ON INTERSECTIONS OF COMPACTA IN EUCLIDEAN SPACE: THE METASTABLE CASE

By

A.N. DRANIŠNIKOV, D. REPOVŠ[1],[2] and E.V. ŠČEPIN

Abstract. We prove the following theorem: Let $f : X \to R^n$ and $g : Y \to R^n$ be any maps of compacta X and Y into the Euclidean n-space R^n, $n \geq 5$. Suppose that $\dim(X \times Y) < n$ and that $2 \dim X + \dim Y < 2n - 1$. Then for every $\varepsilon > 0$ there exist maps $f' : X \to R^n$ and $g' : Y \to R^n$ such that $d(f, f') < \varepsilon$, $d(g, g') < \varepsilon$ and $f'(X) \cap g'(Y) = \emptyset$.

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Introduction

Dimension theory, a classical discipline of general topology, has wittnessed in recent years an unexpected revival. It started with a paper of D. McCullough and L.R. Rubin [37] in which a counterexample to the major result of [36] was constructed: they proved that for every $k \geq 2$, there exists a k-dimensional compactum X such that every map of X into R^{2k} can be approximated by an *embedding*. (Their proof was based on the beautiful short paper by J. Krasinkiewicz and K. Lorentz [31] who first spotted the gap in [36].) Their examples satisfied the inequality $\dim(X \times X) < 2 \dim X$ and McCullough and Rubin asked whether this property was perhaps *characteristic*.

This initiated two parallel, but independent lines of investigation—one in Moscow (by A. N. Dranišnikov, D. Repovš and E. V. Ščepin, joined at a later stage by J. E. West in Ithaca, cf. [10]-[18]) and the other one in Warsaw (by J. Krasinkiewicz, K. Lorentz, S. Spież and H. Toruńczyk, joined at a later stage by J. Segal in Seattle, cf. [30], [31], [48], [51]-[54]). It was soon verified that the McCullough-Rubin question had an *affirmative* answer: For every $(k \geq 2)$-dimensional compactum X, every map of X into R^{2k} can be approximated by an embedding *if and only if* $\dim(X \times X) < 2k$. This was a corollary of more general result, proved by A. N. Dranišnikov, D. Repovš and E. V. Ščepin [16] and independently, using different methods, by J. Krasinkiewicz [30]—in the *only if* direction and by S. Spież [51]-[53]—in the *if* direction:

THEOREM 1.1 (Dranišnikov-Repovš-Ščepin, Krasinkiewicz-Spież). *Given compacta X and Y such that* $\dim X + \dim Y = n$, *there exists a pair of maps* $f : X \to R^n$ *and* $g : Y \to R^n$ *with stable intersections if and only if* $\dim(X \times Y) = n$. ∎

We say that the maps $f : X \to R^n$ and $g : Y \to R^n$ of compacta X and Y into R^n have *unstable intersection* if for every $\varepsilon > 0$ there exist maps $f' : X \to R^n$ and $g' : Y \to R^n$ such that $d(f, f') < \varepsilon$, $d(g, g') < \varepsilon$, and $f'(X) \cap g'(Y) = \emptyset$.

Recall that for *polyhedra* X and Y, the necessary *and* sufficient condition that every pair of maps $f : X \to R^n$ and $g : Y \to R^n$ intersect *unstably* is that $\dim X + \dim Y < n$. Since the *logarithmic law* for dimension holds in the class of polyhedra, one can replace this condition by $\dim(X \times Y) < n$. On the other hand it is wellknown that for *compacta* the logarithmic law formula fails badly (see e. g. [1], [2], [4], [9], [20], [40], [41], [43], [44]). However, if one considers the inequality $\dim(X \times Y) < n$ instead, then as Theorem (1.1) demonstrates, one gets a result *analogous* to the one for polyhedra.

The following question immediately arises: Does Theorem (1.1) remain valid if one omits the hypothesis that $\dim X + \dim Y = n$? It was recently shown by A. N. Dranišnikov and J. E. West [18] that it is indeed *unnecessary* for the *only if* part of Theorem (1.1):

THEOREM 1.2 (Dranišnikov-West). *For every pair X and Y of compacta such that* $\dim(X \times Y) \geq n$, *there exists a pair of maps* $f : X \to R^n$ *and* $g : Y \to R^n$ *such that f and g intersect stably*. ∎

It is interesting to note that in the first version of our earlier paper [16] we had a slick proof of Theorem (1.2), based on a theorem of D. O. Kiguradze [29] which, in turn, depended on a classical theorem of G. Čogošvili [7] from

1930's. However, in 1989 a serious *gap* was detected in [7] and hence the status of Kiguradze's paper [29] became unclear. As a consequence, in our paper [16], the *complementary dimensions* case of Theorem (1.2), i.e. the *if* part of Theorem (1.1), was then reproved, invoking a *different*, more recent result of Kiguradze, a complete proof of which was at this time included in our paper [16] (see our comments in [16] for more history and details). We discuss Čogošvili's theorem [7] and related problems in the Epilogue.

On the other hand, the general case of the *if* part of Theorem (1.1) remains an open problem:

PROBLEM 1.1. *Suppose that X and Y are compacta such that $\dim(X \times Y) < n$. Does then every pair of maps $f : X \to R^n$ and $g : Y \to R^n$ intersect unstably?*

Note that by [2], the condition $\dim(X \times Y) < n$ implies, in general, only that $\dim X \leq n-2$ and $\dim Y \leq n-2$ hence $\dim X + \dim Y \leq 2n-4$. On the other hand, it follows by [9] that for every $r \leq 2n-4$, there exist compacta X and Y such that $\dim(X \times Y) < n$ and $\dim X + \dim Y = r$ so *all* intermediate cases (between n and $2n-4$) can actually occur. The simplest *unsolved* case of Problem (1.1) is therefore the case of 3-dimensional compacta X^3 and Y^3 such that $\dim(X^3 \times Y^3) = 4$: Do every two maps $f : X^3 \to R^5$ and $g : Y^3 \to R^5$ intersect *unstably*?

Problem (1.1) naturally splits into two problems:

PROBLEM 1.2. *Suppose that X and Y are compacta such that $\dim(X \times Y) < n$, and choose any map $f : X \to R^n$ and any $\varepsilon > 0$. Dose then there exist a map $f' : X \to R^n$ such that $d(f, f') < \varepsilon$ and $\dim(f'(X) \times Y) < n$?*

Using the theory of *regular branched* maps, we have proved in our previous paper that Problem (1.2) has an *affirmative* answer provided $2 \dim X + \dim Y < 2n-1$ (cf. Corollary (3.2) of]16]):

THEOREM 1.3 (Dranišnikov-Repovš-Ščepin). *Suppose that X and Y are compacta such that $\dim(X \times Y) < n$ and $2 \dim X + \dim Y < 2n-1$. Then the set*

$$N = \{f \in C(X, R^n) \mid \dim f(X) \leq \dim X \text{ and } \dim(f(X) \times Y) < n\}$$

contains a dense G_δ subset of $C(X, R^n)$ (where $C(X, R^n)$ is the space of all continuous maps of X into R^n, equipped by the sup-norm metric). ■

PROBLEM 1.3. *Given a compactum $X \subset R^n$ such that for some compactum Y, $\dim(X \times Y) < n$, is then $C(Y, R^n - X)$ dense in $C(Y, R^n)$?*

A. N. Dranišnikov [11], [12] answered Problem (1.3) in the *affirmative*—for the case when compactum $X \subset R^n$ is a *tame* (in the sense of M. A. Štan'ko [56]) and has *codimension three*.

The purpose of this paper is to prove that also Problem (1.3) (hence Problem (1.1)) have *affirmative* answers in the case when $2 \dim X + \dim Y < 2n - 1$.

THEOREM 1.4. *Suppose that X and Y are compacta such that $2 \dim X + \dim Y < 2n - 1$ and $\dim (X \times Y) < n$, for some integer $n \geq 5$. Then every two maps $f : X \to R^n$ and $g : Y \to R^n$ intersect unstably.*

Combining Theorems (1.2) and (1.4) one gets the following generalization of Theorem (1.1):

COROLLARY 1.1. *Suppose that X and Y are compacta such that $2 \dim X + \dim Y < 2n - 1$, for some integer $n \geq 5$. Then every two maps $f : X \to R^n$ and $g : Y \to R^n$ intersect unstably if and only if $\dim (X \times Y) < n$.* ∎

At the 1989 Eger Colloquium on Topology, E. V. Ščepin presented a paper by A. N. Dranišnikov [10], from which the special case of Theorem (1.4)—when $2 \dim X + \dim Y < 2n - 2$ followed. Later that year, at the 1989 Habarovsk Soviet-Japanese Dimension Theory Symposium, S. Spież presented a joint paper with J. Segal [48] which obtained the same result as [10] via a different route.

Following that E. V. Ščepin presented in Habarovsk our present Theorem (1.4) and pointed out how it can be derived from [10] (and [48]) by invoking the so-called *hart part* of the Freudenthal Suspension theorem, the complete proof of which is included in this paper—see Appendix. (The paper [48] was also subsequently rewritten to include Ščepin's trick. Later, a technical improvement of [48] appeared in a joint work of S. Spież and H. Toruńczyk [54].)

We wish to point out that while our approach (as well as of [10]) is more *algebraic*—it is based on the Spanier-Whitehead duality [50] and the theory of regularly branched maps [16], the papers [48] and [54] are more *geometric*—their key ingredient is a classical result of C. Weber [58]. The *only* common tool of all these papers is the Freudenthal Suspension theorem. Recently, the first author [11] obtained yet another proof of Theorem (1.4), using localizations and spectral sequences but *not* invoking the Freudenthal Suspension theorem[3].

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[3] *Added in revision.* After this paper had been submitted we have announced in the Fall of 1991 several new results on this subject [17]. In particular, Theorem (1.4) remains valid if the condition $2 \dim X + \dim Y < 2n - 1$ is replaced by a *weaker* hypotheses that $\operatorname{codim} X \cdot \operatorname{codim} Y \geq n$, $\operatorname{codim} X \geq 3$, and $\operatorname{codim} Y \geq 3$.

In conclusion we wish to point out that results described above might play an important role in the attempts on the 4-dimensional *Cell-like Mapping problem*—the only remaining case of the celebrated problem from *geometric* topology: It was shown by W. J. R. Mitchell, D. Repovš and E. V. Ščepin [39] that problem reduces to a question concerning *general position properties* of maps of certain 2-dimensional compacta, called *Pontrjagin disks*, into 4-manifolds. (For more on this connection see the survey [38].)

The preliminary version of this paper was written during the second author's visit to the Steklov Mathematical Institute in Moscow in 1989 (and announced in [13]-[15]), on the basis of the longterm agreement between the Slovene Academy of Arts and Sciences and the Russian Academy of Sciences (1986-1990). The final version was then prepared during the visit by all three authors to the University of Tsukuba in 1990. We wish to thank Professor Yukihiro Kodama for his kind hospitality. We would also like to acknowledge several useful comments from the referee who, in particular, observed that an argument was missing from the first version of the proof of Theorem (1.4).

2. Preliminaries

We shall work in the category of separable metrizable spaces and continuous maps throughout this paper. A *compactum* is a compact metric space. For a CW complex M and an integer i, let $\Sigma^i M$ denote the i-th *derived suspension*, $\Omega^i M$ the i-th iterated *loop space*, and

$$\Omega^\infty \Sigma^\infty M = \varinjlim_{i \in N} \{\Omega^i \Sigma^i M\}$$

(see e. g. [59]). Recall that $\Pi_k(\Omega^\infty \Sigma^\infty X) \cong \Pi_k^{stable}(X)$. The following is well-known for polyhedra [59] and is also true for compacta [10]:

THEOREM 2.1 (Spanier-Whitehead Duality). *For every compactum Z and every compact subset $X \subset S^{n+1}$, there exists a natural isomorphism:*

$$[Z, \Omega^\infty \Sigma^\infty (S^{n+1} - X)] \xrightarrow{\ \beta\ } [Z \wedge X, \Omega^\infty \Sigma^\infty S^n].$$

Furthermore, if $\dim Z < \infty$ then $\Omega^\infty \Sigma^\infty$ can be replaced by $\Omega^m \Sigma^m$ for some large enough integer m. ∎

PROPOSITION 2.1. *Let $n \geq 5$ and let $X \subset R^n$ and Y be compacta such that $\dim(X \times Y) < n$. Then for every closed subset $Y_0 \subset Y$ and every map $g: Y_0 \to B^n - X$ there exists an extension $\bar{g}: Y \to \Omega^\infty \Sigma^\infty(B^n - X)$.*

PROOF. For some large enough m, consider the following commutative diagram:

$$[Y_0, \Omega^m \Sigma^m (Int\ B^n - X)] \xrightarrow{\gamma_1} [Y_0 \wedge X', \Omega^m \Sigma^m S^{n-1}] \xrightarrow{\gamma_2} [\Sigma^m (Y_0 \wedge X'), S^{m+n-1}]$$

$$\uparrow \alpha_1 \qquad\qquad\qquad \uparrow \alpha_2 \qquad\qquad\qquad \uparrow \alpha_3$$

$$[Y, \Omega^m \Sigma^m (Int\ B^n - X)] \xrightarrow{\beta_1} [Y \wedge X', \Omega^m \Sigma^m S^{n-1}] \xrightarrow{\beta_2} [\Sigma^m (Y \wedge X'), S^{m+n-1}]$$

where $X' = (X \cap B^n)/(X \cap \partial B^n)$. Homomorphism α_3 is onto due to the inequality $\dim \Sigma^m (Y \wedge X') < n + m$. Theorem (2.1) implies that β_1 and γ_1 are isomorphisms. Homomorphisms β_2 and γ_2 are bijective since

$$[Y_0 \wedge X', \Omega^m \Sigma^m S^{n-1}] = [Y_0 \wedge X', \Omega^m S^{m+n-1}] = [\Sigma^m (Y_0 \wedge X'), S^{m+n-1}].$$

Here α_1 is an epimorphism. ∎

Recall the *Kuratowski* notation $A\tau B$ [32]: it means that for every closed subset $A_0 \subset A$ and every map $\alpha : A_0 \to B$ there exists an extension $\bar{\alpha} : A \to B$.

In Chapter 3 we shall also need the following result:

PROPOSITION 2.2. *Let $f : X \to Y$ be a surjective map between ANR's X and Y such that $f_* : \Pi_i(X) \to \Pi_i(Y)$ induces an isomorphism for every $i \leq n$. Let Z be a compactum of dimension $\leq n+1$ and suppose that $Z\tau Y$. Then $Z\tau X$.*

PROOF. First, we shall consider the case when Z is a polyhedron. So suppose that we are given a map $\alpha : Z_0 \to X$ where $Z_0 \subset Z$ is a subpolyhedron of Z. Let $\beta = f \circ \alpha$. Then by hypothesis, there exists an extension $\bar{\beta} : Z \to Y$ such that $\bar{\beta} | Z_0 = \beta$.

We construct an extension $\bar{\alpha} : Z \to X$ of α, inductively over the skeleta of Z. Since by hypothesis, f is an n-equivalence, there exists an extension $\tilde{\alpha} : Z^{(n)} \cup Z_0 \to X$ and that the diagram below commutes [47],

$$
\begin{array}{ccc}
Z_0 & \xrightarrow{\ incl\ } & Z^{(n)} \cup Z_0 \\
\alpha \downarrow & & \downarrow \bar{\beta} | (Z^{(n)} \cup Z_0) \\
X & \xrightarrow{\quad f \quad} & Y
\end{array}
$$

so it remains to extend $\tilde{\alpha}$ over the $(n+1)$-simplices $\sigma \in Z^{(n+1)}$. Since $[\bar{\beta}(\partial \sigma)] = [0] \in \Pi_n(Y)$ and since, by hypothesis, $f_* : \Pi_n(X) \to \Pi_n(Y)$ is a monomorphism, it follows from the diagram above that $\tilde{\alpha}$ extends over σ. Doing this for all σ we get the desired map $\bar{\alpha} : Z \to X$ such that $\bar{\alpha} | Z_0 = \alpha$.

We now consider the general case: let Z be a compactum of dimension $\leq n+1$ and let $Z_0 \subset Z$ be an arbitrary closed subset of Z. Then

$$(Z, Z_0) = \underleftarrow{\lim_{k \in N}} \{(Q_k, P_k), p_{k,k+1}\}$$

where for every k, (Q_k, P_k) is a polyhedral pair, $p_{k,k+1} : (Q_{k+1}, P_{k+1}) \to (P_k, Q_k)$ are the bonding maps and $\dim Q_k \leq n+1$ [35].

Suppose we are given a map $\alpha : Z_0 \to X$. Since X is an ANR there exists $k_0 \in N$ and a map $\gamma_k : P_k \to X$ such that for every $k \geq k_0$, α and $\gamma_k \circ (p_k^\infty | Z_0)$ are homotopic.

Use now the special case above to get an extension $\bar{\gamma}_k : Q_k \to X$ such that $\bar{\gamma}_k | P_k = \gamma_k$. Then $\tilde{\alpha} = \bar{\gamma}_k \circ p_k^\infty : Z \to X$ has the property that

$$\tilde{\alpha} | Z_0 = \bar{\gamma}_k \circ p_k^\infty | Z_0 = (\bar{\gamma}_k | P_k) \circ (p_k^\infty | Z_0) = \gamma_k \circ (p_k^\infty | Z_0)$$

is homotopic to α. Therefore, by the Borsuk Tophat theorem [5], $\tilde{\alpha}$ is homotopic to a map $\bar{\alpha} : Z \to X$ such that $\bar{\alpha} | Z_0 = \alpha$. ∎

The next auxiliary result for our proof of Theorem (1.4) can be obtained by an adaptation of the higher dimensional version of the well-known Casson *finger moves* [6], [22]: for details see the proof of a similar low dimensional result—Theorem (5.1) of our previous paper [16] (compare also with the proof of Proposition (1.6) in [54]):

PROPOSITION 2.3. *Let* $Z \subset M$ *be a compactum in a PL n-manifold with boundary ∂M and assume that* $\dim Z \leq n-2$, $n \geq 5$. *Then for every pair of compact polyhedra* (P, Q) *where* $k = \dim P \in \{[n/2], \cdots, n-1\}$ *and every PL immersion* $f : (P, Q) \to (M-Z, \partial M-Z)$ *there is a PL homotopy* $G : P \times I \to M-Z$ *such that:*

(1) $G_0 = f$;

(2) *For every* $t \in I$, $G_t : P \to M-Z$ *is an immersion and* $G_t | Q = f | Q$;

(3) $G_1 : P \to M-Z$ *is a general position map; and*

(4) *The Whitehead product*

$$\Pi_{n-k-1}(M-G_1(P)) \times \Pi_{n-k-1}(M-G_1(P)) \longrightarrow \Pi_{2n-2k-3}(M-G_1(P))$$

is trivial on all elements of $\Pi_{n-k-1}(M-G_1(P))$. ∎

3. Proof of Theorem (1.4)

First, let us observe that we can avoid describing some of the technicalities of the proof since the interested reader can easily reconstruct them by consult-

ing our earlier paper [16] where similar methods are presented in all details. We have also omitted the proofs of some standard facts of cohomological dimension theory which can be found in the literature, e. g. in [9].

Let $n \geq 5$ and pick any $\varepsilon > 0$. Suppose we are given maps $f : X \to R^n$ and $g : Y \to R^n$, where X and Y are compacta, satisfying the hypotheses of Theorem (1.4), i. e. $\dim(X \times Y) < n$ and $2 \dim X + \dim Y < 2n - 1$. Because of Theorem (1.1), we may assume that $\dim X + \dim Y > n$. Let $k = \dim X$ and $l = \dim Y$.

Case 1. $k \geq [n/2]$. Since $\dim(X \times Y) < n$ it follows that $k \leq n - 2$ and $l \leq n - 2$. Observe that for our proof we may assume by Theorem (1.3) that X already lies in R^n and that $f : X \to R^n$ is an inclusion. Also, we may replace R^n by B^n (cf. [16]). We thus have to solve the following problem : For some closed subset $Y_0 \subset Y$, the image of the restriction of the map g, $g | Y_0 : Y_0 \to B^n$, already lies in $B^n - X$. In order to complete the proof of Theorem (1.4) we must show that $g | Y_0$ extends to a map $\tilde{g} : Y \to B^n - X$.

By Proposition (2.1), the restriction $g | Y_0$ extends to a map

$$\tilde{g} : Y \longrightarrow \Omega^\infty \Sigma^\infty (B^n - X).$$

Since $B^n - X$ is clearly $(n - k - 2)$-connected, it follows by the Freudenthal Suspension theorem that the inclusion map

$$\varphi : B^n - X \longrightarrow \Omega^\infty \Sigma^\infty (B^n - X)$$

induces an isomorphism

$$\varphi_* : \Pi_i(B^n - X) \longrightarrow \Pi_i(\Omega^\infty \Sigma^\infty (B^n - X))$$

for every $i < 2(n - k - 1) - 1 = 2n - 2k - 3$ whereas

$$\varphi_* : \Pi_{2n-2k-3}(B^n - X) \longrightarrow \Pi_{2n-2k-3}(\Omega^\infty \Sigma^\infty (B^n - X))$$

is an epimorphism, and by Theorem (5.1), its kernel is generated by Whitehead products.

Glue to $B^n - X$ cells which kill the Whitehead products

$$\Pi_{n-k-1}(B^n - X) \times \Pi_{n-k-1}(B^n - X) \longrightarrow \Pi_{2n-2k-3}(B^n - X)$$

of elements of $\Pi_{n-k-1}(B^n - X)$ to get the space $(B^n - X) \cup (\bigcup_{i \geq 1} C_i)$.

Since $\Omega^\infty \Sigma^\infty (B^n - X)$ is an H-space, the Whitehead products

$$\Pi_{n-k-1}(\Omega^\infty \Sigma^\infty (B^n - X)) \times \Pi_{n-k-1}(\Omega^\infty \Sigma^\infty (B^n - X)) \longrightarrow \Pi_{2n-2k-3}(\Omega^\infty \Sigma^\infty (B^n - X))$$

are all trivial, so we can extend the inclusion map

$$\varphi : B^n - X \longrightarrow \Omega^\infty \Sigma^\infty (B^n - X)$$

to a map

$$\bar{\varphi} : (B^n - X) \cup (\bigcup_{i \geq 1} C_i) \longrightarrow \Omega^\infty \Sigma^\infty (B^n - X).$$

Apply now Proposition (2.2) to conclude that the map

$$g|Y_0 : Y_0 \longrightarrow B^n - X$$

extends over Y to a map

$$\bar{g} : Y \longrightarrow (B^n - X) \cup (\bigcup_{i \geq 1} C_i).$$

Due to the compactness, $\bar{g}(Y)$ intersects only finitely many of the cells C_i, i. e. we get an extension of $g|Y_0$ to a map

$$\bar{g} : Y \longrightarrow (B^n - X) \cup (\bigcup_{i \in \{1, \cdots, s\}} C_{\sigma(i)})$$

for some integer $s \geq 1$.

Approximate X by a compact polyhedron $K \subset R^n$ such that $\dim K \leq k$ [2]. (Note that now $\dim (K \times Y)$ need no longer be $< n$. However, we do not need this condition anymore.) By Proposition (2.3) we many assume that the Whitehead products in $B^n - K$ are all trivial. (The finger moves are done on K in the complement of $\bar{g}(Y) \cup (\bigcup_{i \in \{1, \cdots, s\}} C_{\sigma(i)})$.) Therefore we can push the cells $C_{\sigma(i)}$ back into $B^n - K$ and hence also $\bar{g}(Y) \cap (\bigcup_{i \in \{1, \cdots, s\}} C_{\sigma(i)})$ so the image of Y under the map $\bar{g}$ falls into $B^n - K$. Therefore we can finally get the desired extension of $g|Y_0$ over all of Y, $\tilde{g} : Y \to B^n - X$. This completes the proof of Theorem (1.4) for Case 1.

Case 2. $k < [n/2]$. By Theorem (3.1) below, it suffices to verify the following:

ASSERTION: *Every subset $S \subset R^n$ such that $\dim S < n - 1$ and S lies in a hyperplane $H \subset R^n$ is antiwhitehead.*

PROOF. Observe that $R^n - S$ is naturally homotopy equivalent to the suspension of $H - S$. By the Freudenthal Suspension theorem, all elements of the first nontrivial homotopy group of the suspension are suspensions. The assertion now follows immediately, since the Whitehead product of suspensions is trivial. ∎

A compact subset $C \subset R^n$ is said to be *antiwhitehead* if all the Whitehead products of the elements of the first nontrivial homotopy group of $R^n - C$ are zero. Note that an antiwhitehead embedding of a compactum into R^n cannot be *wild*.

THEOREM 3.1. *Every mapping of a compactum X into R^n can be approximated arbitrarily closely by a mapping whose image is antiwhitehead, provided that* $2 \dim X < n$.

PROOF. Fix coordinates on R^n, let R^{n-1} be a fixed coordinate hyperplane, let $p: R^n \to R^{n-1}$ be the canonical projection of R^n onto R^{n-1}, and let $q: R^n \to (R^{n-1})^\perp$ be the canonical projection of R^n onto the orthogonal complement $(R^{n-1})^\perp$ of R^{n-1}. Pick any map $f: X \to R^n$. Then f can be represented as the ordered pair $f = (f_1, f_2)$, where $f_2 = p \circ f$ and $f_1 = q \circ f$.

Approximate f_2 by a regularly branched map g_2. Therefore, g_2 has a 0-dimensional set of points in the image whose multiplicity is equal to 2. Denote the image of the map $g = (f_1, g_2)$ by Y. It clearly suffices to prove that Y is antiwhitehead. To this end, let's extend Y by adding to it the union of the 0-dimensional family of segments, orthogonal to the hyperplane (this is the family of segments with ends projecting into the same point of the hyperplane). Denote the extended Y by Z.

Since the dimension of X is greater than one, we have that $\dim X = \dim Y = \dim Z$. Since the difference $\dim(Z - Y) = 0$ we have an isomorphism of the first nontrivial homotopy groups of the complements. Therefore it suffices to verify that Z is antiwhitehead.

The decomposition of R^n, generated by the segments which were added to Y, is shrinkable (since it consists of points and parallel segments). In our case, the shrinking pseudo-isotopy of R^n can be chosen so that its composition with the projection p yields the identity. As a result, we get an image of Z in R^n, denoted by W which projects onto the hyperplane, homeomorphically onto $p(Z) = p(Y) = p(W)$. Therefore there is an isotopy of R^n which translates W into $p(W)$. Hence $R^n - W$ is homeomorphic to $R^n - p(W)$. On the other hand, $R^n - W$ is homeomorphic to $R^n - Z$. ∎

4. Epilogue: On Čogošvili's Theorem

Let $X \subset R^n$ be a compactum and let $-1 \leq k \leq n$. It is easy to see that if $\dim X \leq n - k - 1$ then every intersection with any k-plane $L \subset R^n$ is unstable. Indeed, approximate X by a polyhedron $P \subset R^n$ of dimension $\leq n$ and apply general position in R^n to move P off L.

G. Čogošvili published a paper in the 1930's [7] in which the main result asserted that the *converse* is also true, i.e. that if $X \subset R^n$ unstably intersects every k-plane $L \subset R^n$ then it follows that $\dim X \leq n - k - 1$. This statement is

clearly equivalent to the affirmative answer to the following problem:

PROBLEM 4.1. *Does there exist for every $(n-k)$-dimensional compactum $X \subset R^n$, a k-dimensional plane $L \subset R^n$ which stably intersects X?*

Originally, [7] claimed that even for the case when $X \subset R^n$ is *noncompact*. Surprisingly enough, it was not until 1950's that a counterexample to the noncompact version of [7] was found—by K. A. Sitnikov [49]: Let $W \subset R^3$ be the union of a countable dense family of pairwise disjoint lines in R^3 and define $X = R^3 - W$. By the Alexander-Sitnikov duality, dim $X = 2$. It is clear that the set X unstably intersects every line in R^3. Nevertheless, dim $X \nleq 3-1-1=1$ which clearly contradicts [7].

Problem (4.1), i.e. the case of [7] when $X \subset R^n$ is *compact* was long believed to have been proven correctly in [7] and it appears in most monographs on dimension theory as one of the dimension theory classics (e. g. [2], [20]). That the proof in [7] does not go through even in this case was discovered only recently, by A. N. Dranišnikov and, independently, by R. Pol. In spite of several attempts by various people on this problem, the status of Čogošvili's theorem remains open[4].

Let's analyze briefly how [7] proposed to solve Problem (4.1). Recall that a classical result of P. S. Aleksandrov (see e. g. [2] or [20]) asserts that if compactum $X \subset R^n$ has dimension dim $X = n-k$, then there exists a $(k-1)$-dimensional cycle in $[\gamma] \in H_{k-1}(R^n - X; Z)$, *locally* linking X. The cycle γ can be realized in $R^n - X$ as a $(k-1)$-dimensional compact polyhedron $P_0 \subset R^n - X$ and P_0 can be assumed to lie in a k-dimensional compact polyhedron $P \subset R^n$ which *stably* intersects X. Therefore, Aleksandrov's theorem implies:

PROPOSITION 4.1. *Let $X \subset R^n$ be an $(n-k)$-dimensional compactum. Then X stably intersects some k-dimensional compact polyhedron $P \subset R^n$.* ∎

Now, every compact k-polyhedron $P \subset R^n$ lies in the union of *finitely* many k-dimensional *planes* in R^n, so only one more step was missing to complete the solution of Problem (4.1), i.e. the proof in [7]: to conclude that the following problem also has an *affirmative* answer:

PROBLEM 4.2. *Suppose that an $(n-k)$-dimensional compactum $X \subset R^n$ stably*

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[4] *Added in revision.* Y. Sternfeld [55] has recently obtained a partial solution—he has constructed a 2-dimensional compactum in R^{21} which unstably intersects every 19-dimensional plane, parallel to some coordinate plane.

intersects the union $\cup\{L_i\,|\,1\leq i\leq s\}$ *of finitely many k-planes* $L_i\subset R^n$. *Does it follow that for some* $i_0\in\{1,\ \cdots,\ s\}$, X *stably intersects* $L_{i_0}?$

It is asserted in [7] that this is *obvious*. It turns out it isn't and therefore Problem (4.2) remains an open question. The best approximation to the result claimed in [7] so far is the recent theorem due to D. O. Kiguradze (stated below) whose complete proof is included in our earlier paper [16]—he proved Čogošvili's theorem for the class of so-called *irrational* compacta.

For every point $x\in R^n$, let $r(x)$ be the number of *rational* coordinates of x. For every subset $X\subset R^n$, define $r(X)=\max\{r(x)\,|\,x\in X\}$. It is wellknown that $r(X)\geq\dim X$. We call $X\subset R^n$ *irrational* if $r(X)=\dim X$. For example, on the real line, the irrational compacta are precisely the irrational points.

A surjective map $f:X\to B^n$ of a compactum X onto the n-cell B^n is said to be *essential* if there is no map $g:X\to\partial B^n$ such that $g\,|\,f^{-1}(\partial B^n)=f\,|\,f^{-1}(\partial B^n)$ [2].

THEOREM 4.1 (Kiguradze). *Let* $X\subset R^n$ *be a k-dimensional irrational compactum*, $k\leq n$. *Then there is a k-dimensional plane* $L\subset R^n$ *and a closed k-ball* $C\subset L$ *such that the restriction* $p\,|\,p^{-1}(C)\cap X:p^{-1}(C)\cap X\to C$ *of the orthogonal projection* $p:R^n\to L$ *is an essential map.* ∎

As it was mentioned in Chapter 1, the first version of our paper [16] had a proof of Theorem (1.2), based on [29]—that was precisely Theorem (4.1) *without* the irrationality condition. That sets forth the following natural question:

PROBLEM 4.3. *Can one prove theorem* (4.1) *without the irrationality condition?*

If the answer is *affirmative*, then [16] offers a very simple geometric proof of Theorem (1.2). (This would be somewhat surprising since the proof of Dranišnikov and West [18] uses strong algebraic tools, e. g. the Spanier-Whitehead Duality.) Also, an affirmative answer would clearly imply that Čogošvili's theorem [7]—holds in the *compact case*. In particular, it would imply the classical Nöbeling theorem on projections of compacta on hyperplanes [42] (reproved in [34]).

In conclusion note that, by the Nöbeling-Hurewicz theorem [2] every map of a k-dimensional compactum X into R^{2k+1}, can be approximated arbitrarily closely by a map whose image is an irrational k-dimensional compactum. Also, by [57], every embedding of a compactum into R^n can be approximated arbi-

trarily closely by an irrational embedding. Moreover, irrational compacta are *tame* (in the sense of Štan'ko [56]) and their *embedding dimension*, dem X, is equal to their dimension $\dim X = r(X)$, i. e. in terms of general position they behave like $r(X)$-dimensional polyhedra (see [19]).

5. Appendix: The Hard Part of the Freudenthal Suspension Theorem

Let X be a topological space and let ΣX denote the *suspension* of X. One of the classical results of homotopy theory is the *Freudenthal Suspension theorem* which asserts that for every $n \geq 1$ and for every $(n-1)$-connected CW complex X, the suspension homomorphism $E : \Pi_r(X) \to \Pi_{r+1}(\Sigma X)$ is an *isomorphism* for every $r \in \{1, \cdots, 2n-2\}$ and an *epimorphism* for $r = 2n-1$.

What seems to not be so well-known is that this is only the so-called *easy part* of the theorem. The *hard part* describes the *kernel* of the suspension homomorphism E in the top dimension:

THEOREM 5.1. *For every $(n-1)$-connected CW complex X, the kernel of the Freudenthal suspension homomorphism $E : \Pi_{2n-1}(X) \to \Pi_{2n}(\Sigma X)$ is generated by the set $\{[\alpha, \beta] \mid \alpha, \beta \in \Pi_n(X)\}$, where $[-, -] : \Pi_n(X) \times \Pi_n(X) \to \Pi_{2n-1}(X)$ is the Whitehead product.*

We have made an extensive search of the existing literature (including [8], [21], [24], [25], [26], [27], [46], [47], [59]) to find a proof of Theorem (5.1) which we suspected to be true. Since it plays a pivotal role in our proof of Theorem (1.3) (as well as in the ones in [48], [54]) and since we couldn't locate it anywhere, we have decided to include a short proof of it:

PROOF. Consider the James Exact Suspension sequence

$$\cdots \longrightarrow \Pi_{2n+1}(\Sigma X) \overset{H}{\longrightarrow} \Pi_{2n}(JX, X) \overset{p}{\longrightarrow} \Pi_{2n-1}(X) \overset{E}{\longrightarrow} \Pi_{2n}(\Sigma X) \longrightarrow \cdots$$

In order to prove the assertion of the theorem, it clearly suffices, by exactness, to study the image of the homomorphism $p : \Pi_{2n}(JX, X) \to \Pi_{2n-1}(X)$.

Since (JX, X) can be replaced by (J_2X, X) we can reduce the problem to the consideration of the boundary homomorphism $\partial : \Pi_{2n}(J_2X, X) \to \Pi_{2n-1}(X)$. Recall that $J_2X = X \times X / (e, x) \sim (x, e)$, where $e \in X$ is a distinguished point in X.

Next, since the map $J_2X \to J_2X/X$ is $(n-1)+(2n-1)+1 = 3n-1 > 2n$-connected, it follows by the Blakers-Massey theorem that $\Pi_{2n}(J_2X, X) \cong \Pi_{2n}(J_2X/X)$. Furthermore, $J_2X/X \cong X \wedge X$ and the group $\Pi_{2n}(X \wedge X)$ is evidently generated by the elements of the type $\sigma \wedge \tau$ which make up the $2n$-skeleton of $X \wedge X$.

Therefore the preimages of these elements under the isomorphism $\Pi_{2n}(J_2X, X)$ $\to \Pi_{2n}(X \wedge X)$ generate $\Pi_{2n}(J_2X, X)$. On the other hand, these preimages are clearly generated by the elements $\sigma \times \tau$ where σ and τ are n-cells of X. Finally, the image of $\sigma \times \tau$ under the boundary homomorphism $\partial : \Pi_{2n}(J_2X, X)$ $\to \Pi_{2n-1}(X)$ is the Whitehead product $[\partial\sigma, \partial\tau]$. Hence $\mathrm{Ker}\, E = \mathrm{Im}\, \partial$ is indeed generated by the set $\{[\alpha, \beta] | \alpha, \beta \in \Pi_n(X)\}$. ∎

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Steklov Mathematical Institute
Vavilova St. 42
Moscow 117966
Russia