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CHARACTERIZATIONS OF SIMPLY CONNECTED COMPLETE UNITARY-SYMMETRIC KAEHLER MANIFOLDS

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

By

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Abstract. We accomplish the Kaehler version of Choi's characterizations of rotationally symmetric manifolds.

0. Introduction.

A Kaehler manifold M of complex dimension n is said to be unitary-symmetric at a point m of M if the linear isotropy group of automorphisms (that is, holomorphic isometries) of M is the unitary group $U(n)$.

A unitary-symmetric Kaehler manifold is a Kaehler version of a rotationally symmetric manifold (cf. Choi [1], Greene-Wu [2]). The second author [8] has given a characterization of such a Kaehler manifold. Using the result, the present authors have constructed a one parameter family of complete Kaehler metrics on CP^n, the complex projective n-space, which are compatible with the canonical complex structure on it, and have studied the geometry of unitary-symmetric Kaehler manifolds (cf. Watanabe [8], Mori-Watanabe [4], [5], [6]).

Let us fix some notations. Let M be a Kaehler manifold with Kaehler structure (ds^2, J). We denote by ∇ the Levi-Civita connection. The curvature tensor R is defined to be

$$R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

for any vector fields X, Y, Z on M and the Ricci tensor is denoted by *Ric*. Further, we denote by Ω the fundamental 2-form, that is, $\Omega(X, Y)=ds^2(JX, Y)$. Let $m\in M$. We define δ to be the distance from the origin O of the tangent space $T_m(M)$ at m to the first conjugate locus $\tilde{Q}_m$ in $T_m(M)$. Define $\tilde{B}_\delta=\{X\in T_m(M) \mid |X|<\delta\}$. Then it is clear that $\tilde{B}_\delta$ becomes a Riemannian manifold equipped with the metric $exp_m^* ds^2$, since $exp_m: \tilde{B}_\delta\to M$ is non-singular. Now we consider the following four conditions.

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(I) (M, ds^2, J) is unitary-symmetric at m.

(II) The metric $exp_m^* ds^2$ and the fundamental 2-form $exp_m^* \Omega$, pulled back under the exponential mapping exp_m, are given by

$$exp_m^* ds^2 = dr^2 + f(r)^2 d\Theta^2 + f(r)^2 (f'(r)^2 - 1)\eta \otimes \eta ,$$

$$exp_m^* \Omega = 2f(r)f'(r)\eta \wedge dr + f(r)^2 \Psi$$

on the punctured ball $\tilde{B}_\delta - \{O\}$ of radius δ in $T_m(M)$, where f is a C^∞ odd function on $(-\delta, \delta)$ such that $f'(0)=1$ and $f'(r)>0$. Here we assume that δ is infinite when M is non-compact, and we denote by (r, Θ) the usual polar coordinate system of $C^n \equiv T_m(M)$, by $(d\Theta^2, \phi, \xi, \eta)$ the standard Sasakian structure on the unit sphere S^{2n-1} in $T_m(M)$, and set $\Psi(X, Y) = d\Theta^2(\phi X, Y)$.

(III) The Riemannian curvature tensor R satisfies

$$R(J\gamma', \gamma')\gamma' = h(r)J\gamma' , \qquad R(E(r), \gamma')\gamma' = k(r)E(r) ,$$

where γ' is the tangent vector field of a radial geodesic γ starting from m, $h(r)$, $k(r)$ are functions depending only on the *geodesic distance* r from the origin O, and $E(r)$ is a parallel vector field along γ which is perpendicular to both γ' and $J\gamma'$.

(IV) The exponential image of any complex linear subspace (resp. real subspace spanned by u, w) of $T_m(M)$ is a closed, totally geodesic, complex (resp. real) submanifold of M, where u, Ju and w are orthonormal.

Then our assertion is as follows.

THEOREM. *Let (M, ds^2, J) be a complete, connected, simply-connected, Kaehler manifold of complex dimension $n \geqq 2$ and m be a point of M. Then the above conditions I, II, III and IV are equivalent.*

1. Proof of Theorem.

We have already known that (I) is equivalent to (II) (see Watanabe [8]).

We shall show that (III) implies (II). Let γ be a geodesic issuing from m and $E=E(r)$ a parallel vector field along γ such that $E(0)$ is perpendicular to both $\gamma'(0)$ and $J\gamma'(0)$. Then we have the following two kinds of Jacobi fields V and Ξ along γ,

$$V(r) = f(r)E(r) , \qquad \Xi(r) = g(r)J\gamma'$$

for some functions f, g, which satisfy the differential equations

$$f''(r) + k(r)f(r) = 0 , \qquad g''(r) + h(r)g(r) = 0$$

with initial conditions

$$f(0)=0, \quad f'(0)=1, \quad g(0)=0, \quad g'(0)=1,$$

respectively. By applying the Jacobi field argument, a long calculation shows that the Riemannian metric $exp_m^* ds^2$ is given by the form

$$exp_m^* ds^2 = dr^2 + f^2(d\Theta^2 - \eta\otimes\eta) + g^2\eta\otimes\eta$$

(cf. Nakashima-Watanabe [7]). Since (ds^2, J) is a Kaehler structure, we can see that $g=ff'$. Further, since $f>0$ and $f'>0$ on $(0, \delta)$, it follows from the assumption on δ that if $\delta<\infty$, then $f'(\delta)=0$. Thus we have the condition (II).

We shall show that (I) and (II) imply (IV). Let W be a complex linear subspace of $T_m(M)$. Then there exists a unitary matrix φ which leaves W pointwise fixed but $\varphi(X)\neq X$ for every $X\in T_m(M)-W$. From the assumption that (M, ds^2, J) is unitary-symmetric at m, it follows that there exists an automorphims Φ of (M, ds^2, J) such that $(\Phi_*)_m=\varphi$. From this the image of W under the exponential map exp_m is the fixed point set of the isometry Φ of (M, ds^2), which implies that the image set $exp_m(W)$ is a totally geodesic submanifold of (M, ds^2). By restricting the structures (ds^2, J) to the vectors tangent to $exp_m(W)$ we see that $exp_m(W)$ is an almost Hermitian submanifold of (M, ds^2, J). Thus the first assertion is true (see Kobayashi-Nomizu [3], p. 171).

From the first assertion, it suffices to prove the second assertion in the case $n=2$. We adopt a polar coordinate system $\psi(t, \theta_1, \theta_2, \theta_3)=(t\cos\theta_1\cos\theta_2\cos\theta_3,$ $t\cos\theta_1\cos\theta_2\sin\theta_3,\ t\cos\theta_1\sin\theta_2,\ t\sin\theta_1)$, $0<t<\infty$, $-\pi/2<\theta_1,\ \theta_2<\pi/2$, $-\pi<\theta_3<\pi$ for $T_m(M)\equiv \boldsymbol{C}^2$. Then we shall show that the submanifold $exp_m\{\psi(t, 0, \theta_2, 0)\ |t\in R,\ -\pi/2\leqq\theta_2\leqq\pi/2\}$ is a closed, totally geoderic submanifold. We find that with respect to local coordinates $w_1=t$, $w_{i+1}=\theta_i$, $i=1, 2, 3$, the components g_{ij} of the Riemannian metric g are given by $g_{1j}=\delta_{1j}$, $j=1, 2, 3, 4$, $g_{i+1i+1}=f(t)^2(\lambda_i+(f'(t)^2-1)\eta_i^2)$, $i=1, 2, 3$, $g_{i+1j+1}=f(t)^2(f'(t)^2-1)\eta_i\eta_j$, $i\neq j$, where $\lambda_1=1$, $\lambda_2=\cos^2 w_2$, $\lambda_3=\eta_3=\cos^2 w_2\cos^2 w_3$, $\eta_1=\sin w_3$, $\eta_2=-\sin w_2\cos w_2\cos w_3$. From this observation it follows that the Christoffel's symbols satisfy $\Gamma^i_{jk}=0$ for $i=2, 4$ and $j, k=1, 3$, when $0<|w_1|<\infty$, $-\pi/2<w_3<\pi/2$ and $w_2=w_4=0$. Thus, the second assertion is true.

Finally, we shall show that (IV) implies (III). Let u, Ju and w be orthonormal vectors in $T_m(M)$ and consider the geodesic $\gamma(r)=exp_m ru$, $r\in R$. Set $P=exp_m\, span\{u, Ju\}$, $Q=exp_m\, span\{u, w\}$. Denote by $E(r)$ a unit vector field (in Q) along $\gamma(r)$ which is perpendicular to $\gamma'(r)$ and satisfies $E(0)=w$. Since P and Q are 2-dimensional totally geodesic submanifolds of M, we find that $E(r)$ is a (uniquely determined) parallel field along γ which is perpendicular to both γ' and $J\gamma'$ and that

$$R(J\gamma'(r), \gamma'(r))\gamma'(r)=h(r)J\gamma'(r), \qquad R(E(r), \gamma'(r))\gamma'(r)=k(r)E(r),$$

for some functions $h(r)$ and $k(r)$. Thus we have the condition (III).

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