

COVERING PROPERTIES IN COUNTABLE PRODUCTS

By

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1. Introduction.

A space X is said to be *subparacompact* if every open cover of X has a σ -discrete closed refinement, and *metacompact* (*countably metacompact*) if every open cover (countable open cover) of X has a point finite open refinement. A space X is said to be *metalindelöf* if every open cover of X has a point countable open refinement. A collection \mathcal{U} of subsets of a space X is said to be *interior-preserving* if $\text{int}(\bigcap \mathcal{U}) = \bigcap \{\text{int } V : V \in \mathcal{U}\}$ for every $\mathcal{U} \subset \mathcal{U}$. Clearly, an open collection \mathcal{U} is interior-preserving if and only if $\bigcap \mathcal{U}$ is open for every $\mathcal{U} \subset \mathcal{U}$. A space X is said to be *orthocompact* if every open cover of X has an interior-preserving open refinement. Every paracompact Hausdorff space is subparacompact and metacompact, and every metacompact space is countably metacompact, metalindelöf and orthocompact. The reader is referred to D.K. Burke [4] for a complete treatment of these covering properties and some informations of their role in general topology.

Let \mathcal{DC} be the class of all spaces which have a discrete cover by compact sets. The topological game $G(\mathcal{DC}, X)$ was introduced and studied by R. Telgársky [19]. The games are played by two persons called Players I and II. Players I and II choose closed subsets of II's previous play (or of X , if $n=0$): Player I's choice must be in the class \mathcal{DC} and II's choice must be disjoint from I's. We say that Player I *wins* if the intersection of II's choices is empty. Recall from [19] that a space X is said to be *\mathcal{DC} -like* if Player I has a winning strategy in $G(\mathcal{DC}, X)$. The class of \mathcal{DC} -like spaces includes all spaces which admit a σ -closure-preserving closed cover by compact sets, and regular subparacompact, σ - C -scattered spaces.

Paracompactness and Lindelöf property of countable products have been studied by several authors. In particular, if X is a separable metric space or X is a regular Čech-complete Lindelöf space or X is a regular C -scattered Lindelöf space, then $X^\omega \times Y$ is Lindelöf for every regular hereditarily Lindelöf space Y . The first result is due to E. Michael (cf. [14]) and the second one

is due to Z. Frolik [9] and the third one is due to K. Alster [1]. K. Alster [2] also proved that if Y is a perfect paracompact Hausdorff space and X_n is a scattered paracompact Hausdorff space for each $n \in \omega$, then $Y \times \prod_{n \in \omega} Y_n$ is paracompact. Furthermore, the author [17] proved that if Y is a perfect paracompact Hausdorff (regular hereditarily Lindelöf) space and X_n is a paracompact Hausdorff (regular Lindelöf) \mathcal{DC} -like space for each $n \in \omega$, then $Y \times \prod_{n \in \omega} X_n$ is paracompact (Lindelöf).

The aim of this paper is to consider subparacompactness, metacompactness, metalindelöf property and orthocompactness of countable products. We show that if Y is a perfect subparacompact space and X_n is a regular subparacompact \mathcal{DC} -like space for each $n \in \omega$, then $Y \times \prod_{n \in \omega} X_n$ is subparacompact. We also prove that if X_n is a regular metacompact \mathcal{DC} -like (C -scattered) space for each $n \in \omega$, then $\prod_{n \in \omega} X_n$ is metacompact. Furthermore, let Y be a hereditarily metacompact space and X_n be a regular metacompact \mathcal{DC} -like (C -scattered) space for each $n \in \omega$. Then the following statements are equivalent: (a) $Y \times \prod_{n \in \omega} X_n$ is metacompact; (b) $Y \times \prod_{n \in \omega} X_n$ is countably metacompact and (c) $Y \times \prod_{n \in \omega} X_n$ is orthocompact. For metalindelöf property, it will be shown that if Y is a hereditarily metalindelöf space and X_n is a regular metalindelöf \mathcal{DC} -like (C -scattered) space for each $n \in \omega$, then $Y \times \prod_{n \in \omega} X_n$ is metalindelöf.

In this paper, we deal with infinite spaces. No separation axioms are assumed. However, regular spaces are assumed to be T_1 . Let $|A|$ denote the cardinality of a set A . The letter ω denotes the set of natural numbers.

Given a cover \mathcal{U} of a space X , and $Y \subset X$, let $\mathcal{U}|Y = \{U \cap Y : U \in \mathcal{U}\}$. For each $x \in X$, let $\mathcal{U}_x = \{U \in \mathcal{U} : x \in U\}$ and let $\text{ord}(x, \mathcal{U}) = |\mathcal{U}_x|$. Let \mathcal{U}^f be the collection of all finite unions of elements of \mathcal{U} .

We use the finite sequences in the proofs. So we adopt the following notations for them: Let A be a set, and let $\mathcal{P}(A)$ be the set of all nonempty subsets of A . Let $A^0 = \{\emptyset\}$. For each $n \geq 1$, A^n denotes the set of all n -sequences of elements of A and $A^{<\omega} = \bigcup_{n \in \omega} A^n$. If $\tau = (a_0, \dots, a_n) \in A^{<\omega}$ and $a \in A$, then $\tau \oplus a$ denotes the sequence (a_0, \dots, a_n, a) and $\tau_- = (a_0, \dots, a_{n-1})$ if $n \geq 1$ and $\tau_- = \emptyset$ if $n = 0$.

2. Topological games.

For the class \mathcal{DC} and a space X , the *topological game* $G(\mathcal{DC}, X)$ is defined as follows: There are two players I and II (the pursuer and evader). They alternatively choose consecutive terms of a sequence $\langle E_0, F_0, E_1, F_1, \dots, E_n, F_n,$

\dots) of subsets in X . When each player chooses his term, he knows \mathcal{DC} , X and their previous choices.

For a space X , let 2^X denote the set of all closed subsets of X . A sequence $\langle E_0, F_0, E_1, F_1, \dots, E_n, F_n, \dots \rangle$ of subsets in X is a *play* of $G(\mathcal{DC}, X)$ if it satisfies the following conditions: For each $n \in \omega$,

- (1) E_n is the choice of Player I,
- (2) F_n is the choice of Player II,
- (3) $E_n \in 2^X \cap \mathcal{DC}$,
- (4) $F_n \in 2^X$,
- (5) $E_n \cup F_n \subset F_{n-1}$, where $F_{-1} = X$,
- (6) $E_n \cap F_n = \emptyset$.

Player I wins if $\bigcap_{n \in \omega} F_n = \emptyset$ (Player II has no place to run away). Otherwise Player II wins.

A finite sequence $\langle E_0, F_0, E_1, F_1, \dots, E_m, F_m \rangle$ is said to be *admissible* if it satisfies the above conditions (1)-(6) for each $n \leq m$.

Let s' be a function from $\bigcup_{n \in \omega} (2^X)^{n+1}$ into $2^X \cap \mathcal{DC}$. Let

$$S_0 = \{F : \langle s'(X), F \rangle \text{ is admissible for } G(\mathcal{DC}, X)\}.$$

Moreover, we can inductively define

$$S_n = \{(F_0, F_1, \dots, F_n) : \langle E_0, F_0, E_1, F_1, \dots, E_n, F_n \rangle \text{ is admissible for } G(\mathcal{DC}, X), \text{ where } F_{-1} = X \text{ and } E_i = s'(F_0, F_1, \dots, F_{i-1}) \text{ for each } i \leq n\}.$$

Then the restriction s of s' to $\bigcup_{n \in \omega} S_n$ is said to be a *strategy* for Player I in $G(\mathcal{DC}, X)$. We say that the strategy s is a *winning* one if Player I wins every play $\langle E_0, F_0, E_1, F_1, \dots, E_n, F_n, \dots \rangle$ such that $E_n = s(F_0, F_1, \dots, F_{n-1})$ for $n \in \omega$.

Next, we define another (winning) strategy for Player I in $G(\mathcal{DC}, X)$, which depends only on the preceding choice of Player II.

A function s from 2^X into $2^X \cap \mathcal{DC}$ is said to be a *stationary strategy* for Player I in $G(\mathcal{DC}, X)$ if $s(F) \subset F$ for each $F \in 2^X$. We say that the s is *winning* if he wins every play $\langle s(X), F_0, s(F_0), F_1, s(F_1), \dots \rangle$. That is, a function s from 2^X into $2^X \cap \mathcal{DC}$ is a stationary winning strategy if and only if it satisfies

- (i) $s(F) \subset F$ for each $F \in 2^X$,
- (ii) if $\{F_n : n \in \omega\}$ is a decreasing sequence of closed subsets of X such that $s(F_n) \cap F_{n+1} = \emptyset$ for each $n \in \omega$, then $\bigcap_{n \in \omega} F_n = \emptyset$.

The following lemma shows that there is no essential difference between the winning strategy and the stationary winning strategy.

LEMMA 2.1 (F. Galvin and R. Telgársky [10]). *Player I has a winning strategy in $G(\mathcal{DC}, X)$ if and only if he has a stationary winning strategy in it.*

As described in the introduction, a space X is \mathcal{DC} -like if Player I has a winning strategy in $G(\mathcal{DC}, X)$.

LEMMA 2.2 (R. Telgársky [19]). *If a space X has a countable closed cover by \mathcal{DC} -like sets, then X is a \mathcal{DC} -like space.*

Recall that a space X is *scattered* if every non-empty subset A of X has an isolated point of A , and *C-scattered* if for every non-empty closed subset A of X , there is a point of A which has a compact neighborhood in A . Then scattered spaces and locally compact Hausdorff spaces are C -scattered. Let X be a space. For each $F \in 2^X$, let

$$F^{(1)} = \{x \in F : x \text{ has no compact neighborhood in } F\}.$$

Let $X^{(0)} = X$. For each successor ordinal α , let $X^{(\alpha)} = (X^{(\alpha-1)})^{(1)}$. If α is a limit ordinal, let $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$. Notice that a space X is C -scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α . If X is C -scattered, let $\varepsilon(X) = \inf\{\alpha : X^{(\alpha)} = \emptyset\}$. We say that $\varepsilon(X)$ is the *C-scattered height* of X . For each $x \in X$, we denote by $\alpha_x(x)$ the ordinal such that $x \in X^{(\alpha_x(x))} - X^{(\alpha_x(x)+1)}$. Let X be a regular C -scattered space. If A is either open or closed in X , then A is C -scattered. More precisely, if A is an open subset of X , then $A^{(\alpha)} = X^{(\alpha)} \cap A$ for each $\alpha < \varepsilon(X)$ and if A is a closed subset of X , then $A^{(\alpha)} \subset A \cap X^{(\alpha)}$ for each $\alpha < \varepsilon(X)$. Therefore, if $x \in A$, then $\alpha_A(x) \leq \alpha_x(x)$ and hence, $\varepsilon(A) \leq \varepsilon(X)$. A space X is said to be *σ -scattered* (*σ -C-scattered*) if X is the union of countably many closed scattered (C -scattered) subspaces.

LEMMA 2.3 (R. Telgársky [19]). (a) *If a space X has a σ -closure-preserving closed cover by compact sets, then X is a \mathcal{DC} -like space.*

(b) *If X is a regular subparacompact, σ -C-scattered space, then X is \mathcal{DC} -like space.*

LEMMA 2.4 (G. Gruenhagen and Y. Yajima [11], Y. Yajima [21]). (a) *If X is a regular subparacompact (metacompact) \mathcal{DC} -like space, then $X \times Y$ is subparacompact (metacompact) for every subparacompact (metacompact) space Y .*

(b) *If X is a regular C -scattered metacompact space, then $X \times Y$ is metacompact for every metacompact space Y .*

For topological games, the reader is referred to R. Telgársky [18], [19] and Y. Yajima [21].

3. Preliminaries.

Let Z be a space and $\{Y_i : i \in \omega\}$ be a countable collection of spaces. For $Z \times \prod_{i \in \omega} Y_i$, we denote by \mathcal{B} the collection of all basic open subsets of $Z \times \prod_{i \in \omega} Y_i$. Let us denote by \mathcal{R} the collection of closed subsets of $Z \times \prod_{i \in \omega} Y_i$ consisting of sets of the form $R = E_R \times \prod_{i \in \omega} R_i$, where E_R is a closed subset of Z and there is an $n \in \omega$ such that for each $i \leq n$, R_i is a closed subset of Y_i and for each $i > n$, $R_i = Y_i$. For each $B = U_B \times \prod_{i \in \omega} B_i \in \mathcal{B}$ and $R = E_R \times \prod_{i \in \omega} R_i \in \mathcal{R}$, we define $n(B) = \inf \{i \in \omega : B_j = Y_j \text{ for } j \geq i\}$ and $n(R) = \inf \{i \in \omega : R_j = Y_j \text{ for } j \geq i\}$. We call $n(B)$ and $n(R)$ the *length* of B and R respectively. Let $\mathcal{K} = \{\prod_{i \in \omega} K_i : K_i \text{ is a compact subset of } Y_i \text{ for each } i \in \omega\}$. For each $z \in Z$ and $K \in \mathcal{K}$, let $K_{(z, K)} = \{z\} \times K$.

LEMMA 3.1 (D. K. Burke [3], [4]). *The following are equivalent for a space X .*

- (a) X is subparacompact,
- (b) Every open cover of X has a σ -locally finite closed refinement,
- (c) For every open cover \mathcal{U} of X , there is a sequence $\{\mathcal{V}_n\}_{n \in \omega}$ of open refinements of \mathcal{U} such that for each $x \in X$, there is an $n \in \omega$ with $\text{ord}(x, \mathcal{V}_n) = 1$.

It is well known that a space X is metacompact (metalindelöf) if and only if for every open cover \mathcal{U} of X , \mathcal{U}^F has a point finite (point countable) open refinement. In order to study subparacompactness of $Z \times \prod_{i \in \omega} Y_i$, we need the following lemma.

LEMMA 3.2. *Let Z be a space and $\{Y_i : i \in \omega\}$ be a countable collection of spaces. Assume that all finite subproducts of $Z \times \prod_{i \in \omega} Y_i$ are subparacompact. If, for every open cover \mathcal{O} of $Z \times \prod_{i \in \omega} Y_i$, \mathcal{O}^F has a σ -locally finite refinement consisting of elements of \mathcal{R} , then $Z \times \prod_{i \in \omega} Y_i$ is subparacompact.*

PROOF. Let \mathcal{O} be an open cover of $Z \times \prod_{i \in \omega} Y_i$. We may assume that $\mathcal{O} \subset \mathcal{B}$. By the assumption, there is a σ -locally finite refinement $\bigcup_{m \in \omega} \mathcal{R}_m$ of \mathcal{O}^F , consisting of elements of \mathcal{R} . Fix $m \in \omega$. For each $R = E_R \times \prod_{i \in \omega} R_i \in \mathcal{R}_m$, let $\{O(R, k) :$

$k=0, \dots, j(R)$ be a finite subcollection of \mathcal{O} such that $R \subset \bigcup_{k=0}^{j(R)} O(R, k)$. Let $O(R, k) = U_{R, k} \times \prod_{i \in \omega} O(R, k)_i$ for each $k \leq j(R)$, and let $n = \max\{n(R), n(O(R, k)) : k \leq j(R)\}$. Put $R(n) = E_R \times \prod_{i=0}^n R_i$ and $O(R, k, n) = U_{R, k} \times \prod_{i=0}^n O(R, k)_i$ for each $k \leq j(R)$. Let $\mathcal{O}(R) = \{O(R, k, n) : k \leq j(R)\}$. Then $R(n) \subset \bigcup \mathcal{O}(R)$. Since $Z \times \prod_{i=0}^n Y_i$ is subparacompact and $R(n)$ is a closed subspace of $Z \times \prod_{i=0}^n Y_i$, $R(n)$ is subparacompact. Thus there is a σ -discrete closed refinement $\bigcup_{t \in \omega} \mathcal{D}_t(R)$ of $\mathcal{O}(R)|R(n)$. For each $t \in \omega$, let $\mathcal{D}'_t(R) = \{D \times \prod_{i > n} Y_i : D \in \mathcal{D}_t\}$. Put $\mathcal{G}_{m, t} = \bigcup \{D'_t(R) : R \in \mathcal{R}_m\}$ for each $m, t \in \omega$. Then $\bigcup_{m, t \in \omega} \mathcal{G}_{m, t}$ is a σ -locally finite closed refinement of \mathcal{O} . It follows from Lemma 3.1 that $Z \times \prod_{i \in \omega} Y_i$ is subparacompact. The proof is completed.

In order to study metacompactness and metalindelöf property of countable products of C -scattered spaces, we need the following.

LEMMA 3.3. *Let X be a regular C -scattered metacompact (metalindelöf) space. For every open cover \mathcal{U} of X , there is a point finite (point countable) open cover \mathcal{V} of X such that: For each $V \in \mathcal{V}$,*

- (a) clV is contained in some member of \mathcal{U} ,
- (b) $(clV)^{(\alpha)}$ is compact for some $\alpha < \varepsilon(X)$.

PROOF. We prove this lemma by induction on the C -scattered height $\varepsilon(X)$ for the sake of completeness. Let X be a locally compact metacompact (metalindelöf) Hausdorff space (i. e. $\varepsilon(X) = 1$). Thus there is a point finite (point countable) open cover \mathcal{V} of X satisfying the condition (a) such that for each $V \in \mathcal{V}$, clW is compact. Clearly \mathcal{V} satisfies the condition (b). Let X be a regular C -scattered metacompact (metalindelöf) space and $\varepsilon = \varepsilon(X)$, and assume that for each $\alpha < \varepsilon$, the lemma holds. Then there is a point finite (point countable) open cover \mathcal{W} of X such that (cf. R. Telgársky [18, Theorem 1.6]): Let $W \in \mathcal{W}$.

- (i) clW is contained in some member of \mathcal{U} ,
- (ii) If ε is a successor ordinal, then $(clW)^{(\varepsilon-1)}$ is compact,
- (iii) If ε is a limit ordinal, then $(clW)^{(\alpha)} = \emptyset$ for some $\alpha < \varepsilon$.

Case 1. ε is a limit ordinal. By induction hypothesis, for each $W \in \mathcal{W}$, there is a point finite (point countable) open collection $\mathcal{V}'(W)$ in clW such that $\mathcal{V}'(W)$ covers clW and for each $V \in \mathcal{V}'(W)$, $(clV)^{(\alpha)}$ is compact for some $\alpha < \varepsilon$. Put $\mathcal{V}(W) = \mathcal{V}'(W)|W$ for each $W \in \mathcal{W}$ and $\mathcal{V} = \bigcup \{\mathcal{V}(W) : W \in \mathcal{W}\}$. Then \mathcal{V} satisfies the conditions (a) and (b).

Case 2. ε is a successor ordinal. Let $\mathcal{W}_0 = \{W \in \mathcal{W} : \varepsilon(clW) = \varepsilon\}$, and $\mathcal{W}_1 = \mathcal{W} - \mathcal{W}_0$. Take a $W \in \mathcal{W}_1$. Then $\varepsilon(clW) < \varepsilon$. By induction hypothesis, there is a point finite (point countable) open collection $\mathcal{C}'(W)$ in clW such that $\mathcal{C}'(W)$ covers clW and for each $V \in \mathcal{C}'(W)$, $(clV)^{(\alpha)}$ is compact for some $\alpha < \varepsilon$. Put $\mathcal{C}(W) = \mathcal{C}'(W)|_W$ for each $W \in \mathcal{W}_1$. Take a $W \in \mathcal{W}_0$. Since $\varepsilon(clW) = \varepsilon$, $(clW)^{(\varepsilon-1)}$ is compact. Let $\mathcal{C} = \mathcal{W}_0 \cup (\cup \{\mathcal{C}(W) : W \in \mathcal{W}_1\})$. Then \mathcal{C} satisfies the conditions (a) and (b).

The proof is completed.

4. Subparacompactness.

We firstly study subparacompactness of $Z \times \prod_{i \in \omega} Y_i$.

THEOREM 4.1. *If Z is a perfect subparacompact space and Y_i is a regular subparacompact \mathcal{DC} -like space for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is subparacompact.*

PROOF. Without loss of generality, we may assume that $Y_i = X$ for each $i \in \omega$ and there is an isolated point a in X . Indeed, put $X = \bigoplus_{i \in \omega} Y_i \cup \{a\}$, where $a \notin \bigcup_{i \in \omega} Y_i$. The topology of X is as follows: Every Y_i is an open-and-closed subspace of X and a is an isolated point in X . Since every Y_i is a regular subparacompact \mathcal{DC} -like space, by Lemma 2.2, X is a regular subparacompact \mathcal{DC} -like space. $Z \times \prod_{i \in \omega} Y_i$ is a closed subspace of $Z \times X^\omega$. Therefore, if $Z \times X^\omega$ is subparacompact, then $Z \times \prod_{i \in \omega} Y_i$ is subparacompact.

Let \mathcal{O} be an open cover of $Z \times X^\omega$. Put $\mathcal{O}' = \{B \in \mathcal{B} : B \subset O \text{ for some } O \in \mathcal{O}^F\}$. For each $z \in Z$ and $K \in \mathcal{K}$, there is an $O \in \mathcal{O}^F$ such that $K_{(z, K)} \subset O$. Then, by Wallace theorem in R. Engelking [8], there is a $B \in \mathcal{B}$ such that $K_{(z, K)} \subset B \subset O$. Thus we have $B \in \mathcal{O}'$. Define $n(K_{(z, K)}) = \inf \{n(O) : O \in \mathcal{O}' \text{ and } K_{(z, K)} \subset O\}$.

Let s be a stationary winning strategy for Player I in $G(\mathcal{DC}, X)$. Let $R = E_R \times \prod_{i \in \omega} R_i \in \mathcal{R}$ such that for each $i \leq n(R)$, we have already obtained a compact set $C_{\lambda(R, i)}$ of R_i . ($C_{\lambda(R, n(R))} = \emptyset$. $C_{\lambda(R, i)} = \emptyset$ may be occur for $i < n(R)$.) Fix $i \leq n(R)$. If $C_{\lambda(R, i)} \neq \emptyset$, let $F_{\gamma(R, i, m)} = R_i$ for each $m \in \omega$. Put $\Lambda(R, i) = \{\lambda(R, i)\}$ and $\Gamma(R, i, m) = \{\gamma(R, i, m)\}$ for each $m \in \omega$. Let $\mathcal{C}(R, i) = \{C_\lambda : \lambda \in \Lambda(R, i)\} = \{C_{\lambda(R, i)}\}$ and $\mathcal{F}(R, i, m) = \{F_\gamma : \gamma \in \Gamma(R, i, m)\} = \{F_{\gamma(R, i, m)}\}$ for each $m \in \omega$. Put $\mathcal{F}(R, i) = \bigcup_{m \in \omega} \mathcal{F}(R, i, m)$. Assume that $C_{\lambda(R, i)} = \emptyset$. Then there is a discrete collection $\mathcal{C}(R, i) = \{C_\lambda : \lambda \in \Lambda(R, i)\}$ of compact subsets of X such that $s(R_i) = \bigcup \mathcal{C}(R, i)$. Since R_i is a closed subspace of X , R_i is a subparacompact space.

Then there is a family $\mathcal{F}(R, i) = \bigcup_{m \in \omega} \mathcal{F}(R, i, m)$, where $\mathcal{F}(R, i, m) = \{F_\gamma : \gamma \in \Gamma(R, i, m)\}$, of collections of closed subsets in R_i (and hence, in X), satisfying

- (1) $\mathcal{F}(R, i)$ covers R_i ,
- (2) Every member of $\mathcal{F}(R, i)$ meets at most one member of $\mathcal{C}(R, i)$,
- (3) $\mathcal{F}(R, i, m)$ is discrete in X for each $m \in \omega$.

In each case, for $\gamma \in \bigcup_{m \in \omega} \Gamma(R, i, m)$, let $K_\gamma = F_\gamma \cap C_\lambda$ if $F_\gamma \cap C_\lambda \neq \emptyset$ for some (unique) C_λ . If $F_\gamma \cap (\bigcup \mathcal{C}(R, i)) = \emptyset$, then we take a point $p_\gamma \in F_\gamma$ and let $K_\gamma = \{p_\gamma\}$. Thus, if $C_{\lambda(R, i)} \neq \emptyset$, then $K_{\gamma(R, i, m)} = F_{\gamma(R, i, m)} \cap C_{\lambda(R, i)} = C_{\lambda(R, i)}$ for each $m \in \omega$. For $\eta = (m_0, \dots, m_{n(R)}) \in \omega^{n(R)+1}$, let $\Delta_{R, \eta} = \Gamma(R, 0, m_0) \times \dots \times \Gamma(R, n(R), m_{n(R)})$. For each $\eta \in \omega^{n(R)+1}$ and $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(R))) \in \Delta_{R, \eta}$, let $K(\delta) = K_{\gamma(\delta, 0)} \times \dots \times K_{\gamma(\delta, n(R))} \times \{a\} \times \dots \times \{a\} \times \dots$, and let $\mathcal{K}_{R, \eta} = \{K(\delta) : \delta \in \Delta_{R, \eta}\}$. Then $\mathcal{K}_{R, \eta} \subset \mathcal{K}$. For each $z \in E_R$, $\eta \in \omega^{n(R)+1}$ and $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(R))) \in \Delta_{R, \eta}$, let $r(K_{(z, K(\delta))}) = \max\{n(K_{(z, K(\delta))}), n(R)\}$. Fix $z \in E_R$, $\eta \in \omega^{n(R)+1}$ and $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(R))) \in \Delta_{R, \eta}$. Take an $O_{z, \delta} = U_{z, \delta} \times \prod_{i \in \omega} O_{z, \delta, i} \in \mathcal{O}'$ such that $K_{(z, K(\delta))} \subset O_{z, \delta}$ and $n(K_{(z, K(\delta))}) = n(O_{z, \delta})$. Then we can take a subset $H_{(z, K(\delta))} = H_{z, \delta} \times \prod_{i \in \omega} H_{(z, K(\delta)), i}$ of $Z \times X^\omega$ such that

- (4) $H_{z, \delta}$ is an open neighborhood of z in E_R such that $H_{z, \delta} \subset U_{z, \delta}$,
- (5) $H_{z, \delta} \times \prod_{i=0}^{n(K_{(z, K(\delta))})-1} cl H_{(z, K(\delta)), i} \times X \times \dots \times X \times \dots \subset O_{z, \delta}$,
- (6-1) For each i with $n(K_{(z, K(\delta))}) \leq i \leq r(K_{(z, K(\delta))})$, let $H_{(z, K(\delta)), i} = F_{\gamma(\delta, i)}$,
- (6-2) For each $i < n(K_{(z, K(\delta))})$ with $i \leq n(R)$, $H_{(z, K(\delta)), i}$ be an open subset of $F_{\gamma(\delta, i)}$ such that $K_{\gamma(\delta, i)} \subset H_{(z, K(\delta)), i} \subset cl H_{(z, K(\delta)), i} \subset O_{z, \delta, i}$,
- (6-3) For each i with $n(R) < i < n(K_{(z, K(\delta))})$, let $H_{(z, K(\delta)), i} = \{a\}$,
- (6-4) In case of that $r(K_{(z, K(\delta))}) = n(R)$, let $H_{(z, K(\delta)), i} = X$ for $n(R) < i$. In case of that $r(K_{(z, K(\delta))}) = n(K_{(z, K(\delta))}) > n(R)$, let $H_{(z, K(\delta)), i} = X$ for $n(K_{(z, K(\delta))}) \leq i$.

Then we have $K_{(z, K(\delta))} \subset H_{(z, K(\delta))}$. For each $j \in \omega$, let $V_j(K(\delta)) = \{z \in E_R : n(K_{(z, K(\delta))}) = j\}$ and $\mathcal{H}_j(K(\delta)) = \{H_{z, \delta} : n(K_{(z, K(\delta))}) = j\}$. Fix $j \in \omega$. Then $\bigcup_{k=0}^j V_k(K(\delta)) = \bigcup \{H_{z, \delta} : n(K_{(z, K(\delta))}) \leq j\} = \bigcup_{k=0}^j (\bigcup \mathcal{H}_k(K(\delta)))$. Since Z is a perfect space, $V_j(K(\delta))$ is an F_σ -set in E_R . Since E_R is subparacompact, there is a family $\mathcal{D}_{\eta, \delta, j} = \bigcup_{k \in \omega} \mathcal{D}_{\eta, \delta, j, k}$, where $\mathcal{D}_{\eta, \delta, j, k} = \{D_\xi : \xi \in \Xi_{\eta, \delta, j, k}\}$, of collections of closed subsets in E_R (and hence, in Z) satisfying

- (7) Every member of $\mathcal{D}_{\eta, \delta, j}$ is contained in some member of $\mathcal{H}_j(K(\delta)) \mid V_j(K(\delta))$,
- (8) $\mathcal{D}_{\eta, \delta, j}$ covers $V_j(K(\delta))$,
- (9) $\mathcal{D}_{\eta, \delta, j, k}$ is discrete in Z for each $k \in \omega$.

For $k \in \omega$ and $\xi \in \Xi_{\eta, \delta, j, k}$, take a $z(\xi) \in V_j(K(\delta))$ such that $D_\xi \subset H_{z(\xi), \delta} \cap V_j(K(\delta))$.

Put $F_\delta = \prod_{i=0}^{n(R)} F_{\gamma(\delta, i)} \times X \times \cdots \times X \times \cdots$ and $D_{\xi, \delta} = D_\xi \times F_\delta$. Then $\{D_{\xi, \delta} : \eta \in \omega^{n(R)+1}, \delta \in \Delta_{R, \eta}, j, k \in \omega \text{ and } \xi \in \Xi_{\eta, \delta, j, k}\}$ is a collection of elements of \mathcal{R} such that for each $\eta \in \omega^{n(R)+1}, \delta \in \Delta_{R, \eta}, j, k \in \omega$ and $\xi \in \Xi_{\eta, \delta, j, k}, D_{\xi, \delta} \subset R$ and $\{D_{\xi, \delta} : \eta \in \omega^{n(R)+1}, \delta \in \Delta_{R, \eta}, j, k \in \omega \text{ and } \xi \in \Xi_{\eta, \delta, j, k}\}$ covers R .

(10) For each $\eta \in \omega^{n(R)+1}$ and $j, k \in \omega, \{D_{\xi, \delta} : \delta \in \Delta_{R, \eta} \text{ and } \xi \in \Xi_{\eta, \delta, j, k}\}$ is discrete in $Z \times X^\omega$.

Fix $\eta \in \omega^{n(R)+1}$ and $j, k \in \omega$. Let $(z, x) \in Z \times X^\omega$ and $x = (x_i)_{i \in \omega}$. For each $i \leq n(R)$, since R_i is a closed subset of X , we may assume that $x_i \in R_i$. Then, for each $i \leq n(R)$, there is an open neighborhood $B(x_i)$ of x_i in X such that $|\{\delta \in \Delta_{R, \eta} : \prod_{i=0}^{n(R)} B(x_i) \cap F_\delta(n(R)) \neq \emptyset\}| \leq 1$, where $F_\delta(n(R)) = \prod_{i=0}^{n(R)} F_{\gamma(\delta, i)}$ for each $\delta \in \Delta_{R, \eta}$. Put $B'(x) = \prod_{i=0}^{n(R)} B(x_i)$ and $B(x) = B'(x) \times \prod_{i > n(R)} X_i$, where X_i is a copy of X for $i > n(R)$. If $B'(x) \cap F_\delta(n(R)) = \emptyset$ for each $\delta \in \Delta_{R, \eta}$, then $Z \times B(x) \in \mathcal{B}$ and $(Z \times B(x)) \cap D_{\xi, \delta} = \emptyset$ for each $\delta \in \Delta_{R, \eta}$ and $\xi \in \Xi_{\eta, \delta, j, k}$. Otherwise, take a unique $\delta \in \Delta_{R, \eta}$ such that $B'(x) \cap F_\delta(n(R)) \neq \emptyset$. Since $\mathcal{D}_{\eta, \delta, j, k}$ is discrete in Z , there is an open neighborhood U of z in Z such that $|\{\xi \in \Xi_{\eta, \delta, j, k} : U \cap D_\xi \neq \emptyset\}| \leq 1$. Then $U \times B(x) \in \mathcal{B}$ and $|\{D_{\xi, \delta'} : D_{\xi, \delta'} \cap (U \times B(x)) \neq \emptyset, \delta' \in \Delta_{R, \eta} \text{ and } \xi \in \Xi_{\eta, \delta', j, k}\}| \leq 1$. Thus $\{D_{\xi, \delta} : \delta \in \Delta_{R, \eta} \text{ and } \xi \in \Xi_{\eta, \delta, j, k}\}$ is discrete in $Z \times X^\omega$.

For each $\eta \in \omega^{n(R)+1}, \delta \in \Delta_{R, \eta}, j, k \in \omega$ and $\xi \in \Xi_{\eta, \delta, j, k}$, let $G_{\xi, \delta} = D_\xi \times \prod_{i \in \omega} clH_{(z(\xi), K(\delta)), i} \subset D_{\xi, \delta}$ and $\mathcal{G}_{\eta, \delta, j, k}(R) = \{G_{\xi, \delta} : \xi \in \Xi_{\eta, \delta, j, k}\}$. Define $\mathcal{G}_{\eta, j, k}(R) = \cup \{\mathcal{G}_{\eta, \delta, j, k}(R) : \delta \in \Delta_{R, \eta}\}$ for each $\eta \in \omega^{n(R)+1}$ and $j, k \in \omega$. Then we have

(11) For each $\eta \in \omega^{n(R)+1}, j, k \in \omega$, every member of $\mathcal{G}_{\eta, j, k}(R)$ is contained in some member of \mathcal{O}' .

(12) For each $\eta \in \omega^{n(R)+1}, j, k \in \omega, \mathcal{G}_{\eta, j, k}(R)$ is discrete in $Z \times X^\omega$.

This is clear from (10).

(13) For each $\eta \in \omega^{n(R)+1}, j, k \in \omega$, every element of $\mathcal{G}_{\eta, j, k}$ has the length $\max\{j, n(R)+1\}$.

Fix $\eta \in \omega^{n(R)+1}, \delta = (\gamma(\delta), 0), \dots, \gamma(\delta, n(R)) \in \Delta_{R, \eta}, j, k \in \omega$ and $\xi \in \Xi_{\eta, \delta, j, k}$. Then $n(K_{(z(\xi), K(\delta))}) = j$ and hence, $r(K_{(z(\xi), K(\delta))}) = \max\{j, n(R)\}$. Let $A \in \mathcal{P}(\{0, 1, \dots, r(K_{(z(\xi), K(\delta))})\})$. In case of that $r(K_{(z(\xi), K(\delta))}) = n(R)$, i.e., $n(R) \geq j$. For each $i \in A$, let $R_{\xi, A, i} = F_{\gamma(\delta, i)} - H_{(z(\xi), K(\delta)), i}$. For each $i \notin A$ with $i \leq n(R)$, let $R_{\xi, A, i} = clH_{(z(\xi), K(\delta)), i}$. For each $i > n(R)$, let $R_{\xi, A, i} = X$. Put $R_{\xi, A} = D_\xi \times \prod_{i \in \omega} R_{\xi, A, i}$. In case of that $j > n(R)$. For each $i \in A$ with $i \leq n(R)$, let $R_{\xi, A, i} = F_{\gamma(\delta, i)} - H_{(z(\xi), K(\delta)), i}$. For each $i \notin A$ with $i \leq n(R)$, let $R_{\xi, A, i} = clH_{(z(\xi), K(\delta)), i}$. Let $n(R) < i < j$. If $i \in A$, let $R_{\xi, A, i} = X - H_{(z(\xi), K(\delta)), i} = X - \{a\}$. If $i \notin A$, let $R_{\xi, A, i} = clH_{(z(\xi), K(\delta)), i} = \{a\}$. For $i \geq j$, let $R_{\xi, A, i} = X$. Put $R_{\xi, A} = D_\xi \times \prod_{i \in \omega} R_{\xi, A, i}$. In each

case, $R_{\xi, A, i} \subset R_i$ for each $i \in \omega$. Notice that if $R_{\xi, A} \neq \emptyset$, then $n(R) < n(R_{\xi, A})$. By the definition, $D_{\xi, \delta} = G_{\xi, \delta} \cup (\cup \{R_{\xi, A} : A \in \mathcal{P}(\{0, 1, \dots, \max\{j, n(R)\})\})$. For each $A \in \mathcal{P}(\{0, 1, \dots, \max\{j, n(R)\})$, let $\mathcal{R}_{\eta, \delta, j, k, A}(R) = \{R_{\xi, A} : \xi \in \Xi_{\eta, \delta, j, k} \text{ and } R_{\xi, A} \neq \emptyset\}$. For $j, k \in \omega$ and $A \in \mathcal{P}(\{0, 1, \dots, \max\{j, n(R)\})$, define $\mathcal{R}_{\eta, j, k, A}(R) = \cup \{\mathcal{R}_{\eta, \delta, j, k, A}(R) : \delta \in \Delta_{R, \eta}\}$. Then, by (10), we have

(14) Every $\mathcal{R}_{\eta, j, k, A}(R)$ is discrete in $Z \times X^\omega$.

Let $\mathcal{R}_{\eta, j, k}(R) = \cup \{\mathcal{R}_{\eta, j, k, A}(R) : A \in \mathcal{P}(\{0, 1, \dots, \max\{j, n(R)\})\}$. Then, by (14),

(15) For each $\eta \in \omega^{n(R)+1}$, $j, k \in \omega$, $\mathcal{R}_{\eta, j, k}(R)$ is locally finite in $Z \times X^\omega$.

(16) For each $\eta \in \omega^{n(R)+1}$ and $j, k \in \omega$ with $\mathcal{R}_{\eta, j, k} \neq \emptyset$, every element of $\mathcal{R}_{\eta, j, k}$ has the length $\max\{j, n(R)\} + 1$.

Fix a $R_{\xi, A} = D_\xi \times \prod_{i \in \omega} R_{\xi, A, i} \in \mathcal{R}_{\eta, \delta, j, k, A}(R)$ for $\eta \in \omega^{n(R)+1}$, $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_{R, \eta}$, $j, k \in \omega$, $\xi \in \Xi_{\eta, \delta, j, k}$ and $A \in \mathcal{P}(\{0, 1, \dots, \max\{j, n(R)\})$.

(17) For each $i \in A$ with $i \leq n(R)$ such that $C_{\lambda(R, i)} = \emptyset$, $s(R_i) \cap R_{\xi, A, i} = \emptyset$.

Since $R_{\xi, A, i} = F_{\gamma(\delta, i)} - H_{(z(\xi), K(\delta)), i}$, $s(R_i) \cap R_{\xi, A, i} = (\cup \mathcal{C}(R, i)) \cap (F_{\gamma(\delta, i)} - H_{(z(\xi), K(\delta)), i}) = K_{\gamma(\delta, i)} - H_{(z(\xi), K(\delta)), i} = \emptyset$.

For each $i \notin A$ with $i \leq n(R)$, a compact set $K_{\gamma(\delta, i)}$ is contained in $R_{\xi, A, i} = cl H_{(z(\xi), K(\delta)), i}$. Let $C_{\lambda(R_{\xi, A, i})} = K_{\gamma(\delta, i)}$. For each $i \notin A$ with $n(R) < i < j$, let $C_{\lambda(R_{\xi, A, i})} = \{a\}$. For each $i \in A$, let $C_{\lambda(R_{\xi, A, i})} = \emptyset$.

For $t \in \omega$, we shall inductively construct an index set Φ_t and two collections \mathcal{G}_τ and \mathcal{R}_τ for each $\tau \in \Phi_t$ satisfying

(18) For $t \geq 1$ and $\tau \in \Phi_t$, $\tau_- \in \Phi_{t-1}$,

(19) For $t \in \omega$ and $\tau \in \Phi_t$, \mathcal{G}_τ and \mathcal{R}_τ are collections of elements of \mathcal{R} ,

(20) For $t \in \omega$ and $\tau \in \Phi_t$ with $\mathcal{R}_\tau \neq \emptyset$, elements of \mathcal{R}_τ have the same length.

Let $\Phi_0 = \omega^3$. For each $\tau = (m, j, k) \in \Phi_0$, let $\mathcal{G}_\tau = \mathcal{G}_\tau(Z \times X^\omega) = \mathcal{G}_{m, j, k}(Z \times X^\omega)$ and $\mathcal{R}_\tau = \mathcal{R}_\tau(Z \times X^\omega) = \mathcal{R}_{m, j, k}(Z \times X^\omega)$. Let $\tau = (m, j, k) \in \Phi_0$. By the construction, \mathcal{G}_τ and \mathcal{R}_τ are collections of elements of \mathcal{R} . Assume that $\mathcal{R}_\tau \neq \emptyset$. By (16), elements of \mathcal{R}_τ have the same length. Thus \mathcal{G}_τ and \mathcal{R}_τ , $\tau \in \Phi_0$, satisfy the conditions (19) and (20). Assume that for $t \in \omega$, we have already obtained an index set Φ_i , for $i \leq t$, and families $\{\mathcal{G}_\tau : \tau \in \bigcup_{i=0}^t \Phi_i\}$, $\{\mathcal{R}_\tau : \tau \in \bigcup_{i=0}^t \Phi_i\}$ satisfying the conditions (18), (19) and (20). Take a $\tau \in \Phi_t$ with $\mathcal{R}_\tau \neq \emptyset$. By (20), elements of \mathcal{R}_τ have the same length. So we denote this length by $n(\tau)$. Let $\Phi_\tau = \{\tau \oplus (\eta, j, k) : \eta \in \omega^{n(\tau)+1}, j, k \in \omega\}$. For each $R \in \mathcal{R}_\tau$ and $\eta \in \omega^{n(\tau)+1}$, $j, k \in \omega$, we denote $\mathcal{G}_{\eta, j, k}(R)$ and $\mathcal{R}_{\eta, j, k}(R)$ by $\mathcal{G}_{\tau \oplus (\eta, j, k)}(R)$ and $\mathcal{R}_{\tau \oplus (\eta, j, k)}(R)$ respectively. Define $\mathcal{G}_{\tau \oplus (\eta, j, k)} = \cup \{\mathcal{G}_{\tau \oplus (\eta, j, k)}(R) : R \in \mathcal{R}_\tau\}$ and $\mathcal{R}_{\tau \oplus (\eta, j, k)} = \cup \{\mathcal{R}_{\tau \oplus (\eta, j, k)}(R) : R \in \mathcal{R}_\tau\}$. Let $\Phi_{t+1} = \cup \{\Phi_\tau : \tau \in \Phi_t \text{ and } \mathcal{R}_\tau \neq \emptyset\}$. Then, by (16) and the construction, Φ_{t+1} , families $\{\mathcal{G}_\mu : \mu \in \Phi_{t+1}\}$ and $\{\mathcal{R}_\mu : \mu \in \Phi_{t+1}\}$ satisfy the conditions (18),

(19) and (20). Thus, for each $t \in \omega$, we have an index set Φ_t , families $\{\mathcal{G}_\tau : \tau \in \Phi_t\}$ and $\{\mathcal{R}_\tau : \tau \in \Phi_t\}$ satisfying the conditions (18), (19) and (20). Let $\Phi = \bigcup \{\Phi_t : t \in \omega\}$. Then $|\Phi| \leq \omega$.

By Lemmas 2.4 and 3.2, our proof is complete if we show

CLAIM. $\bigcup \{\mathcal{G}_\tau : \tau \in \Phi\}$ is a σ -locally finite closed refinement of \mathcal{O}' .

PROOF OF CLAIM. Let $\tau \in \Phi$. By (19), $\mathcal{G}_\tau \subset \mathcal{R}$. By (11), every member of \mathcal{G}_τ is contained in some member of \mathcal{O}' . By (12), (15) and induction, \mathcal{G}_τ is locally finite in $Z \times X^\omega$. Assume that $\bigcup \{\mathcal{G}_\tau : \tau \in \Phi\}$ does not cover $Z \times X^\omega$. Take a point $(z, x) \in Z \times X^\omega - \bigcup \{\mathcal{G}_\tau : \tau \in \Phi\}$. Let $x = (x_i)_{i \in \omega}$. Take an $\eta(0) = m(0) \in \omega$ and $\delta(0) = \gamma(\delta(0), 0) \in \mathcal{A}_{Z \times X^\omega, \eta(0)} = \Gamma(Z \times X^\omega, 0, m(0))$ such that $x \in F_{\delta(0)}$. Put $\mathcal{F}(0) = \{F_{\gamma(\delta(0), 0)}\}$. Let $K(0) = K(\delta(0)) \in \mathcal{K}_{Z \times X^\omega, \eta(0)}$ and let $j(0) = n(K_{(z, K(0))})$. Choose a $k(0) \in \omega$ such that $(z, x) \in \bigcup \mathcal{G}_{\eta(0), j(0), k(0)}(Z \times X^\omega) \cup (\bigcup \mathcal{R}_{\eta(0), j(0), k(0)}(Z \times X^\omega))$. Let $\tau(0) = (\eta(0), j(0), k(0)) \in \Phi_0$. Take a $\xi(0) \in \bar{\mathcal{E}}_{\eta(0), \delta(0), j(0), k(0)}$ such that $z \in D_{\xi(0)}$. Put $\mathcal{H}(0) = \{H_{(z(\xi(0)), K(0)), i} : i \leq j(0)\}$. Since $(z, x) \notin \bigcup \mathcal{G}_{\tau(0)}$, there is an $A(0) \in \mathcal{P}(\{0, 1, \dots, j(0)\})$ such that $(z, x) \in R_{\xi(0), A(0)}$, $R_{\xi(0), A(0)} \in \mathcal{R}_{\tau(0)}(Z \times X^\omega)$. By the definition, if $0 \in A(0)$, then $R_{\xi(0), A(0), 0} = F_{\gamma(\delta(0), 0)} - H_{(z(\xi(0)), K(0)), 0}$. We have $0 = n(Z \times X^\omega) < n(R_{\xi(0), A(0)})$. For $R_{\xi(0), A(0)}$, take $\eta(1) \in \omega^{n(R_{\xi(0), A(0)})+1}$, $\delta(1) = (\gamma(\delta(1)), 0), \dots, \gamma(\delta(1), n(R_{\xi(0), A(0)})) \in \mathcal{A}_{R_{\xi(0), A(0)}, \eta(1)}$ such that $x \in F_{\delta(1)}$. Put $\mathcal{F}(1) = \{F_{\gamma(\delta(1), i)} : i \leq n(R_{\xi(0), A(0)})\}$. Let $K(1) = K(\delta(1)) \in \mathcal{K}_{R_{\xi(0), A(0)}, \eta(1)}$ and $j(1) = n(K_{(z, K(1))})$. Take a $k(1) \in \omega$ such that $(z, x) \in \bigcup \mathcal{G}_{\eta(1), j(1), k(1)}(R_{\xi(0), A(0)}) \cup (\bigcup \mathcal{R}_{\eta(1), j(1), k(1)}(R_{\xi(0), A(0)}))$. Let $\tau(1) = ((\eta(0), j(0), k(0)), (\eta(1), j(1), k(1))) \in \Phi_1$. Take a $\xi(1) \in \bar{\mathcal{E}}_{\eta(1), \delta(1), j(1), k(1)}$ such that $z \in D_{\xi(1)}$. Put $\mathcal{H}(1) = \{H_{(z(\xi(1)), K(1)), i} : i \leq \max\{j(1), n(R_{\xi(0), A(0)})\}\}$. Since $(z, x) \notin \bigcup \mathcal{G}_{\tau(1)}$, there is an $A(1) \in \mathcal{P}(\{0, 1, \dots, \max\{j(1), n(R_{\xi(0), A(0)})\}\})$ such that $(z, x) \in R_{\xi(1), A(1)}$, $R_{\xi(1), A(1)} \in \mathcal{R}_{\tau(1)}(R_{\xi(0), A(0)})$. Then, if $i \in A(1)$ with $i \leq n(R_{\xi(0), A(0)})$, then $R_{\xi(1), A(1), i} = F_{\gamma(\delta(1), i)} - H_{(z(\xi(1)), K(1)), i}$. We have $n(R_{\xi(0), A(0)}) < n(R_{\xi(1), A(1)})$. Continuing this matter, we can choose a sequence $\{\eta(t) : t \in \omega\}$ of elements of $\omega^{< \omega}$, a sequence $\{\delta(t) : t \in \omega\}$, a sequence $\{\mathcal{F}(t) : t \in \omega\}$ of collections, a sequence $\{K(t) : t \in \omega\}$ of compact subsets in X^ω , where $K(t) = \prod_{i \in \omega} K(t)_i \in \mathcal{K}$, sequences $\{j(t) : t \in \omega\}$, $\{k(t) : t \in \omega\}$ of natural numbers, a sequence $\{\tau(t) : t \in \omega\}$ of elements of Φ , where $\tau(t) = ((\eta(0), j(0), k(0)), \dots, (\eta(t), j(t), k(t)))$, a sequence $\{\xi(t) : t \in \omega\}$, a sequence $\{\mathcal{H}(t) : t \in \omega\}$ of collections, a sequence $\{A(t) : t \in \omega\}$ of finite subsets of ω , a sequence $\{R_{\xi(t), A(t)} : t \in \omega\}$ of elements of \mathcal{R} containing (z, x) , where $R_{\xi(t), A(t)} = D_{\xi(t)} \times \prod_{i \in \omega} R_{\xi(t), A(t), i}$, satisfying the following: Let $t \in \omega$. Assume that we have already obtained sequences $\{\eta(i) : i \leq t\}$, $\{\delta(i) : i \leq t\}$, $\{\mathcal{F}(i) : i \leq t\}$, $\{K(i) : i \leq t\}$, $\{j(i) : i \leq t\}$, $\{k(i) : i \leq t\}$, $\{\tau(i) : i \leq t\}$, $\{\xi(i) : i \leq t\}$, $\{\mathcal{H}(i) : i \leq t\}$, $\{A(i) : i \leq t\}$ and $\{R_{\xi(i), A(i)} : i \leq t\}$. Then

$$(21) \quad \eta(t+1) \in \omega^{n(R_{\xi(t), A(t)})+1},$$

(22) $\delta(t+1) = (\gamma(\delta(t+1), 0), \dots, \gamma(\delta(t+1), n(R_{\xi(t), A(t)}))) \in \mathcal{A}_{R_{\xi(t), A(t)}, \eta(t+1)}$ such that $x \in F_{\delta(t+1)}$, and $\mathcal{F}(t+1) = \{F_{\gamma(\delta(t+1), i)} : i \leq n(R_{\xi(t), A(t)})\}$,

$$(23) \quad K(t+1) = K(\delta(t+1)) \in \mathcal{K}_{R_{\xi(t), A(t)}, \eta(t+1)},$$

(24) $j(t+1) = n(K_{(z, K(t+1))})$, $k(t+1) \in \omega$ and $\tau(t+1) = ((\eta(0), j(0), k(0)), \dots, (\eta(t+1), j(t+1), k(t+1))) \in \Phi_{t+1}$,

(25) $\xi(t+1) \in \mathcal{E}_{\xi(t+1), \delta(t+1), j(t+1), k(t+1)}$, $\mathcal{H}(t+1) = \{H_{(z(\xi(t+1)), K(t+1)), i} : i \leq \max\{j(t+1), n(R_{\xi(t), A(t)})\}\}$ and $A(t+1) \in \mathcal{P}(\{0, 1, \dots, \max\{j(t+1), n(R_{\xi(t), A(t)})\}\})$,

(26) If $i \in A(t+1)$ with $i \leq n(R_{\xi(t), A(t)})$, then $R_{\xi(t+1), A(t+1), i} = F_{\gamma(\delta(t+1), i)} - H_{(z(\xi(t+1)), K(t+1)), i}$,

(27) $(z, x) \in R_{\xi(t+1), A(t+1)} = D_{\xi(t+1)} \times \prod_{i \in \omega} R_{\xi(t+1), A(t+1), i}$, $R_{\xi(t+1), A(t+1)} \in \mathcal{R}_{\tau(t+1)}(R_{\xi(t), A(t)})$, and $n(R_{\xi(t), A(t)}) < nR_{\xi(t+1), A(t+1)}$,

(28) For each $i \leq n(R_{\xi(t), A(t)})$ with $i \in A(t+1)$ such that $C_{\lambda(R_{\xi(t), A(t)}, i)} = \emptyset$, $s(R_{\xi(t), A(t), i}) \cap R_{\xi(t+1), A(t+1), i} = \emptyset$,

(29) For each $i \leq n(R_{\xi(t), A(t)})$ with $i \notin A(t+1)$ such that $C_{\lambda(R_{\xi(t), A(t)}, i)} \neq \emptyset$, $K(t+1)_i = C_{\lambda(R_{\xi(t), A(t)}, i)}$.

The rest of the proof is similar to that of Theorem 3.2 in the author [17]. However we include it here, because the method of it plays the fundamental role in this paper.

Assume that for each $i \in \omega$, $|\{t \in \omega : i \in A(t)\}| < \omega$. Then for each $i \in \omega$, there is a $t_i \in \omega$ such that $i \leq t_i$ and if $t \geq t_i$, then $i \notin A(t)$. Then, by (29),

(30) For each $i \in \omega$ and $t \geq t_i$, $K(t)_i = K(t_i)_i$.

Let $K = \prod_{i \in \omega} K(t_i)_i \in \mathcal{K}$. There is an $O \in \mathcal{O}'$ such that $K_{(z, K)} \subset O$. By (27) and (30), take a $t \geq 1$ such that $n(O) \leq n(R_{\xi(t-1), A(t-1)})$ and if $i \leq n(O)$, then $K(t)_i = K(t_i)_i$. Then we have $K_{(z, K(t))} \subset O$ and hence, $j(t) = n(K_{(z, K(t))}) \leq n(O)$. Since $\xi(t) \in \mathcal{E}_{\eta(t), \delta(t), j(t), k(t)}$, $n(K_{(z(\xi(t)), K(t))}) = j(t)$. For i with $n(O) \leq i \leq n(R_{\xi(t-1), A(t-1)})$, by the definition, $H_{(z(\xi(t)), K(t)), i} = F_{\gamma(\delta(t), i)}$. Hence $A(t) \cap \{n(O), \dots, n(R_{\xi(t-1), A(t-1)})\} = \emptyset$. Since $(z, x) \in R_{\xi(t), A(t)}$ and $R_{\xi(t), A(t)} \in \mathcal{R}_{\tau(t)}(R_{\xi(t-1), A(t-1)})$, there is an $i \in A(t)$ such that $x_i \notin H_{(z(\xi(t)), K(t)), i}$. Thus $i < n(O)$ and $x_i \in R_{\xi(t), A(t), i} = F_{\gamma(\delta(t), i)} - H_{(z(\xi(t)), K(t)), i}$. Since $i \in A(t)$, $t < t_i$. For each $t' > t$, $K(t')_i \subset R_{\xi(t), A(t), i}$. Thus $K(t_i)_i \subset R_{\xi(t), A(t), i}$. Since $K(t)_i \subset H_{(z(\xi(t)), K(t)), i}$, we have $K(t)_i \neq K(t_i)_i$. This is a contradiction. Therefore there is an $i \in \omega$ such that $|\{t \in \omega : i \in A(t)\}| = \omega$. Let $\{t \in \omega : i \in A(t) \text{ and } i \leq n(R_{\xi(t), A(t)})\} = \{t_\rho : \rho \in \omega\}$. Let $\rho \in \omega$. Since $C_{\lambda(R_{\xi(t_\rho), A(t_\rho)}, i)} = \emptyset$, if $t_{\rho+1} = t_\rho + 1$, then, by (28), $s(R_{\xi(t_\rho), A(t_\rho), i}) \cap R_{\xi(t_{\rho+1}), A(t_{\rho+1}), i} = \emptyset$. Assume that $t_{\rho+1} > t_\rho + 1$. Since $K_{\gamma(\delta(t_{\rho+1}), i)} = C_{\lambda(R_{\xi(t_{\rho+1}), A(t_{\rho+1}), i)} = C_{\lambda(R_{\xi(t_{\rho+1}-1), A(t_{\rho+1}-1), i)} \subset H_{(z(\xi(t_{\rho+1})), K(t_{\rho+1})), i}$, by the definition, we have $s(R_{\xi(t_\rho), A(t_\rho), i}) \cap R_{\xi(t_{\rho+1}), A(t_{\rho+1}), i} = \emptyset$. Since s is a stationary winning strategy

for Player I in $G(\mathcal{DC}, X)$, $\bigcap_{\rho \in \omega} R_{\xi(t_\rho), A(t_\rho), i} = \emptyset$. But $x_i \in \bigcap_{\rho \in \omega} R_{\xi(t_\rho), A(t_\rho), i}$, which is a contradiction. It follows that $\bigcup \{\mathcal{G}_\tau : \tau \in \Phi\}$ is a cover of $Z \times X^\omega$. The proof is completed.

COROLLARY 4.2. *If Z is a perfect subparacompact space and Y_i is a regular subparacompact space with a σ -closure-preserving cover by compact sets for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is subparacompact.*

PROOF. This immediately follows from Theorem 4.1 and Lemma 2.3(a).

Similarly, by Theorem 4.1 and Lemma 2.3(b), we have

COROLLARY 4.3. *If Z is a perfect subparacompact space and Y_i is a regular subparacompact, σ - C -scattered space for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is subparacompact.*

REMARK 4.4. Let M be the Michael line and let P be the space of irrationals. P is homeomorphic to ω^ω . The following are well-known (see D. K. Burke [4]).

(a) M is hereditarily paracompact but $M \times P$ is not normal and hence, not paracompact.

(b) $M \times P$ is hereditarily subparacompact and hereditarily metacompact (see also P. Nyikos [15]).

5. Metacompactness, orthocompactness and metalindelöf property.

THEOREM 5.1. *If Y_i is a regular metacompact \mathcal{DC} -like space for each $i \in \omega$, then $\prod_{i \in \omega} Y_i$ is metacompact.*

PROOF. We may assume that $Y_i = X$ for each $i \in \omega$ and there is an isolated point a in X . Let \mathcal{O} be an open cover of $Z \times X^\omega$. Similarly, let $\mathcal{O}' = \{B \in \mathcal{B} : B \subset O \text{ for some } O \in \mathcal{O}^F\}$. For $K \in \mathcal{K}$, there is an $O \in \mathcal{O}^F$ such that $K \subset O$. Then there is a $B \in \mathcal{B}$ such that $K \subset B \subset O$. Define $n(K) = \inf \{n(O) : O \in \mathcal{O}' \text{ and } K \subset O\}$. It suffices to prove that \mathcal{O}' has a point finite open refinement.

Let s be a stationary winning strategy for Player I in $G(\mathcal{DC}, X)$. Let $B = \prod_{i \in \omega} B_i \in \mathcal{B}$ such that for each $i \leq n(B)$, we have already obtained a compact set $C_{\lambda(B, i)}$ of $cl B_i$. ($C_{\lambda(B, n(B))} = \emptyset$. $C_{\lambda(B, i)} = \emptyset$ may be occur for $i < n(B)$.) We define $\mathcal{G}(B)$ and $\mathcal{B}(B)$ of collections of elements of \mathcal{B} . Fix $i \leq n(B)$. If $C_{\lambda(B, i)}$

$\neq \emptyset$, let $W_{\gamma(B,i)} = B_i$. Put $\Lambda(B, i) = \{\lambda(B, i)\}$ and $\Gamma(B, i) = \{\gamma(B, i)\}$. Let $\mathcal{C}(B, i) = \{C_\lambda : \lambda \in \Lambda(B, i)\} = \{C_{\lambda(B,i)}\}$, and $\mathcal{W}(B, i) = \{W_\gamma : \gamma \in \Gamma(B, i)\} = \{W_{\gamma(B,i)}\}$. Assume that $C_{\lambda(B,i)} = \emptyset$. Then there is a discrete collection $\mathcal{C}(B, i) = \{C_\lambda : \lambda \in \Lambda(B, i)\}$ of compact subsets of X such that $s(clB_i) = \cup \mathcal{C}(B, i)$. Since X is a regular metacompact space, there is a collection $\mathcal{W}(B, i) = \{W_\gamma : \gamma \in \Gamma(B, i)\}$ of open subsets in B_i (and hence, in X) satisfying

- (1) $\mathcal{W}(B, i)$ covers B_i ,
- (2) For each $\gamma \in \Gamma(B, i)$, clW_γ meets at most one member of $\mathcal{C}(B, i)$,
- (3) $\mathcal{W}(B, i)$ is point finite in B_i and hence, point finite in X .

In each case, for $\gamma \in \Gamma(B, i)$, $K_\gamma = clW_\gamma \cap C_\lambda$ if $clW_\gamma \cap C_\lambda \neq \emptyset$ for some (unique) C_λ . If $clW_\gamma \cap (\cup \mathcal{C}(B, i)) = \emptyset$, then we take a point $p \in W_\gamma$ and let $K_\gamma = \{p_\gamma\}$. Thus, if $C_{\lambda(B,i)} \neq \emptyset$, then $K_{\gamma(B,i)} = clW_{\gamma(B,i)} \cap C_{\lambda(B,i)} = C_{\lambda(B,i)}$. Put $\Delta_B = \Gamma(B, 0) \times \cdots \times \Gamma(B, n(B))$. For each $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B$, let $K(\delta) = K_{\gamma(\delta, 0)} \times \cdots \times K_{\gamma(\delta, n(B))} \times \{a\} \times \cdots \times \{a\} \times \cdots$, and let $\mathcal{K}_B = \{K(\delta) : \delta \in \Delta_B\}$. Then $\mathcal{K}_B \subset \mathcal{K}$. For each $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B$, let $r(K(\delta)) = \max\{n(K(\delta)), n(B)\}$. Fix a $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B$. Take an $O(\delta) = \prod_{i \in \omega} O(\delta)_i \in \mathcal{O}'$ such that $K(\delta) \subset O(\delta)$ and $n(K(\delta)) = n(O(\delta))$. Since X is a regular space, there is an $H(\delta) = \prod_{i \in \omega} H(\delta)_i \in \mathcal{B}$ such that:

$$(4) \quad \prod_{i=0}^{n(K(\delta))-1} clH(\delta)_i \times X \times \cdots \times X \times \cdots \subset O(\delta),$$

(5-1) For each i with $n(K(\delta)) \leq i \leq r(K(\delta))$, let $H(\delta)_i = X$,

(5-2) For each $i < n(K(\delta))$ with $i \leq n(B)$, let $H(\delta)_i$ be an open subset of X such that $K_{\gamma(\delta, i)} \subset H(\delta)_i \subset clH(\delta)_i \subset O(\delta)_i$,

(5-3) For each i with $n(B) < i < n(K(\delta))$, let $H(\delta)_i = \{a\}$,

(5-4) In case of that $r(K(\delta)) = n(B)$, let $H(\delta)_i = X$ for $n(B) < i$. In case of that $r(K(\delta)) = n(K(\delta)) > n(B)$, let $H(\delta)_i = X$ for $n(K(\delta)) \leq i$.

Then we have $K(\delta) \subset H(\delta)$. Put $W(\delta) = \prod_{i=0}^{n(B)} W_{\gamma(\delta, i)} \times X \times \cdots \times X \times \cdots$. Then $\{W(\delta) : \delta \in \Delta_B\}$ is a collection of elements of \mathcal{B} such that for each $\delta \in \Delta_B$, $W(\delta) \subset B$ and $\{W(\delta) : \delta \in \Delta_B\}$ covers B . By the definition, we have

(6) $\{W(\delta) : \delta \in \Delta_B\}$ is point finite in X^ω .

Fix a $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B$. In case of that $r(K(\delta)) = n(B)$. For each $i \leq n(B)$, let $G(\delta)_i = O(\delta)_i \cap W_{\gamma(\delta, i)}$. For each $i > n(B)$, let $G(\delta)_i = X$. Put $G(\delta) = \prod_{i \in \omega} G(\delta)_i$. In case of that $r(K(\delta)) = n(K(\delta)) > n(B)$. For each $i \leq n(B)$, let $G(\delta)_i = O(\delta)_i \cap W_{\gamma(\delta, i)}$. For each i with $n(B) < i < n(K(\delta))$, let $G(\delta)_i = H(\delta)_i = \{a\}$. For each $i > n(K(\delta))$, let $G(\delta)_i = X$. Put $G(\delta) = \prod_{i \in \omega} G(\delta)_i$. Then we have $G(\delta) \subset W(\delta)$. Define $\mathcal{G}(B) = \{G(\delta) : \delta \in \Delta_B\}$. Then

(7) Every member of $\mathcal{G}(B)$ is contained in some member of \mathcal{O}' .

(8) $\mathcal{G}(B)$ is point finite in X^ω .

This is clear from (6).

Fix $\delta=(\gamma(\delta, 0), \dots, \gamma(\delta, n(B)))\in\Delta_B$. Let $A\in\mathcal{P}(\{0, 1, \dots, r(K(\delta))\})$. In case of that $r(K(\delta))=n(B)$. For each $i\in A$, let $B_{\delta,A,i}=W_{\gamma(\delta,i)}-clH(\delta)_i$. For each $i\notin A$ with $i\leq n(B)$, let $B_{\delta,A,i}=O(\delta)_i\cap W_{\gamma(\delta,i)}$. For each $i>n(B)$, let $B_{\delta,A,i}=X$. Put $B_{\delta,A}=\prod_{i\in\omega} B_{\delta,A,i}$. In case of that $r(K(\delta))=n(K(\delta))>n(B)$. For each $i\in A$ with $i\leq n(B)$, let $B_{\delta,A,i}=W_{\gamma(\delta,i)}-clH(\delta)_i$. For each $i\notin A$ with $i\leq n(B)$, let $B_{\delta,A,i}=O(\delta)_i\cap W_{\gamma(\delta,i)}$. Let $n(B)<i<n(K(\delta))$. If $i\in A$, let $B_{\delta,A,i}=X-clH(\delta)_i=X-\{a\}$. If $i\notin A$, let $B_{\delta,A,i}=H(\delta)_i=\{a\}$. For $i\geq n(K(\delta))$, let $B_{\delta,A,i}=X$. Put $B_{\delta,A}=\prod_{i\in\omega} B_{\delta,A,i}$. In each case, $B_{\delta,A,i}\subset B_i$ for each $i\in\omega$. We have that if $B_{\delta,A}\neq\emptyset$, then $n(B)<n(B_{\delta,A})$. Let $\mathcal{B}_\delta(B)=\{B_{\delta,A}: A\in\mathcal{P}(\{0, 1, \dots, r(K(\delta))\})\}$ and $B_{\delta,A}\neq\emptyset$. By the definition, $W(\delta)=G(\delta)\cup(\cup\mathcal{B}_\delta(B))$. Define $\mathcal{B}(B)=\cup\{\mathcal{B}_\delta(B): \delta\in\Delta_B\}$. Then, by (6), we have

(9) $\mathcal{B}(B)$ is point finite in X^ω .

Fix a $B_{\delta,A}=\prod_{i\in\omega} B_{\delta,A,i}\in\mathcal{B}_\delta(B)$ for $\delta=(\gamma(\delta, 0), \dots, \gamma(\delta, n(B)))\in\Delta_B$ and $A\in\mathcal{P}(\{0, 1, \dots, r(K(\delta))\})$.

(10) For each $i\in A$ with $i\leq n(B)$ such that $C_{\lambda(B,i)}=\emptyset$, $s(clB_i)\cap clB_{\delta,A,i}=\emptyset$.

Since $B_{\delta,A,i}=W_{\gamma(\delta,i)}-clH(\delta)_i$, $s(clB_i)\cap clB_{\delta,A,i}\subset(\cup C(B,i))\cap(clW_{\gamma(\delta,i)}-H(\delta)_i)=K_{\gamma(\delta,i)}-H(\delta)_i=\emptyset$.

For each $i\notin A$ with $i\leq n(B)$, since $clB_{\delta,A,i}=cl(O(\delta)_i\cap W_{\gamma(\delta,i)})\supset O(\delta)_i\cap clW_{\gamma(\delta,i)}$, a compact set $K_{\gamma(\delta,i)}$ is contained in $clB_{\delta,A,i}$. Let $C_{\lambda(B_{\delta,A,i})}=K_{\gamma(\delta,i)}$. For each $i\notin A$ with $n(B)<i<n(K(\delta))$, let $C_{\lambda(B_{\delta,A,i})}=\{a\}$. For each $i\in A$, let $C_{\lambda(B_{\delta,A,i})}=\emptyset$.

Now we define \mathcal{G}_j and \mathcal{B}_j for each $j\in\omega$. Let $\mathcal{G}_0=\mathcal{G}_0(X^\omega)=\mathcal{G}(X^\omega)$ and $\mathcal{B}_0=\mathcal{B}_0(X^\omega)=\mathcal{B}(X^\omega)$. Assume that for $j\in\omega$, we have already obtained \mathcal{G}_j and \mathcal{B}_j . For each $B\in\mathcal{B}_j$, we denote $\mathcal{G}(B)$ and $\mathcal{B}(B)$ by $\mathcal{G}_{j+1}(B)$ and $\mathcal{B}_{j+1}(B)$ respectively. Define $\mathcal{G}_{j+1}=\cup\{\mathcal{G}_{j+1}(B): B\in\mathcal{B}_j\}$ and $\mathcal{B}_{j+1}=\cup\{\mathcal{B}_{j+1}(B): B\in\mathcal{B}_j\}$.

Our proof is complete if we show

CLAIM. $\cup\{\mathcal{G}_j: j\in\omega\}$ is a point finite open refinement of \mathcal{O}' .

PROOF OF CLAIM. Let $j\in\omega$. By the construction, $\mathcal{G}_j\subset\mathcal{B}$. By (7), every member of \mathcal{G}_j is contained in some member of \mathcal{O}' . By (8), (9) and induction, \mathcal{G}_j is point finite in X^ω . Take a $x=(x_i)_{i\in\omega}\in X^\omega$. Let $\Delta(0)=\{\delta\in\Delta_{X^\omega}: x\in W(\delta)\}$. Then, by (6), $1\leq|\Delta(0)|<\omega$. Let $\mathcal{K}(0)=\{K(\delta): \delta\in\Delta(0)\}$. Put $\mathcal{H}(0)=\{H(\delta): \delta\in\Delta(0)\}$, $\mathcal{W}(0)=\{W(\delta): \delta\in\Delta(0)\}$ and $\mathcal{G}(0)=\{G(\delta): \delta\in\Delta(0)\}\subset\mathcal{G}_0$. For each $\delta\in\Delta(0)$, let $\mathcal{A}(\delta)=\mathcal{P}(\{0, 1, \dots, r(K(\delta))\})$, and let $\mathcal{A}(0)=\cup\{\mathcal{A}(\delta): \delta\in\Delta(0)\}$. Let $\mathcal{B}(0)=\cup\{\mathcal{B}_\delta(X^\omega): \delta\in\Delta(0)\}$. Then $\mathcal{B}(0)\subset\mathcal{B}_0$. By the definition, for each $\delta=\gamma(\delta, 0)\in$

$\mathcal{A}(0)$ and $A \in \mathcal{A}(\delta)$ with $0 \in A$, $B_{\delta, A, 0} = W_{\gamma(\delta, 0)} - clH(\delta)_0$. Since $W(\delta) = G(\delta) \cup (\cup \mathcal{B}_\delta(X^\omega))$ for each $\delta \in \mathcal{A}(0)$, $1 \leq |\mathcal{G}(0) \cup \mathcal{B}(0)| < \omega$. Observe that $(\mathcal{G}_0 \cup \mathcal{B}_0)_x \subset \mathcal{G}(0) \cup \mathcal{B}(0)$. Take a $B \in \mathcal{B}(0)$. Let $\mathcal{A}(B) = \{\delta' \in \mathcal{A}_B : x \in W(\delta')\}$ and let $\mathcal{A}(1) = \cup \{\mathcal{A}(B) : B \in \mathcal{B}(0)\}$. Let $\mathcal{K}(1) = \{K(\delta) : \delta \in \mathcal{A}(1)\}$. Put $\mathcal{H}(1) = \{H(\delta) : \delta \in \mathcal{A}(1)\}$, $\mathcal{W}(1) = \{W(\delta) : \delta \in \mathcal{A}(1)\}$ and $\mathcal{G}(1) = \{G(\delta) : \delta \in \mathcal{A}(1)\} \subset \mathcal{G}_1$. Define $\mathcal{A}(\delta)$ for each $\delta \in \mathcal{A}(1)$, and $\mathcal{A}(1)$ as before. Let $\mathcal{B}(1) = \cup \{\mathcal{B}_\delta(B) : B \in \mathcal{B}(0) \text{ and } \delta \in \mathcal{A}(B)\} \subset \mathcal{B}_1$. Let $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \mathcal{A}(B)$ and $B \in \mathcal{B}(0)$. For each $A \in \mathcal{A}(\delta)$, if $i \in A$ with $i \leq n(B)$, then $B_{\delta, A, i} = W_{\gamma(\delta, i)} - clH(\delta)_i$. We have $|\mathcal{G}(1) \cup \mathcal{B}(1)| < \omega$ and $(\mathcal{G}_1 \cup \mathcal{B}_1)_x \subset \mathcal{G}(1) \cup \mathcal{B}(1)$. Continuing this matter, we can choose a collection $\{\mathcal{A}(j) : j \in \omega\}$, a family $\{\mathcal{K}(j) : j \in \omega\}$ of collections of compact subsets of X^ω , where for each $K \in \mathcal{K}(j)$ and $j \in \omega$, $K = \prod_{i \in \omega} K_i \in \mathcal{K}$, families $\{\mathcal{H}(j) : j \in \omega\}$, $\{\mathcal{W}(j) : j \in \omega\}$, $\{\mathcal{G}(j) : j \in \omega\}$ of collections of elements of \mathcal{B} , a family $\{\mathcal{A}(j) : j \in \omega\}$ of collections of finite subsets of ω and a family $\{\mathcal{B}(j) : j \in \omega\}$ of collections of elements of \mathcal{B} such that for $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \mathcal{A}(B)$, $B \in \mathcal{B}(j-1)$, where $B_{\delta(-1), \mathcal{A}(-1)} = X^\omega$, and $\mathcal{B}_{-1} = \mathcal{B}(-1) = \{X^\omega\}$, and $A \in \mathcal{A}(\delta)$, if $i \in A$ with $i \leq n(B)$, then $B_{\delta, A, i} = W_{\gamma(\delta, i)} - clH(\delta)_i$, and for each $j \in \omega$, $|\mathcal{G}(j) \cup \mathcal{B}(j)| < \omega$ and $(\mathcal{G}_j \cup \mathcal{B}_j)_x \subset \mathcal{G}(j) \cup \mathcal{B}(j)$. Assume that $x \in \cup \mathcal{B}_j$ for each $j \in \omega$. Then, by the construction, $x \in \cup \mathcal{B}(j)$ for each $j \in \omega$. Since $\mathcal{B}(j)_x$ is non-empty and finite for each $j \in \omega$, it follows from König's lemma (cf. K. Kunen [13]) that there are a sequence $\{\delta(j) : j \in \omega\}$, a sequence $\{K(j) : j \in \omega\}$ of compact subsets of X^ω , sequences $\{H(\delta(j)) : j \in \omega\}$, $\{W(\delta(j)) : j \in \omega\}$ of elements of \mathcal{B} , a sequence $\{\mathcal{A}(j) : j \in \omega\}$ of finite subsets of ω , a sequence $\{B_{\delta(j), \mathcal{A}(j)} : j \in \omega\}$ of elements of \mathcal{B} such that:

$$(11) \quad \delta(j) = (\gamma(\delta(j), 0), \dots, \gamma(\delta(j), n(B_{\delta(j-1), \mathcal{A}(j-1)}))) \in \mathcal{A}(j),$$

$$(12) \quad K(j) = K(\delta(j)),$$

$$(13) \quad \mathcal{A}(j) \in \mathcal{A}(\delta(j)),$$

$$(14) \quad \text{For each } i \in \mathcal{A}(j) \text{ with } i \leq n(B_{\delta(j-1), \mathcal{A}(j-1)}), \quad B_{\delta(j), \mathcal{A}(j), i} = W_{\gamma(\delta(j), i)} - clH(\delta(j))_i,$$

$$(15) \quad x \in B_{\delta(j), \mathcal{A}(j)} \text{ and } B_{\delta(j), \mathcal{A}(j)} \in \mathcal{B}(B_{\delta(j-1), \mathcal{A}(j-1)}).$$

Furthermore we have

$$(16) \quad n(B_{\delta(j), \mathcal{A}(j)}) < n(B_{\delta(j+1), \mathcal{A}(j+1)}) \text{ for each } j \in \omega,$$

$$(17) \quad \text{For each } i \leq n(B_{\delta(j), \mathcal{A}(j)}) \text{ with } i \in \mathcal{A}(j+1) \text{ such that } C_{\lambda(B_{\delta(j), \mathcal{A}(j)}, i)} = \emptyset, \\ s(clB_{\delta(j), \mathcal{A}(j)}) \cap clB_{\delta(j+1), \mathcal{A}(j+1)} = \emptyset,$$

$$(18) \quad \text{For each } i \leq n(B_{\delta(j), \mathcal{A}(j)}) \text{ with } i \notin \mathcal{A}(j+1) \text{ such that } C_{\lambda(B_{\delta(j), \mathcal{A}(j)}, i)} \neq \emptyset, \\ K(j+1) = C_{\lambda(B_{\delta(j), \mathcal{A}(j)}, i)}.$$

By the similar proof of Claim in Theorem 4.1, we can show that there is an $i \in \omega$ such that $|\{j \in \omega : i \in \mathcal{A}(j)\}| = \omega$. Let $\{j \in \omega : i \in \mathcal{A}(j) \text{ and } i \leq n(B_{\delta(j), \mathcal{A}(j)})\}$

$=\{j_k : k \in \omega\}$. Then we can prove that $s(clB_{\delta(j_k), A(j_k)}) \cap clB_{\delta(j_{k+1}), A(j_{k+1})} = \emptyset$ for each $k \in \omega$. Since s is a stationary winning strategy for Player I in $G(\mathcal{DC}, X)$, $\bigcap_{k \in \omega} clB_{\delta(j_k), A(j_k)} = \emptyset$. But $x_i \in \bigcap_{k \in \omega} B_{\delta(j_k), A(j_k)}$, which is a contradiction. Thus there is a $k \in \omega$ such that $x \notin \bigcup \mathcal{B}_k$. Let $j = \inf\{k \in \omega : x \notin \bigcup \mathcal{B}_k\}$. Since $x \in \bigcup \mathcal{B}_{j-1}$, we have $x \in \bigcup \mathcal{G}_j$. For each $k > j$, every element of \mathcal{G}_k is contained in some member of \mathcal{B}_j . Therefore $(\bigcup \{\mathcal{G}_k : k \in \omega\})_x \subset \bigcup \{\mathcal{G}_k : k \leq j\}$. Since every \mathcal{G}_k is point finite in X^ω , it follows that $\bigcup \{\mathcal{G}_k : k \in \omega\}$ is a point finite open refinement of \mathcal{O}' . The proof is completed.

COROLLARY 5.2. *If Y_i is a regular metacompact space with a σ -closure-preserving cover by compact sets for each $i \in \omega$, then $\prod_{i \in \omega} Y_i$ is metacompact.*

PROOF. This follows from Theorem 5.1 and Lemma 2.3(a).

For a T_1 -space X , let $\mathcal{F}[X]$ denote the Pixley-Roy hyperspace of X (cf. E. K. van Douwen [7]). Every Pixley-Roy hyperspace is a hereditarily metacompact Tychonoff space and has a closure-preserving cover by finite sets. In [17], the author proved that if Z is a perfect paracompact Hausdorff space and Y_i is a T_1 -space such that $\mathcal{F}[Y_i]$ is paracompact for each $i \in \omega$, then $Z \times \prod_{i \in \omega} \mathcal{F}[Y_i]$ is paracompact.

COROLLARY 5.3. *If Y_i is a T_1 -space for each $i \in \omega$, then $\prod_{i \in \omega} \mathcal{F}[Y_i]$ is metacompact.*

By D. K. Burke [4] and M. M. Čoban [6], every perfect metacompact (metalindelöf) space is hereditarily metacompact (hereditarily metalindelöf). Next, we show the following result.

THEOREM 5.4. *Let Z be a hereditarily metacompact space and Y_i be a regular metacompact \mathcal{DC} -like space for each $i \in \omega$. Then the following are equivalent.*

- (a) $Z \times \prod_{i \in \omega} Y_i$ is metacompact,
- (b) $Z \times \prod_{i \in \omega} Y_i$ is countably metacompact,
- (c) $Z \times \prod_{i \in \omega} Y_i$ is orthocompact.

PROOF. (a)→(c) Obvious.

(c)→(b) We shall modify the proof of Theorem 2.1 in N. Kemoto and Y. Yajima [12]. Assume that $Z \times \prod_{i \in \omega} Y_i$ is orthocompct. Let $\mathcal{O} = \{O_j : j \in \omega\}$ be a

countable open cover of $Z \times \prod_{i \in \omega} Y_i$. By their proof, it suffices to prove that there is a countable open refinement \mathcal{U} of \mathcal{O} such that for every infinite subcollection \mathcal{U}' of \mathcal{U} , $\text{int}(\bigcap \mathcal{U}') = \emptyset$. Applying their technique to $Z \times \prod_{i \in \omega} Y_i$, we have a countable collection $\{G_{j,t} : j \in \omega \text{ and } t=0, 1\}$, where $G_{j,t} = Z \times H_{j,t}$ for each $j \in \omega$ $t=0, 1$, of open subsets of $Z \times \prod_{i \in \omega} Y_i$ such that

- (i) For each $j \in \omega$, $\prod_{i \in \omega} Y_i = H_{j,0} \cup H_{j,1}$ and hence, $Z \times \prod_{i \in \omega} Y_i = G_{j,0} \cup G_{j,1}$,
- (ii) For each infinite subset M of ω and each $t=0, 1$, $\text{int}(\bigcap \{H_{j,t} : j \in M\}) = \emptyset$ and hence, $\text{int}(\bigcap \{G_{j,t} : j \in M\}) = Z \times \text{int}(\bigcap \{H_{j,t} : j \in M\}) = \emptyset$.

Let $\mathcal{U} = \{O_j \cap G_{j,t} : j \in \omega \text{ and } t=0, 1\}$. Then \mathcal{U} is a countable open refinement of \mathcal{O} such that for every infinite subcollection \mathcal{U}' of \mathcal{U} , $\text{int}(\bigcap \mathcal{U}') = \emptyset$.

(b)→(a) Assume that $Z \times \prod_{i \in \omega} Y_i$ is countably metacompact. For each $i \in \omega$, take a point a_i in Y_i . Let \mathcal{O} be an open cover of $Z \times \prod_{i \in \omega} Y_i$ and let $\mathcal{O}' = \{B \in \mathcal{B} : B \subset O \text{ for som } O \in \mathcal{O}^F\}$. For each $z \in Z$ and $K \in \mathcal{K}$, define $n(K_{(z,K)})$ as the proof of Theorem 4.1.

Let s_i be a stationary winning strategy for Player I in $G(\mathcal{D}\mathcal{C}, Y_i)$ for $i \in \omega$. As Theorem 5.1, take a $B = U_B \times \prod_{i \in \omega} B_i \in \mathcal{B}$ satisfying the following condition: For each $i \leq n(B)$, we have already obtained a compact set $C_{\lambda(B,i)}$ of clB_i . ($C_{\lambda(B,n(B))} = \emptyset$. $C_{\lambda(B,i)} = \emptyset$ may be occur for $i < n(B)$.) Fix $i \leq n(B)$. If $C_{\lambda(B,i)} \neq \emptyset$, take the same $W_{\Gamma(B,i)}$, $\Lambda(B,i)$, $\Gamma(B,i)$, $\mathcal{C}(B,i)$ and $\mathcal{W}(B,i)$ in Theorem 5.1. Assume that $C_{\lambda(B,i)} = \emptyset$. Then we take a discrete collection $\mathcal{C}(B,i) = \{C_\lambda : \lambda \in \Lambda(B,i)\}$ of compact subset of Y_i such that $s_i(clB_i) = \bigcup \mathcal{C}(B,i)$, and a collection $\mathcal{W}(B,i) = \{W_\gamma : \gamma \in \Gamma(B,i)\}$ of open subsets in B_i (and hence, in Y_i) satisfying the condition (1')=(1), (2')=(2) in the proof of Theorem 5.1 and

- (3') $\mathcal{W}(B,i)$ is point finite in B_i and hence, point finite in Y_i .

Define the same K_γ for $\gamma \in \Gamma(B,i)$ and Δ_B in Theorem 5.1. For $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B$, let $K(\delta) = K_{\gamma(\delta,0)} \times \dots \times K_{\gamma(\delta,n(B))} \times \{a_{n(B)+1}\} \times \dots \times \{a_k\} \times \dots$. Define \mathcal{K}_B as before. For each $z \in U_B$ and $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B$, let $r(K_{(z,K(\delta))}) = \max\{n(K_{(z,K(\delta))}), n(B)\}$. Fix $z \in U_B$ and $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B$. Take an $O_{z,\delta} = U_{z,\delta} \times \prod_{i \in \omega} O_{z,\delta,i} \in \mathcal{O}'$ such that $K_{(z,K(\delta))} \subset O_{z,\delta}$ and $n(K_{(z,K(\delta))}) = n(O_{z,\delta})$. Since Y_i is a regular space, there is an $H_{(z,K(\delta))} = H_{z,\delta} \times \prod_{i \in \omega} H_{(z,K(\delta)),i} \in \mathcal{B}$ such that:

$$(4') \quad H_{z,\delta} \times \prod_{i=0}^{n(K_{(z,K(\delta))})-1} clH_{(z,K(\delta)),i} \times Y_{n(K_{(z,K(\delta))})} \times \dots \times Y_k \times \dots \subset O_{z,\delta} \text{ and } z \in$$

$$H_{z,\delta} \subset U_B \cap U_{z,\delta},$$

- (5'-1) For each i with $n(K_{(z,K(\delta))}) \leq i \leq r(K_{(z,K(\delta))})$, let $H_{(z,K(\delta)),i} = Y_i$,
- (5'-2) For each $i < n(K_{(z,K(\delta))})$ with $i \leq n(B)$, let $H_{(z,K(\delta)),i}$ be an open subset

of Y_i such that $K_{\gamma(\delta, i)} \subset H_{(z, K(\delta)), i} \subset clH_{(z, K(\delta)), i} \subset O_{z, \delta, i}$,

(5'-3) For each i with $n(B) < i < n(K_{(z, K(\delta))})$, let $H_{(z, K(\delta)), i}$ be an open subset of Y_i such that $a_i \in H_{(z, K(\delta)), i} \subset clH_{(z, K(\delta)), i} \subset O_{z, \delta, i}$,

(5'-4) In case of that $r(K_{(z, K(\delta))}) = n(B)$, let $H_{(z, K(\delta)), i} = Y_i$ for $n(B) < i$. In case of that $r(K_{(z, K(\delta))}) = n(K_{(z, K(\delta))}) > n(B)$, let $H_{(z, K(\delta)), i} = Y_i$ for $n(K_{(z, K(\delta))}) \leq i$.

Then we have $K_{(z, K(\delta))} \subset H_{(z, K(\delta))}$. For each $j \in \omega$, let $\mathcal{H}_{\delta, j} = \{H_{z, \delta} : n(K_{(z, K(\delta))}) \leq j\}$. Fix $j \in \omega$ and let $V_j(K(\delta)) = \{z \in U_B : n(K_{(z, K(\delta))}) \leq j\}$. Then $V_j(K(\delta)) = \cup \mathcal{H}_{\delta, j}$. Since Z is a hereditarily metacompact space, there is a family $\mathcal{V}_{\delta, j} = \{V_\xi : \xi \in \mathcal{E}_{\delta, j}\}$, of collections of open sets in $V_j(K(\delta))$ (and hence, in Z) satisfying

(6') Every member of $\mathcal{V}_{\delta, j}$ is contained in some member of $\mathcal{H}_{\delta, j}$,

(7') $\mathcal{V}_{\delta, j}$ covers $V_j(K(\delta))$,

(8') $\mathcal{V}_{\delta, j}$ is point finite in $V_j(K(\delta))$ and hence, point finite in Z .

For each $\xi \in \mathcal{E}_{\delta, j}$, take a $z(\xi) \in V_j(K(\delta))$ such that $V_\xi \subset H_{z(\xi), \delta}$. Put $W_\delta = \prod_{i=0}^{n(B)} W_{\gamma(\delta, i)} \times Y_{n(B)+1} \times \dots \times Y_k \times \dots$ and $V_{\xi, \delta} = V_\xi \times W_\delta$. Then $\{V_{\xi, \delta} : \delta \in \Delta_B, j \in \omega \text{ and } \xi \in \mathcal{E}_{\delta, j}\}$ is a collection of elements of \mathcal{B} such that for each $\delta \in \Delta_B, j \in \omega$ and $\xi \in \mathcal{E}_{\delta, j}$, $V_{\xi, \delta} \subset B$ and $\{V_{\xi, \delta} : \delta \in \Delta_B, j \in \omega \text{ and } \xi \in \mathcal{E}_{\delta, j}\}$ covers B . Clearly we have

(9') For each $j \in \omega$, $\{V_{\xi, \delta} : \delta \in \Delta_B \text{ and } \xi \in \mathcal{E}_{\delta, j}\}$ is point finite in $Z \times \prod_{i \in \omega} Y_i$.

Fix a $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B, j \in \omega$ and $\xi \in \mathcal{E}_{\delta, j}$. In case of that $r(K_{(z(\xi), K(\delta))}) = n(B)$. For each $i \leq n(B)$, let $G_{(z(\xi), K(\delta)), i} = O_{z(\xi), \delta, i} \cap W_{\gamma(\delta, i)}$. For each $i > n(B)$, let $G_{(z(\xi), K(\delta)), i} = Y_i$. Put $G_{(z(\xi), K(\delta))} = V_\xi \times \prod_{i \in \omega} G_{(z(\xi), K(\delta)), i}$. In case of that $r(K_{(z(\xi), K(\delta))}) = n(K_{(z(\xi), K(\delta))}) > n(B)$. For each $i \leq n(B)$, let $G_{(z(\xi), K(\delta)), i} = O_{z(\xi), \delta, i} \cap W_{\gamma(\delta, i)}$. For each i with $n(B) < i < n(K_{(z(\xi), K(\delta))})$, let $G_{(z(\xi), K(\delta)), i} = O_{z(\xi), \delta, i}$. For each $i \geq n(K_{(z(\xi), K(\delta))})$, let $G_{(z(\xi), K(\delta)), i} = Y_i$. Put $G_{(z(\xi), K(\delta))} = V_\xi \times \prod_{i \in \omega} G_{(z(\xi), K(\delta)), i}$. Then we have $G_{(z(\xi), K(\delta))} \subset V_{\xi, \delta}$. Define $\mathcal{Q}_{\delta, j}(B) = \{G_{(z(\xi), K(\delta))} : \xi \in \mathcal{E}_{\delta, j}\}$ and $\mathcal{Q}_j(B) = \cup \{\mathcal{Q}_{\delta, j}(B) : \delta \in \Delta_B\}$. Then, by (9') and definition,

(10') For each $j \in \omega$, every member of $\mathcal{Q}_j(B)$ is contained in some member of \mathcal{O}' .

(11') For each $j \in \omega$, $\mathcal{Q}_j(B)$ is point finite in $Z \times \prod_{i \in \omega} Y_i$.

Fix $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B, j \in \omega$ and $\xi \in \mathcal{E}_{\delta, j}$. Let $A \in \mathcal{P}(\{0, 1, \dots, r(K_{(z(\xi), K(\delta))})\})$. In case of that $r(K_{(z(\xi), K(\delta))}) = n(B)$. For each $i \in A$, let $B_{\xi, A, i} = W_{\gamma(\delta, i)} - clH_{(z(\xi), K(\delta)), i}$. For each $i \notin A$ with $i \leq n(B)$, let $B_{\xi, A, i} = O_{z, \delta, i} \cap W_{\gamma(\delta, i)}$. For each $i > n(B)$, let $B_{\xi, A, i} = Y_i$. Put $B_{\xi, A} = V_\xi \times \prod_{i \in \omega} B_{\xi, A, i}$. In case of that $r(K_{(z(\xi), K(\delta))}) = n(K_{(z(\xi), K(\delta))}) > n(B)$. For each $i \in A$ with $i \leq n(B)$, let $B_{\xi, A, i} = W_{\gamma(\delta, i)} - clH_{(z(\xi), K(\delta)), i}$. For each $i \notin A$ with $i \leq n(B)$, let $B_{\xi, A, i} = O_{z(\xi), \delta, i} \cap W_{\gamma(\delta, i)}$.

Let $n(B) < i < n(K_{(z(\xi), K(\delta))})$. If $i \in A$, let $B_{\xi, A, i} = Y_i - clH_{(z(\xi), K(\delta)), i}$. If $i \notin A$, let $B_{\xi, A, i} = O_{z(\xi), \delta, i}$. For $i > n(K_{(z(\xi), K(\delta))})$, let $B_{\xi, A, i} = Y_i$. Put $B_{\xi, A} = V_{\xi} \times \prod_{i \in \omega} B_{\xi, A, i}$. We have that $B_{\xi, A, i} \subset B_i$ for each $i \in \omega$ and if $B_{\xi, A} \neq \emptyset$, then $n(B) < n(B_{\xi, A})$. Since $n(K_{(z(\xi), K(\delta))}) \leq j$, for a subset $A \in \mathcal{P}(\{0, 1, \dots, \max\{j, n(B)\}\})$, let $\mathcal{B}_{\delta, j, A}(B) = \{B_{\xi, A} : \xi \in \Xi_{\delta, j}, B_{\xi, A} \text{ is defined and } B_{\xi, A} \neq \emptyset\}$. For $j \in \omega$, let $\mathcal{B}_j(B) = \cup \{\mathcal{B}_{\delta, j, A}(B) : \delta \in \Delta_B \text{ and } A \in \mathcal{P}(\{0, 1, \dots, \max\{j, n(B)\}\})\}$. Then we have

$$(12') \text{ Every } \mathcal{B}_j(B) \text{ is point finite in } Z \times \prod_{i \in \omega} Y_i.$$

Fix a $B_{\xi, A} = V_{\xi} \times \prod_{i \in \omega} B_{\xi, A, i} \in \mathcal{B}_{\delta, j, A}(B)$ for $\delta = (\gamma(\delta, 0), \dots, \gamma(\delta, n(B))) \in \Delta_B$, $j \in \omega$, $\xi \in \Xi_{\delta, j}$ and $A \in \mathcal{P}(\{0, 1, \dots, \max\{j, n(B)\}\})$. Then

$$(13') \text{ For each } i \in A \text{ with } i \leq n(B) \text{ such that } C_{\lambda(B, i)} = \emptyset, s_i(clB_i) \cap clB_{\xi, A, i} = \emptyset.$$

For each $i \leq n(B_{\xi, A})$, define a compact set $C_{\lambda(B_{\xi, A}, i)}$ in $clB_{\xi, A, i}$ as Theorem 5.1.

Now we define \mathcal{G}_{τ} and \mathcal{B}_{τ} for each $\tau \in \omega^{<\omega}$ with $\tau \neq \emptyset$. For each $j \in \omega$, let $\mathcal{G}_j = \mathcal{G}_j(Z \times \prod_{i \in \omega} Y_i)$ and $\mathcal{B}_j = \mathcal{B}_j(Z \times \prod_{i \in \omega} Y_i)$. Assume that for $\tau \in \omega^{<\omega}$ with $\tau \neq \emptyset$, we have already obtained \mathcal{G}_{τ} and \mathcal{B}_{τ} . For each $B \in \mathcal{B}_{\tau}$ and $j \in \omega$, we denote $\mathcal{G}_j(B)$ and $\mathcal{B}_j(B)$ by $\mathcal{G}_{\tau \oplus j}(B)$ and $\mathcal{B}_{\tau \oplus j}(B)$ respectively. Define $\mathcal{G}_{\tau \oplus j} = \cup \{\mathcal{G}_{\tau \oplus j}(B) : B \in \mathcal{B}_{\tau}\}$ and $\mathcal{B}_{\tau \oplus j} = \cup \{\mathcal{B}_{\tau \oplus j}(B) : B \in \mathcal{B}_{\tau}\}$.

Firstly we show that $\cup \{\mathcal{G}_{\tau} : \tau \in \omega^{<\omega} \text{ and } \tau \neq \emptyset\}$ is a σ -point finite open refinement of \mathcal{O}' . Let $\tau \in \omega^{<\omega}$ and $\tau \neq \emptyset$. By (10'), every element of \mathcal{G}_{τ} is contained in some member of \mathcal{O}' . By (11'), (12') and induction, for each $\tau \in \omega^{<\omega}$ and $\tau \neq \emptyset$, \mathcal{G}_{τ} is point finite. Thus, it suffices to prove that $\cup \{\mathcal{G}_{\tau} : \tau \in \omega^{<\omega} \text{ and } \tau \neq \emptyset\}$ is a cover of $Z \times \prod_{i \in \omega} Y_i$. However, the proof is similar to that of Claim in Theorem 4.1. Let $G_{\tau} = \cup \mathcal{G}_{\tau}$ for each $\tau \in \omega^{<\omega}$ with $\tau \neq \emptyset$. Then $\{G_{\tau} : \tau \in \omega^{<\omega} \text{ and } \tau \neq \emptyset\}$ is a countable open cover of $Z \times \prod_{i \in \omega} Y_i$. Since $Z \times \prod_{i \in \omega} Y_i$ is countably metacompact, there is a point finite open refinement $\{G'_{\tau} : \tau \in \omega^{<\omega} \text{ and } \tau \neq \emptyset\}$ such that $G'_{\tau} \subset G_{\tau}$ for each $\tau \in \omega^{<\omega}$ with $\tau \neq \emptyset$. Then $\{G'_{\tau} \cap G : G \in \mathcal{G}_{\tau}, \tau \in \omega^{<\omega} \text{ and } \tau \neq \emptyset\}$ is a point finite open refinement of \mathcal{O}' . It follows that $Z \times \prod_{i \in \omega} Y_i$ is metacompact. The proof is completed.

REMARK 5.5. B. Scott [16] showed that if Y is orthocompact and Z is compact, metric and infinite, then $Y \times Z$ is orthocompact if and only if Y is countably metacompact. J. Chaber [5] constructed a scattered hereditarily orthocompact space Y which is not countably metacompact. Thus, for J. Chaber's space Y , $Y \times (\omega + 1)$ is not orthocompact, even though both factors are hereditarily orthocompact and scattered (cf. Lemma 2.4).

COROLLARY 5.6. *Let Z be a hereditarily metacompact space and Y_i be a regular metacompact space with a σ -closurepreserving cover by compact sets for each $i \in \omega$. Then the following are equivalent.*

- (a) $Z \times \prod_{i \in \omega} Y_i$ is metacompact,
- (b) $Z \times \prod_{i \in \omega} Y_i$ is countably metacompact,
- (c) $Z \times \prod_{i \in \omega} Y_i$ is orthocompact.

Since every σ -point countable collection of $Z \times \prod_{i \in \omega} Y_i$ is point countable, by the proof of the implication (b) \rightarrow (a) in Theorem 5.4, we have

THEOREM 5.7. *If Z is a hereditarily metalindelöf space and Y_i is a regular metalindelöf \mathcal{DC} -like space for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is metalindelöf.*

COROLLARY 5.8. *If Z is a hereditarily metalindelöf space and Y_i is a regular metalindelöf space with a σ -closure-preserving cover by compact sets for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is metalindelöf.*

We consider metacompactness, orthocompactness and metalindelöf property of countable products using C -scattered spaces.

THEOREM 5.9. *If Y_i is a regular C -scattered metacompact space for each $i \in \omega$, then $\prod_{i \in \omega} Y_i$ is metacompact.*

PROOF. We also assume that $Y_i = X$ for each $i \in \omega$ and there is an isolated point a in X . We shall modify the proof of Theorem 5.1. Let \mathcal{O} be an open cover of X^ω . Define the same \mathcal{O}' and $n(K)$ for each $K \in \mathcal{K}$. We take a $B = \prod_{i \in \omega} B_i \in \mathcal{B}$ satisfying the condition of the proof of Theorem 5.1. Fix $i \leq n(B)$. If $C_{\lambda(B, i)} \neq \emptyset$, then we take the same $W_{\gamma(B, i)}$, $A(B, i)$, $\Gamma(B, i)$, $\mathcal{C}(B, i)$, and $\mathcal{W}(B, i)$. Assume that $C_{\lambda(B, i)} = \emptyset$. Since clB_i is a regular C -scattered metacompact space, by Lemma 3.3, there is a collection $\mathcal{W}(B, i) = \{W_\gamma : \gamma \in \Gamma(B, i)\}$ of open subsets in B_i satisfying the conditions (1'') = (1) and (2'') = (3) in the proof of Theorem 5.1 and

(3'') For each $\gamma \in \Gamma(B, i)$, $(clW_\gamma)^{(\alpha(\gamma))}$ is compact for some $\alpha(\gamma)$.

Let $A(B, i) = \Gamma(B, i)$ and $\mathcal{C}(B, i) = \{(clW_\lambda)^{(\alpha(\lambda))} : \lambda \in A(B, i)\}$.

Let $K_\gamma = (clW_\gamma)^{(\alpha(\gamma))}$ for $\gamma \in \Gamma(B, i)$ and take Δ_B , $K(\delta)$ for $\delta \in \Delta_B$, \mathcal{K}_B , $r(K(\delta))$, $H(\delta)$, $W(\delta)$ and $G(\delta)$ for $\delta \in \Delta_B$, $\mathcal{G}(B)$, $B_{\delta, A}$, $\mathcal{B}_\delta(B)$ and $\mathcal{B}(B)$ for $\delta \in \Delta(B)$, $A \in \mathcal{P}(\{0, 1, \dots, r(K(\delta))\})$ as before satisfying the conditions (4'') = (4), (5''- i) = (5- i) for $i = 1, 2, 3$ and 4, (6'') = (6), (7'') = (7), (8'') = (8) and (9'') = (9). Furthermore, we

take the same \mathcal{G}_j and \mathcal{B}_j for each $j \in \omega$, and show that $\cup \{\mathcal{G}_j : j \in \omega\}$ is a point finite open refinement of \mathcal{O}' . Let $x = (x_i)_{i \in \omega}$. Take the same $\{\mathcal{A}(j) : j \in \omega\}$, $\{\mathcal{K}(j) : j \in \omega\}$, $\{\mathcal{H}(j) : j \in \omega\}$, $\{\mathcal{W}(j) : j \in \omega\}$, $\{\mathcal{G}(j) : j \in \omega\}$, $\{\mathcal{A}(j) : j \in \omega\}$ and $\{\mathcal{B}(j) : j \in \omega\}$. Assuming $x \in \cup \mathcal{B}_j$ for each $j \in \omega$, we similarly choose a sequence $\{\delta(j) : j \in \omega\}$, a sequence $\{K(j) : j \in \omega\}$ of compact subsets of X^ω , where for each $j \in \omega$, $K(j) = \prod_{i \in \omega} K(j)_i \in \mathcal{K}$, sequences $\{H(\delta(j)) : j \in \omega\}$, $\{W(\delta(j)) : j \in \omega\}$ of elements of \mathcal{B} , a sequence $\{A(j) : j \in \omega\}$ of finite subsets of ω , a sequence $\{B_{\delta(j), A(j)} : j \in \omega\}$ of elements of \mathcal{B} satisfying the conditions (10'')=(11), (11'')=(12), (12'')=(13), (13'')=(14), (14'')=(15), (15'')=(16) and (16'')=(18). Then there is an $i \in \omega$ such such that $|\{j \in \omega : i \in A(j)\}| = \omega$. Let $\{j \in \omega : i \in A(j) \text{ and } i \leq n(B_{\delta(j), A(j)})\} = \{j_k : k \in \omega\}$. We have

$$(17'') \text{ For each } k \in \omega, \varepsilon(\text{cl}W_{\gamma(\delta(j_{k+1}+1), i)}) < \varepsilon(\text{cl}W_{\gamma(\delta(j_k+1), i)}).$$

Fix $k \in \omega$ and take a $y \in \text{cl}W_{\gamma(\delta(j_{k+1}+1), i)}$. Since $W_{\gamma(\delta(j_{k+1}+1), i)} \subset W_{\gamma(\delta(j_k+1), i)}$, $\alpha \text{cl}W_{\gamma(\delta(j_{k+1}+1), i)}(y) \leq \alpha \text{cl}W_{\gamma(\delta(j_k+1), i)}(y)$. Assume that $j_{k+1} = j_k + 1$. Then $W_{\gamma(\delta(j_{k+1}+1), i)} \subset B_{\delta(j_{k+1}), A(j_{k+1}), i}$ and

$$K(j_{k+1})_i = K_{\gamma(\delta(j_{k+1}), i)} = (\text{cl}W_{\gamma(\delta(j_{k+1}), i)})^{\alpha(\gamma(\delta(j_{k+1}), i))} \subset H(\delta(j_{k+1}))_i.$$

Assume that $j_{k+1} > j_k + 1$. Then

$$K(j_{k+1})_i = K_{\gamma(\delta(j_{k+1}), i)} = C_{\lambda(B_{\delta(j_{k+1}), A(j_{k+1}), i})} = C_{\lambda(B_{\delta(j_{k+1}-1), A(j_{k+1}-1), i})} \subset H(\delta(j_{k+1}))_i.$$

In each case, we have $\alpha \text{cl}W_{\gamma(\delta(j_{k+1}), i)}(y) < \alpha(\gamma(\delta(j_{k+1}), i))$. Hence $\alpha \text{cl}W_{\gamma(\delta(j_{k+1}+1), i)}(y) < \alpha(\gamma(\delta(j_{k+1}), i))$. Therefore $\varepsilon(\text{cl}W_{\gamma(\delta(j_{k+1}+1), i)}) \leq \alpha(\gamma(\delta(j_{k+1}), i))$. Since $\varepsilon(\text{cl}W_{\gamma(\delta(j_{k+1}), i)}) = \alpha(\gamma(\delta(j_{k+1}), i)) + 1$, we have $\varepsilon(\text{cl}W_{\gamma(j_{k+1}+1), i}) < \varepsilon(\text{cl}W_{\gamma(\delta(j_{k+1}), i)})$.

Thus $\{\varepsilon(\text{cl}W_{\gamma(\delta(j_{k+1}), i)}) : k \in \omega\}$ is an infinite decreasing sequence of ordinals, which is a contradiction. Thus there is a $k \in \omega$ such that $x \notin \cup \mathcal{B}_k$. Similarly, it follows that $\cup \{\mathcal{G}_j : j \in \omega\}$ is a point finite open refinement of \mathcal{O}' . The proof is completed.

Similarly, we have

THEOREM 5.10. *Let Z be a hereditarily metacompact space and Y_i be a regular C -scattered metacompact space for each $i \in \omega$. Then the following are equivalent.*

- (a) $Z \times \prod_{i \in \omega} Y_i$ is metacompact,
- (b) $Z \times \prod_{i \in \omega} Y_i$ is countably metacompact,
- (c) $Z \times \prod_{i \in \omega} Y_i$ is orthocompact.

THEOREM 5.11. *If Z is a hereditarily metalindelöf space and Y_i is a regular C -scattered metalindelöf space for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is metalindelöf.*

Acknowledgement.

The author would like to thank Mr. Seiji Fujii for his kindness.

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