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ON SELF-INJECTIVE DIMENSIONS OF ARTINIAN RINGS

By

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Throughout this note R stands for a left and right artinian ring unless specified otherwise. We denote by $\operatorname{mod} R$ (resp. $\operatorname{mod} R^{\mathrm{op}}$) the category of all finitely generated left (resp. right) R-modules and by $(\ )^*$ both the R-dual functors. For an $X\in\operatorname{mod} R$, we denote by $\varepsilon_X: X\to X^{**}$ the usual evaluation map, by $E(X)$ its injective envelope and by $[X]$ its image in $K_0(\operatorname{mod} R)$, the Grothendieck group of $\operatorname{mod} R$.

In this note, we ask when $\operatorname{inj\,dim}{}_RR=\operatorname{inj\,dim} R_R$. Note that if $\operatorname{inj\,dim}{}_RR<\infty$ and $\operatorname{inj\,dim} R_R<\infty$ then by Zaks [10, Lemma A] $\operatorname{inj\,dim}{}_RR=\operatorname{inj\,dim} R_R$. So we ask when $\operatorname{inj\,dim} R_R<\infty$ implies $\operatorname{inj\,dim}{}_RR<\infty$. There has not been given any example of R with $\operatorname{inj\,dim}{}_RR\neq\operatorname{inj\,dim} R_R$. However, we know only a little about the question. By Eilenberg and Nakayama [5, Theorem 18], ${}_RR$ is injective if and only if so is R_R. In case R is an artin algebra, we know from the theory of tilting modules that $\operatorname{inj\,dim}{}_RR\leqq 1$ if and only if $\operatorname{inj\,dim} R_R\leqq 1$ (see Bongartz [3, Theorem 2.1]). Also, if R is of finite representation type, it is well known and easily checked that $\operatorname{inj\,dim}{}_RR<\infty$ if and only if $\operatorname{inj\,dim} R_R<\infty$.

Suppose $\operatorname{inj\,dim} R_R<\infty$. Then we have a well defined linear map

$$\delta: K_0(\operatorname{mod} R^{\mathrm{op}}) \longrightarrow K_0(\operatorname{mod} R)$$

such that

$$\delta([M])=\sum_{i\geqq 0}(-1)^i[\operatorname{Ext}_R^i(M, R)]$$

for $M\in\operatorname{mod} R^{\mathrm{op}}$. Since R is artinian, both $K_0(\operatorname{mod} R^{\mathrm{op}})$ and $K_0(\operatorname{mod} R)$ are finitely generated free abelian groups of the same rank. Also, for an $M\in\operatorname{mod} R^{\mathrm{op}}$, $[M]=0$ if and only if $M=0$. Thus $\operatorname{inj\,dim}{}_RR<\infty$ if (and only if) the following two conditions are satisfied:

(a) δ is surjective.
(b) There is an integer $d\geqq 1$ such that $\delta([\operatorname{Ext}_R^d(X, R)])=0$ for all $X\in\operatorname{mod} R$.

In this note, along the principle above, we will prove the following

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THEOREM A. *Let* $m, n \geqq 1$. *Suppose that* $\operatorname{inj}\dim R_R \leqq n$ *and that for all* $0 \leqq i \leqq n-2$ (*if* $n \geqq 2$) $\operatorname{Ext}_R^i(\operatorname{Ext}_R^m(-, R), R)$ *vanishes on* $\operatorname{mod} R$. *Then* $\operatorname{inj}\dim {}_R R < \infty$.

REMARK. Let $0 \to R_R \to E_0 \to E_1 \to \cdots$ be a minimal injective resolution of R_R. Suppose $\operatorname{proj}\dim E_i < m$ for all $0 \leqq i \leqq n-2$. Then it follows by Cartan and Eilenberg [4, Chap. VI, Proposition 5.3] that for all $0 \leqq i \leqq n-2$ $\operatorname{Ext}_R^i(\operatorname{Ext}_R^m(-, R), R)$ vanishes on $\operatorname{mod} R$. The converse fails. Namely, there has been given an example of R such that $\operatorname{proj}\dim E_0 = \infty$ and $\operatorname{Ext}_R^1(-, R)^*$ vanishes on $\operatorname{mod} R$ (see Hoshino [7, Example]).

Consider the case $n=1$ in Theorem A. Then the last assumption is empty and we get the following

COROLLARY. $\operatorname{inj}\dim {}_R R \leqq 1$ *if and only if* $\operatorname{inj}\dim R_R \leqq 1$.

As another application of Theorem A, we will prove the following

THEOREM B. *Let* $0 \to R_R \to E_0 \to E_1 \to \cdots$ *be a minimal injective resolution of* R_R. *Suppose* $\operatorname{inj}\dim R_R \leqq 2$. *Then the following statements are equivalent.*

(1) $\operatorname{inj}\dim {}_R R < \infty$.

(2) $\operatorname{proj}\dim E_0 < \infty$.

(3) $\operatorname{proj}\dim E_2 < \infty$.

The following question is raised: Does $\operatorname{inj}\dim R_R < \infty$ imply $\operatorname{proj}\dim E(R_R) < \infty$? If this is the case, it would follow from Theorem B that $\operatorname{inj}\dim {}_R R \leqq 2$ if and only if $\operatorname{inj}\dim R_R \leqq 2$. At least, it would be possible to check directly that $\operatorname{inj}\dim R_R \leqq 1$ implies $\operatorname{proj}\dim E(R_R) \leqq 1$. In connection with this, we notice that $\operatorname{proj}\dim E({}_R R) \leqq 1$ does not imply $\operatorname{proj}\dim E(R_R) < \infty$ (see Hoshino [7, Example]).

1. Proof of Theorem A

We may assume $m > n$. We claim $\operatorname{inj}\dim {}_R R \leqq m+n-2$. Let

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

be an exact sequence in $\operatorname{mod} R$ with the P_i projective. Put $X_i = \operatorname{Cok}(P_{i+1} \to P_i)$ for $i \geqq 0$ and $M_i = \operatorname{Cok}(P_{i-1}^* \to P_i^*)$ for $i \geqq 1$. As remarked in the introduction, we have only to check the following two conditions:

(a)′ $[X]=\delta\Big(\sum_{i=0}^{m+n-3}(-1)^i[P_i{}^*]+(-1)^{m+n-2}[X_{m+n-2}{}^*]\Big)$.

(b)′ $\mathrm{Ext}_R^i(\mathrm{Ext}_R^{m+n-1}(X, R), R)=0$ for all $i\geqq 0$.

We will check these in several steps.

STEP 1: $M_i{}^{**}\cong X_{i+1}{}^*\cong \mathrm{Ker}\,(P_{i+1}{}^*\to P_{i+2}{}^*)$ for all $i\geqq 1$.

PROOF. Let $i\geqq 1$. Since each P_j is reflexive, we have

$$\begin{aligned} M_i{}^* &\cong \mathrm{Ker}\,(P_i{}^{**} \longrightarrow P_{i-1}{}^{**}) \\ &\cong \mathrm{Ker}\,(P_i \longrightarrow P_{i-1}) \\ &\cong \mathrm{Cok}\,(P_{i+2} \longrightarrow P_{i+1})\,. \end{aligned}$$

Applying ( )*, the assertion follows.

STEP 2: For each $i\geqq 1$, there is the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}_R^i(X, R) \arrow[r] \arrow[d, equal] & M_i \arrow[r, "\varepsilon_{M_i}"] \arrow[d, equal] & M_i{}^{**} \arrow[r] \arrow[d] & \mathrm{Ext}_R^{i+1}(X, R) \arrow[r] \arrow[d] & 0 \\ (e_i): 0 \arrow[r] & \mathrm{Ext}_R^i(X, R) \arrow[r] & M_i \arrow[r, "\phi_i"] & P_{i+1}{}^* \arrow[r] & M_{i+1} \arrow[r] & 0\,. \end{tikzcd}$$

PROOF. This is a consequence of Auslander [1, Proposition 6.3]. However, for the benefit of the reader, we provide a direct proof. By Step 1 we have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd} P_{i-1}{}^* \arrow[d] & P_{i+2}{}^* & & \\ P_i{}^* \arrow[r] \arrow[d] & P_{i+1}{}^* \arrow[r] \arrow[u] & M_{i+1} \arrow[r] & 0 \\ M_i \arrow[r, "\varepsilon_{M_i}"] \arrow[d] & M_i{}^{**} \arrow[u] & & \\ 0 & 0\,. \arrow[u] & & \end{tikzcd}$$

Since the $\mathrm{Ext}_R^i(-, R)$ are derived functors of ( )*, the assertion follows.

STEP 3: $\mathrm{Ext}_R^1(M_i, R)=0$ for all $i\geqq 2$.

PROOF. Let $i \geqq 2$. Note that X_{i-1} is torsionless. We have a finite presentation $P_{i-1}^* \to P_i^* \to M_i \to 0$ with $X_{i-1} \cong \mathrm{Cok}\,(P_i^{**} \to P_{i-1}^{**})$. Thus by Step 2 $\mathrm{Ext}_R^1(M_i, R) \cong \mathrm{Ker}\, \varepsilon_{X_{i-1}} = 0$.

STEP 4: Suppose $n \geqq 2$. Then $\mathrm{Ext}_R^{j+1}(M_{i+j}, R) = 0$ for all $i \geqq m$ and $n-1 \geqq j \geqq 1$.

PROOF. Note that for all $i \geqq m$ and $n-2 \geqq j \geqq 0$ $\mathrm{Ext}_R^j(\mathrm{Ext}_R^j(-, R), R)$ vanishes on mod R. Let $i \geqq m$. Applying $(\ )^*$ to exact sequences $(e_i), \cdots, (e_{i+n-2})$ in Step 2, we get a chain of embeddings:

$$\mathrm{Ext}_R^n(M_{i+n-1}, R) \hookrightarrow \cdots \hookrightarrow \mathrm{Ext}_R^1(M_i, R).$$

By Step 3 the assertion follows.

STEP 5: $\mathrm{Ext}_R^j(M_i, R) = 0$ for all $i \geqq m+n-1$ and $j \geqq 1$.

PROOF. Note that for all $j \geqq n+1$ $\mathrm{Ext}_R^j(-, R)$ vanishes on mod R^{op}. By Steps 3 and 4 the assertion follows.

STEP 6: X_i is reflexive for all $i \geqq m+n-2$.

PROOF. Let $i \geqq m+n-2$. Since $m+n-2 \geqq 1$, X_i is torsionless. Also, as in the proof of Step 3, we have $\mathrm{Cok}\, \varepsilon_{X_i} \cong \mathrm{Ext}_R^2(M_{i+1}, R)$. By Step 5 the assertion follows.

STEP 7: $\mathrm{Ext}_R^j(X_i^*, R) = 0$ for all $i \geqq m+n-2$ and $j \geqq 1$.

PROOF. Note that X_i^* is a second syzygy of M_{i+1}. By Step 5 the assertion follows.

STEP 8: $[X] = \delta\Big(\sum_{j=0}^{i-1} (-1)^j [P_j^*] + (-1)^i [X_i^*]\Big)$ for all $i \geqq m+n-2$.

PROOF. Let $i \geqq m+n-2$. By Steps 6 and 7 we have

$$\begin{aligned}
[X] &= \sum_{j=0}^{i-1} (-1)^j [P_j] + (-1)^i [X_i] \\
&= \sum_{j=0}^{i-1} (-1)^j [P_j^{**}] + (-1)^i [X_i^{**}] \\
&= \sum_{j=0}^{i-1} (-1)^j \delta([P_j^*]) + (-1)^i \delta([X_i^*]) \\
&= \delta\Big(\sum_{j=0}^{i-1} (-1)^j [P_j^*] + (-1)^i [X_i^*]\Big).
\end{aligned}$$

STEP 9: $\mathrm{Ext}_R^j(\mathrm{Ext}_R^i(X, R), R)=0$ for all $i \geqq m+n-1$ and $j \geqq 0$.

PROOF. Let $i \geqq m+n-1$. Observe the commutative diagram in Step 2. It is not difficult to see that $\phi_i{}^*$ is epic. Thus, by applying $(\ )^*$ to the exact sequence (e_i), the assertion follows by Step 5.

This finishes the proof of Theorem A.

2. Proof of Theorem B

We will use a result of Cartan and Eilenberg [4, Chap. VI, Proposition 5.3] without any reference.

(1) $\Rightarrow$ (2) and (3). See Iwanaga [8, Proposition 1].

(2) $\Rightarrow$ (1). Let $m \geqq 1$ and $X \in \mathrm{mod}\, R$. Suppose $\mathrm{proj\,dim}\, E_0 < m$. Then $\mathrm{Hom}_R(\mathrm{Ext}_R^m(X, R), E_0) \cong \mathrm{Tor}_m^R(E_0, X)=0$ and thus $\mathrm{Ext}_R^m(X, R)^*=0$. Hence Theorem A applies.

(3) $\Rightarrow$ (1). Let $m \geqq 2$ and suppose $\mathrm{proj\,dim}\, E_2 < m$. We claim that $\mathrm{Ext}_R^m(-, R)^*$ vanishes on $\mathrm{mod}\, R$. Let

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

be an exact sequence in $\mathrm{mod}\, R$ with the P_i projective and put $M=\mathrm{Cok}\,(P_{m-1}{}^* \to P_m{}^*)$. Note first that for all $i \geqq m$, since $\mathrm{Hom}_R(\mathrm{Ext}_R^i(X, R), E_2) \cong \mathrm{Tor}_i^R(E_2, X)=0$, $\mathrm{Ext}_R^2(\mathrm{Ext}_R^i(X, R), R)=0$. By Step 2 of Section 1 we have an exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^m(X, R) \longrightarrow M \xrightarrow{\varepsilon_M} M^{**} \longrightarrow \mathrm{Ext}_R^{m+1}(X, R) \longrightarrow 0\,.$$

Note that by Step 3 of Section 1 $\mathrm{Ext}_R^1(M, R)=0$, that since M^{**} is a second syzygy, $\mathrm{Ext}_R^i(M^{**}, R)=0$ for all $i \geqq 1$, and that $\varepsilon_M{}^*$ is epic. Applying $(\ )^*$ to the above exact sequence, we get

$$\begin{aligned} \mathrm{Ext}_R^m(X, R)^* &\cong \mathrm{Ext}_R^1(\mathrm{Im}\, \varepsilon_M, R) \\ &\cong \mathrm{Ext}_R^2(\mathrm{Ext}_R^{m+1}(X, R), R) \\ &=0\,, \end{aligned}$$

as required.

3. Remarks

In this and the next sections, we will make some remarks on our subject.

PROPOSITION 1. *Let* $\cdots \to P_1 \to P_0 \to X \to 0$ *be an exact sequence in* $\mathrm{mod}\, R$ *with the* P_i *projective. Put* $X_i=\mathrm{Cok}\,(P_{i+1} \to P_i)$ *for* $i \geqq 0$. *Then for each* $n \geqq 1$ *the* *following equality holds in* $K_0(\mathrm{mod}\, R^{\mathrm{op}})$:

$$\sum_{i=0}^{n}(-1)^i[\mathrm{Ext}_R^i(X, R)]=\sum_{i=0}^{n-1}(-1)^i[P_i{}^*]+(-1)^n[X_n{}^*]\,.$$

PROOF. By direct calculation.

PROPOSITION 2. *Suppose* $\mathrm{inj\,dim}\, R_R \leqq 2$ *and* $\mathrm{proj\,dim}\, E({}_RR) \leqq 1$. *Then* $\mathrm{inj\,dim}\, {}_RR \leqq 2$.

PROOF. By Hoshino [7, Proposition D] $\mathrm{Ext}_R^1(-, R)^*$ vanishes on $\mathrm{mod}\, R$. Thus by Theorem A and Zaks [10, Lemma A] the assertion follows.

PROPOSITION 3. *The following statements are equivalent.*

(1) $\mathrm{inj\,dim}\, {}_RR \leqq 1$.

(2) $\mathrm{inj\,dim}\, R_R \leqq 1$.

(3) *Every* $X \in \mathrm{mod}\, R$ *with* $\mathrm{Ext}_R^1(X, R)=0$ *is torsionless.*

(4) *Every* $M \in \mathrm{mod}\, R^{\mathrm{op}}$ *with* $\mathrm{Ext}_R^1(M, R)=0$ *is torsionless.*

PROOF. (1)⇔(2). By Corollary to Theorem A.
(1)⇔(4) and (2)⇔(3). See Hoshino [6, Remark].

4. Appendix

In this section, as an appendix, we deal with the case of R being noetherian.

We remarked in [6] that for a left and right noetherian ring R, $\mathrm{inj\,dim}\, {}_RR \leqq 1$ if and only if every $M \in \mathrm{mod}\, R^{\mathrm{op}}$ with $\mathrm{Ext}_R^1(M, R)=0$ is torsionless. Compare this with the following

PROPOSITION 4. *Let R be a left and right noetherian ring. Then the following statements are equivalent.*

(1) $\mathrm{proj\,dim}\, X \leqq 1$ *for every* $X \in \mathrm{mod}\, R$ *with* $\mathrm{proj\,dim}\, X < \infty$.

(2) $M^* \neq 0$ *for every nonzero* $M \in \mathrm{mod}\, R^{\mathrm{op}}$ *with* $\mathrm{Ext}_R^1(M, R)=0$.

PROOF. (1)⇒(2). Let $M \in \mathrm{mod}\, R^{\mathrm{op}}$ with $\mathrm{Ext}_R^1(M, R)=0=M^*$. We claim $M=0$. Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution in $\mathrm{mod}\, R^{\mathrm{op}}$ and put $X=\mathrm{Cok}\,(P_1{}^* \to P_2{}^*)$. Then we have a projective resolution $0 \to P_0{}^* \to P_1{}^* \to P_2{}^* \to X \to 0$ in $\mathrm{mod}\, R$. Since $\mathrm{proj\,dim}\, X < \infty$, we get $\mathrm{proj\,dim}\, X \leqq 1$. Thus, since each P_i is reflexive, $M \cong \mathrm{Cok}\,(P_1{}^{**} \to P_0{}^{**}) \cong \mathrm{Ext}_R^2(X, R)=0$.

(2)⇒(1). Suppose to the contrary that there is a torsionless $X \in \mathrm{mod}\, R$ with

$\operatorname{proj\,dim} X=1$. Let $0\to P_1\to P_0\to X\to 0$ be a projective resolution in $\operatorname{mod} R$ and put $M=\operatorname{Cok}(P_0{}^*\to P_1{}^*)$. Note that $M\neq 0$. By Auslander [1, Proposition 6.3] $\operatorname{Ext}^1_R(M, R)\cong\operatorname{Ker}\varepsilon_X=0$. On the other hand, since each P_i is reflexive, $M^*\cong\operatorname{Ker}(P_1\to P_0)=0$, a contradiction.

PROPOSITION 5. *Let R be a left and right noetherian ring with* $\operatorname{inj\,dim} R_R\leqq 2$. *Suppose there is an integer $m\geqq 1$ such that* $\operatorname{Ext}^m_R(-, R)^*$ *vanishes on* $\operatorname{mod} R$. *Then the following statements are equivalent.*

(1) $\operatorname{inj\,dim}{}_R R<\infty$.

(2) *There is an integer $n\geqq 0$ such that* $\operatorname{proj\,dim} X\leqq n$ *for every* $X\in\operatorname{mod} R$ *with* $\operatorname{proj\,dim} X<\infty$.

(3) *For an* $M\in\operatorname{mod} R^{\mathrm{op}}$, $\operatorname{Ext}^i_R(M, R)=0$ *for all $i\geqq 0$ implies $M=0$.*

PROOF. (1)$\Rightarrow$(2). See Bass [2, Proposition 4.3].

(2)$\Rightarrow$(3). Let $M\in\operatorname{mod} R^{\mathrm{op}}$ with $\operatorname{Ext}^i_R(M, R)=0$ for all $i\geqq 0$. Then by the same argument as in the proof of (1)$\Rightarrow$(2) in Proposition 4 it follows that $M=0$.

(3)$\Rightarrow$(1). Let $M\in\operatorname{mod} R^{\mathrm{op}}$ with $\operatorname{Ext}^i_R(M, R)=0$ for all $i\geqq 1$. We claim that M is reflexive. We show first that such an M is torsionless. Let $\cdots\to P_1\to P_0\to M\to 0$ be a projective resolution in $\operatorname{mod} R^{\mathrm{op}}$ and put $X=\operatorname{Cok}(P_{m-1}{}^*\to P_m{}^*)$. Then we have an exact sequence $P_0{}^*\to\cdots\to P_m{}^*\to X\to 0$ in $\operatorname{mod} R$ with the $P_i{}^*$ projective. Since $M\cong\operatorname{Cok}(P_1{}^{**}\to P_0{}^{**})$, as in Step 2 of Section 1, $\operatorname{Ker}\varepsilon_M\cong\operatorname{Ext}^m_R(X, R)$. Thus $(\operatorname{Ker}\varepsilon_M)^*=0$. Also, since $\operatorname{Im}\varepsilon_M$ is torsionless, the exact sequence $0\to\operatorname{Ker}\varepsilon_M\to M\to\operatorname{Im}\varepsilon_M\to 0$ yields $\operatorname{Ext}^i_R(\operatorname{Ker}\varepsilon_M, R)\cong\operatorname{Ext}^{i+1}_R(\operatorname{Im}\varepsilon_M, R)=0$ for all $i\geqq 1$. Thus $\operatorname{Ker}\varepsilon_M=0$. Next, let $\alpha: P\to M^*$ be epic in $\operatorname{mod} R$ with P projective. Put $\beta=\alpha^*\circ\varepsilon_M: M\to P^*$ and $N=\operatorname{Cok}\beta$. Then β is monic and β^* is epic. Thus the exact sequence $0\to M\to P^*\to N\to 0$ yields $\operatorname{Ext}^i_R(N, R)=0$ for all $i\geqq 1$. Hence $\operatorname{Ker}\varepsilon_N=0$. Since P^* is reflexive, the exact sequence just above yields also that $\operatorname{Cok}\varepsilon_M\cong\operatorname{Ker}\varepsilon_N$. Therefore M is reflexive and by Hoshino [6, Proposition 2.2] the assertion follows.

According to Bass [2, Proposition 4.3], a result of Jensen [9, Proposition 6] would imply the following

PROPOSITION 6. *Let R be a left noetherian ring with* $\operatorname{inj\,dim}{}_R R=m<\infty$. *Then* $\operatorname{proj\,dim} X\leqq m$ *for every left R-module X with* $\operatorname{weak\,dim} X<\infty$.

PROOF. Let X be a left R-module with $\operatorname{weak\,dim} X<\infty$. According to Bass [2, Proposition 4.3], we have only to prove that $\operatorname{proj\,dim} X<\infty$. Let

$$\cdots\longrightarrow F_1\longrightarrow F_0\longrightarrow X\longrightarrow 0$$

be a free resolution of X and put $X_i=\mathrm{Cok}(F_{i+1}\to F_i)$ for $i\geqq 1$. Let $n=\max\{m+1, \operatorname{weak\,dim} X\}$. We claim that X_n is projective. It suffices to show that $\mathrm{Ext}_R^n(X, X_n)=0$. Note that X_n is flat. Let $\cdots\to P_1\to P_0\to Y\to 0$ be an exact sequence in $\operatorname{mod} R$ with the P_i projective. Since $\mathrm{Hom}_R(P_i, R)\otimes_R X_n \xrightarrow{\sim} \mathrm{Hom}_R(P_i, X_n)$ for all $i\geqq 0$, and since the functor $-\otimes_R X_n$ is exact, it follows that $\mathrm{Ext}_R^i(Y, R)\otimes_R X_n \xrightarrow{\sim} \mathrm{Ext}_R^i(Y, X_n)$ for all $i\geqq 0$. Thus $\operatorname{inj\,dim} X_n \leqq \operatorname{inj\,dim}{}_R R =m<n$ and $\mathrm{Ext}_R^n(X, X_n)=0$.

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