n \otimes \mathcal{L}^{\otimes -2} \otimes \mathcal{F}) = 0$$

by the exact sequence $(B)_1$ because $C$ is one-dimensional. And

$$H^1(C, Q_n \otimes \mathcal{L}^{\otimes -1} \otimes \mathcal{F}) = 0$$

by the sequence $(A)_1$. Therefore we have the result. Q. E. D.

## 3. Application

In this section, we consider the surjectivity of $\Phi_{\mathcal{L}}^{(n)}$. Now we prepare several definitions and lemmas.

DEFINITION 8. Let $\mathcal{L}_1, \cdots, \mathcal{L}_n$ be invertible sheaves on $C$. We define a vector subspace $\mathcal{R}(\mathcal{L}_1, \cdots, \mathcal{L}_n) \subset \Gamma(C, \mathcal{L}_1) \otimes \cdots \otimes \Gamma(C, \mathcal{L}_n)$ by

$$\left\{ \sum_j x_j^{(1)} \otimes \cdots \otimes x_j^{(n)} : \begin{array}{l} \sum_j x_j^{(\sigma(1))}(d^{i_1} x_j^{(\sigma(2))}) \cdots (d^{i_{n-1}} x_j^{(\sigma(n))}) = 0, \\ 0 \leqq i_1 \leqq 1, \cdots, 0 \leqq i_{n-1} \leqq n-1 \\ \text{and } \sum_{s=1}^{n-1} i_s < \frac{n(n-1)}{2}, \text{ for any } \sigma \in \mathfrak{S}_n \end{array} \right\}.$$

LEMMA 15. *The above space* $\mathcal{R}(\mathcal{L}_1, \cdots, \mathcal{L}_n)$ *is well-defined.*

PROOF. Let $f_i$ be a transition function of an invertible sheaf $\mathcal{L}_i$ and let $\tilde{x}_j^{(i)} = x_j^{(i)} f_i$ where $x_j^i$ and $\tilde{x}_j^i$ are local sections of $\mathcal{L}_i$ $(i=1, \cdots)$. We assume that $\sum_j x_j^{(1)} \otimes \cdots \otimes x_j^{(n)}$ satisfies

$$\sum_j x_j^{(\sigma(1))}(d^{i_1} x_j^{(\sigma(2))}) \cdots (d^{i_{n-1}} x_j^{(\sigma(n))}) = 0,$$

for $0 \leqq i_1 \leqq 1, \cdots, 0 \leqq i_{n-1} \leqq n-1$, $\sum_{s=1}^{n-1} i_s < \frac{n(n-1)}{2}$ and $\sigma \in \mathfrak{S}_n$. As

$$d^i(fx) = \sum_{i=0}^{i} \binom{i}{s} (d^s x)(d^{n-s} f),$$

we have that

$$\sum_j \tilde{x}_j^{(\sigma(1))}(d^{i_1} \tilde{x}_j^{(\sigma(2))}) \cdots (d^{i_{n-1}} \tilde{x}_j^{((\sigma(n))}$$

$$= \sum_{s_1=0}^{i_1} \cdots \sum_{s_{n-1}=0}^{i_{n-1}} \left( \sum_j x_j^{(\sigma(1))}(d^{i_1} x_j^{(\sigma(2))}) \cdots (d^{i_{n-1}} x_j^{(\sigma(n))}) \right)$$

$$\cdot \binom{i_1}{s_1} \cdots \binom{i_{n-1}}{s_{n-1}} f_{\sigma(1)}(d^{i_1 - s_1} f_{\sigma(\sigma)}) \cdots (d^{i_{n-1} - s_{n-1}} f_{\sigma(n)})$$

Therefore we have the result. Q.E.D.

REMARK 2. If $n=2$, then $\mathcal{R}(\mathcal{L}_1, \mathcal{L}_2)$ in Definition 8 is the kernel of a cup product map

$$\Gamma(C, \mathcal{L}_1) \otimes \Gamma(C, \mathcal{L}_2) \longrightarrow \Gamma(C, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

because

$$\sum_j x_j^{(1)} x_j^{(2)} = 0$$

in the only relation in Definition 8.

DEFINITION 9. Let $\tau=\sum x_j^{(1)}\otimes\cdots\otimes x_j^{(n)}\in\mathcal{R}(\mathcal{L}_1,\cdots,\mathcal{L}_n)$. We define the $n$-Wahl map

$$\Phi_{\mathcal{L}_1,\cdots,\mathcal{L}_n}:\mathcal{R}(\mathcal{L}_1,\cdots,\mathcal{L}_n)\longrightarrow\Gamma(C,\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_n)$$

by

$$\Phi_{\mathcal{L}_1,\cdots,\mathcal{L}_n}(\tau)=\Phi_{\mathcal{L}_1,\cdots,\mathcal{L}_n}\Big(\sum_j x_j^{(1)}\otimes\cdots\otimes x_j^{(n)}\Big)$$

$$=\frac{1}{n!}\sum_j\Big(\sum_{\sigma\in\mathfrak{S}_n}sgn(\sigma)x_j^{(\sigma(1))}(dx_j^{(\sigma(2))})\cdots(d^{n-1}x_j^{(\sigma(n))})\Big).$$

LEMMA 16. *The above $\Phi_{\mathcal{L}_1,\cdots,\mathcal{L}_n}$ is well-defined.*

PROOF. Let $f_i$ be a transition function of an invertible sheaf $\mathcal{L}_i$ and let $\tilde{x}_j^{(i)}=x_j^{(i)}f_i$ $(i=1,\cdots)$. Then

$$\sum_j\Big(\sum_{\sigma\in\mathfrak{S}_n}sgn(\sigma)\tilde{x}_j^{(\sigma(1))}(d\tilde{x}_j^{(\sigma(2))})\cdots(d^{n-1}\tilde{x}_j^{(\sigma(n))})\Big)$$

$$=\sum_j\Big(\sum_{\sigma\in\mathfrak{S}_n}sgn(\sigma)x_j^{(\sigma(1))}(dx_j^{(\sigma(2))})\cdots(d^{n-1}x_j^{(\sigma(n))})\Big)(f_1\cdots f_n)$$

$$+\sum_j\Big(\sum_{\sigma\in\mathfrak{S}_n}\Big(\sum_{u_2+v_2=1,v_2\neq 1}\cdots\sum_{u_n+v_n=n-1,v_n\neq n-1}sgn(\sigma)$$

$$f_{\sigma(1)}x_j^{(\sigma(1))}\binom{1}{u_2}(d^{u_2}f_{\sigma(2)})(d^{v_2}x_j^{(\sigma(2))})\cdots\binom{n-1}{u_n}(d^{u_n}f_{\sigma(n)})(d^{v_n}x_j^{(\sigma(n))})\Big)\Big)$$

$$=\sum_j\Big(\sum_{\sigma\in\mathfrak{S}_n}sgn(\sigma)x_j^{(\sigma(1))}(dx_j^{(\sigma(2))})\cdots(d^{n-1}x_j^{(\sigma(n))})\Big)(f_1\cdots f_n).$$

Thus we have the result. Q.E.D.

LEMMA 17. *Assume that $x_1,\cdots,x_n\in\Gamma(C,\mathcal{L})$, then $\mathcal{R}(\mathcal{L},\cdots,\mathcal{L})$ contains*

$$\sum_{\sigma\in\mathfrak{S}_n}sgn(\sigma)x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}$$

*and*

$$\Phi_{\mathcal{L}_1,\cdots,\mathcal{L}_n}\Big(\sum_{\sigma\in\mathfrak{S}_n}sgn(\sigma)x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}\Big)=\Phi_{\mathcal{L}}^{(n)}(x_1\wedge\cdots\wedge x_n).$$

PROOF. If $0\leq i_1\leq 1,\cdots,0\leq i_{n-1}\leq n-1$ and $\sum_{s=1}^{n-1}i_s<\frac{n(n-1)}{2}$, then $0, i_1,\cdots,i_{n-1}$ are not all distinct. Therefore $\mathcal{R}(\mathcal{L},\cdots,\mathcal{L})$ contains $\sum_{\sigma\in\mathfrak{S}_n}sgn(\sigma)x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}$ by Definition 8. Moreover

$$\Phi_{\mathcal{L}_1,\cdots,\mathcal{L}_n}\Big(\sum_{\sigma\in\mathfrak{S}_n} sgn(\tau)x_{\tau(1)}\otimes\cdots\otimes x_{\tau(n)}\Big)$$

$$=\frac{1}{n!}\sum_{\tau\in\mathfrak{S}_n} sgn(\tau)\Big(\sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)x_{(\sigma\tau(1))}(dx_{(\sigma\tau(2))})\cdots(d^{n-1}x_{(\sigma\tau(n))})\Big)$$

$$=\sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)x_{(\sigma(1))}(dx_{(\sigma(2))})\cdots(d^{n-1}x_{(\sigma(n))})$$

$$=\Phi_{\mathcal{L}}^{(n)}(x_1\wedge\cdots\wedge x_n).$$

This complete the proof. Q.E.D.

LEMMA 18. *The image of* $\Phi_{\mathcal{L}_1,\cdots,\mathcal{L}}$ *is equal to the image of* $\Phi_{\mathcal{L}}^{(n)}$.

PROOF. Let $\tau=\sum x_j^{(1)}\otimes\cdots\otimes x_n^{(n)}$ be an element of $\mathcal{R}(\mathcal{L},\cdots,\mathcal{L})$. Then

$$\Phi_{\mathcal{L},\cdots,\mathcal{L}}(\tau)=\frac{1}{n!}\sum_j\Big(\sum_{\sigma\in\mathfrak{S}_n} sgn(\sigma)x_j^{(\sigma(1))}(dx_j^{(\sigma(2))})\cdots(d^{n-1}x_j^{(\sigma(n))})$$

$$=\frac{1}{n!}\sum_j\begin{vmatrix} x_j^{(1)} & \cdots & d^{n-1}x_j^{(1)} \\ \vdots & & \vdots \\ x_j^{(n)} & \cdots & d^{n-1}x_j^{(n)}\end{vmatrix}$$

$$=\frac{1}{n!}\sum_j\Phi_{\mathcal{L}}^{(n)}(x_1\wedge\cdots\wedge x_n).$$

Therefore the image of $\Phi_{\mathcal{L},\cdots,\mathcal{L}}$ is contained in the image of $\Phi_{\mathcal{L}}^{(n)}$. The converse is clear by Lemma 17. Q.E.D.

LEMMA 19. *Let* $\mathcal{L}_1,\cdots,\mathcal{L}_n$ *and* $\mathcal{P}$ *be invertible sheaves on* $C$. *Then the following diagram is commutative*:

$$\begin{array}{ccc} \mathcal{R}(\mathcal{L}_1,\cdots,\mathcal{L}_i,\cdots,\mathcal{L}_n)\otimes\Gamma(C,\mathcal{P}) & \xrightarrow{\text{cup product}} & \\ \downarrow{\scriptstyle \Phi_{\mathcal{L}_1,\cdots,\mathcal{L}_n}\otimes id} & & \\ \Gamma(C,\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_i\otimes\cdots\otimes\mathcal{L}_n)\otimes\Gamma(C,\mathcal{P}) & \xrightarrow{\text{cup product}} & \\ & & \mathcal{R}(\mathcal{L}_1,\cdots,\mathcal{L}_i\otimes\mathcal{P},\cdots,\mathcal{L}_n) \\ & & \downarrow{\scriptstyle \Phi_{\mathcal{L}_1,\cdots,\mathcal{L}_i\otimes\mathcal{P},\cdots,\mathcal{L}_n}} \\ & & \Gamma(C,\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_i\otimes\mathcal{P}\otimes\cdots\otimes\mathcal{L}_n) \end{array}$$

PROOF. This lemma is clear by the same calculations as in Lemma 16. Q.E.D.

LEMMA 20. *Let* $\mathcal{L}$ *and* $\mathcal{M}$ *be invertible sheaves on* $C$. *If* $deg(\mathcal{L})\geqq 2g+1$ *and* $deg(\mathcal{M})\geqq$ *then* $\Gamma(C,\mathcal{L})\otimes\Gamma(C,\mathcal{M})\to\Gamma(C,\mathcal{L}\otimes\mathcal{M})$ *is surjective.*

PROOF. See Fujita [6] p. 168 Proposition 1,10, Q. E. D.

PROPOSITION 2. *Let $\mathcal{L}$ be an invertible sheaf on $C$. If*

$$deg(\mathcal{L})\geqq(g-1)(2n^2-2n+3)+2(n^2-1),$$

*then there is an effective divisor $A$ of degree $g+n$ such that $\Gamma(C, \mathcal{O}(A))$ is an $(n-1)$-immersive $(n+1)$-net and*

$$H^1(C, (\omega_C^{\otimes n^2-4n+5}\otimes\mathcal{O}(A)^{\otimes 2n-2})^{\otimes -1}\mathcal{L})=0$$

PROOF. If $A$ is a general effective divisor of degree $g+n$, then $\dim\Gamma(C, \mathcal{O}(A-nP))=1$ for every $P\in C$. This implies that $\Gamma(C, \mathcal{O}(A))$ is an $(n-1)$-immersive $(n+1)$-net. Moreover the condition

$$H^1(C, (\omega_C^{\otimes n^2-4n+5}\otimes\mathcal{O}(A)^{\otimes 2n-2})^{\otimes -1}\otimes\mathcal{L})=0$$

is also a general condition. Therefore we have the result. Q. E. D.

Now we prove the Main theorem.

THEOREM 2. *Let $\mathcal{L}$ be an invertible sheaf on $C$. If*

$$deg(\mathcal{L})\geqq(g-1)(2n^2-2n+3)+2(n^2-1),$$

*then the $n$-Wahl map*

$$\Phi_{\mathcal{L}}^{(n)}: \wedge^n\Gamma(C, \mathcal{L}) \longrightarrow \Gamma(C, \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n})$$

*is surjective.*

PROOF. Let $A$ be an effective divisor in Proposition 2. By Lemma 18, Lemma 19 and Theorem 1, $\Phi_{\mathcal{O}(A),\cdots\mathcal{O}(A),\mathcal{O}(A),\mathcal{L}}$ is surjective. As

$$deg(\mathcal{L}\otimes\mathcal{O}(A))\geqq 2g \text{ and } deg(\omega_C^{\otimes n(n-1)/2}\otimes\mathcal{O}(A)^{\otimes n-1}\otimes\mathcal{L})\geqq 2g+1,$$

a cup product map

$$\mu: \Gamma(C, \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{O}(A)^{\otimes n-1}\otimes\mathcal{L})\otimes\Gamma(C, \mathcal{L}\otimes\mathcal{O}(-A))^{\otimes n-1}$$
$$\longrightarrow \Gamma(C, \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes n})$$

is surjective by Lemma 20. By Lemma 19 we have the following commutative diagram:

$$\begin{array}{ccc}
\Gamma(C, \omega_C^{\otimes n(n-1)/2}\otimes\mathcal{O}(A)^{\otimes n-1}\otimes\mathcal{L})\otimes\Gamma(C, \mathcal{L}\otimes\mathcal{O}(-A))^{\otimes n-1} & \xrightarrow{\mu} & \Gamma(C, \omega_D^{\otimes n(n-1)/2}\otimes\mathcal{L}^{\otimes}) \\
\Phi_{\mathcal{O}(A),\cdots\mathcal{O}(A),\mathcal{O}(A),\mathcal{L}} \uparrow & & \Phi_{\mathcal{L},\cdots,\mathcal{L}} \uparrow \\
\mathcal{R}(\mathcal{O}(A), \cdots, \mathcal{O}(A), \mathcal{L})\otimes\Gamma(C, \mathcal{L}\otimes\mathcal{O}(-A))^{\otimes n-1} & \longrightarrow & \mathcal{R}(\mathcal{L}, \cdots, \mathcal{L}).
\end{array}$$

As $\mu$ and $\Phi_{\mathcal{O}(A),\dots\mathcal{O}(A),\mathcal{O}(A),\mathcal{L}}$ are surjective, we have that $\Phi_{\mathcal{L}\dots,\mathcal{L}}$ is surjective. Therefore $\Phi_{\mathcal{L}}^{(n)}$ is surjective by Lemma 18. Q. E. D.

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