

ON LOCKETT'S CONJECTURE FOR FINITE GROUPS

By

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All groups in this paper are finite. A class of groups F will be called the Fitting's class if there are satisfied the following two conditions:

- 1°. if $G \in F$ and $N \triangleleft G$ then $N \in F$
- 2°. if $G = N_1 N_2$, $N_i \triangleleft G$ and $N_i \in F$; $i = 1, 2$ then $G \in F$.

By the condition 2° it follows that if F is an arbitrary nonempty Fitting's class then the product G_F of all normal F -subgroups of G belongs to F . It has been remarked in the paper [3] that the description of construction of a Fitting class is very difficult problem in general case.

Many well-known results concerning soluble Fitting classes are strictly connected with Lockett's construction given in [5]. Namely, if F is any Fitting class then $F^* = \{G : (G \times G)_F \text{ is subdirect in } G \times G\}$ is also the Fitting class.

Moreover, we denote by F_* the intersection of all such Fitting classes X for which $X^* = F^*$.

In the paper [5] Lockett conjectured that every soluble Fitting class F is the same as $F^* \cap N(F)$, where $N(F)$ denotes the normal Fitting class generated by F .

Nonempty Fitting class X is called normal if a subgroup G_X of G is a maximal X -subgroup of G . In [2] Bryce and Cossey proved that the Lockett's condition is satisfied by soluble Fitting's class iff $F_* = F^* \cap \gamma_*$ where γ_* is a minimal normal Fitting's class and γ denotes the class of all soluble groups. They also proved that primitive saturated formations which are Fitting classes satisfy the Lockett conjecture. In [8] was proved that all soluble local Fitting classes F satisfy Lockett's condition whenever $F \supseteq N_0$, where N_0 denotes the class of all nilpotent groups. Note that the classes XN_0 and $X\gamma\pi\gamma\pi'$, considered in [1] are particular cases of mentioned above. Now, we can consider non-empty class U of finite groups with respect to the operations S, Q, R, Ext_U described in [7].

The class U will be called universal class.

Then we see that fundamental problem concerning the construction of Fit-

ting classes X and U is the following:

PROBLEM 1. For which Fitting's classes X and U we have

$$(*) \quad X_* = X^* \cap U_*.$$

In this paper we show that $(*)$ is true when X is local and every group $G \in U$ satisfies $G/G_{x(p)} \in \gamma$ for all primes $p \in \Pi(X)$, where x denotes the function locally defining the class X .

Our result improves and generalizes the fundamental theorem of [8]. Namely if $U = \gamma$ then we obtain that any soluble and local Fitting class satisfies the Lockett's condition, so the condition $F \supseteq N_0$ given in [8] can be omitted. The Fitting class F will be called local (see [9], [10]) or the Fitting class with radical function f , if $f(p)$ is the Fitting class for all primes p and

$$F = U_{\pi(F)} \cap \left(\bigcap_{p \in \pi(F)} f(p) U_p U_{p'} \right).$$

By the product of two Fitting classes F and H we mean the class of all groups such that $G/G_F \in H$, (Cf. [7]).

LEMMA 1. *If F is local Fitting class then $F = F^*$.*

For the proof see [9].

LEMMA 2. *Let F and H be the Fitting classes. Then we have*

- 1° *If $F \subseteq H$ then $F_* \subseteq H_*$*
- 2° *$(F^*)_* = F_* \subseteq F \subseteq F^* = (F_*)^*$.*

For the proof see [5] and [8].

LEMMA 3. *Let F be a Fitting class and H denotes the Fitting class which is saturated homomorph.*

Then

$$(FH)^* = F^*H.$$

For the proof see [8].

If F is a Fitting class then the subgroup V of the group G will be called F -injector of G if $V \cap N$ is a maximal F -subgroup of N for any subnormal in G of the subgroup N .

LEMMA 4. *If $G \in F\gamma$ then there exist F -injectors in the group G and any two of them are conjugates.*

For the proof see [6].

Let $Y(F, H)$ denotes the class of all groups such that F -injectors exist and are H -subgroups. We have

LEMMA 5. *Let $F=FU_p$ and H be the Fitting classes. If every group $G \in U$ satisfies $G/G_F \in \gamma$ then $Y(F, HU_{p'})$ is also Fitting class and moreover if H is local then*

$$Y(F, (HU_{p'})^*) = (Y(F, HU_{p'}))^* .$$

The proof of Lemma 5 follows by Lemma 4, Theorem 3.3 of [4] Lemma 1 of [1].

LEMMA 6. *Every local Fitting class F is defined by a radical function f such that $f(p)U_p = f(p) \subseteq F$ for all primes p .*

The proof of Lemma 6 follows by slighty modification of the proof of Lemma 1 from [8].

Now, we can prove the main result :

THEOREM. *If X is the Fitting class with radical function x and every group $G \in U$, where U is the universal class, satisfies $G/G_{x(p)} \in \gamma$ for all primes $p \in \Pi(X)$ then*

$$X_* = X^* \cap U_* .$$

PROOF. By Lemma 1 we have $X^* = X$. Thus we can prove that $X_* = X \cap U_*$, From Lemma 2 we have $X_* \subseteq U_*$ and $X_* \subseteq X$ and therefore we obtain $X_* \subseteq X \cap U_*$. It suffices to prove that $X \cap U_* \subseteq X_*$. By Lemma 6 we can assume without loss of generality that for all primes p , $x(p)U_p = x(p) \subseteq X$. Let $G \in U$. Then by Lemma 4 we have that in G there exist $x(p)$ -injectors and any two of them are conjugates in G . From Lemma 3 and Lemma 2 we obtain

$$(XU_{p'})^* = X^*U_{p'} = (X_*)^*U_{p'} = (X_*U_{p'})^* .$$

By last equality it follows that $x(p) \subseteq (XU_{p'})^*$.

Consequently $Y(x(p), (X_*U_{p'})^*) = U$.

Thus by Lemma 5 we obtain $(Y(x(p), (X_*U_{p'})^*))^* = U$. But by the second part of the assertion of Lemma 2 we have $U_* = (Y(x(p), X_*U_{p'}))^*$.

From Lemma 2 and Lemma 5 we obtain that U_* is contained in the Fitting class $Y(x(p), X_*U_{p'})$.

Thus we have

$$x(p) \cap U_* \subseteq x(p) \cap Y(x(p), X_* U_{p'}) = x(p) \cap X_* U_{p'} \subseteq X_* U_{p'}$$

and consequently

$$\left(\bigcap_{p \in \pi(X)} x(p) U_{p'} \right) \cap U_* \left(\bigcap_{p \in \pi(X)} U_{p'} \right) \subseteq X_* \left(\bigcap_{p \in \pi(X)} U_{p'} \right).$$

Hence

$$X \cap U_* U_{\pi(x)} \subseteq X_* U_{\pi(x)} \cap U_{\pi(x)}$$

and therefore

$$X \cap U_* \subseteq X_*$$

and the proof our Theorem is complete.

COROLLARY. *Every soluble and local Fitting class F satisfies*

$$F_* = F^* \cap \gamma_*.$$

REMARK. There exist non-local soluble Fitting classes for which Lockett's condition is true.

Indeed, let F be a normal class of Fitting such that $F^* = \gamma$.

Then by Lemma 1 we have that F is non-local Fitting class. But from Lemma 2 we have $F_* = \gamma_*$ and consequently $F_* = F^* \cap \gamma_*$.

So it would be of some interest to solve the following:

PROBLEM 2. Characterize non-normal and non-local soluble Fitting classes satisfying Lockett's condition.

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