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ON CLASSIFICATION OF SOME SURFACES OF REVOLUTION OF FINITE TYPE

By

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Abstract. In this article, we study the following problem of [5]: Classify all finite type surfaces in a Euclidean 3-space E^3. A surface M in a Euclidean 3-space is said to be of finite type if each of its coordinate functions is a finite sum of eigenfunctions of the Laplacian operator on M with respect to the induced metric (cf. [1, 2]). Minimal surfaces are the simplest examples of surfaces of finite type, in fact, minimal surfaces are of 1-type. The spheres, minimal surfaces and circular cylinders are the only known examples of surfaces of finite type in E^3 and it seems to be the only finite type surfaces in E^3 (cf. [5]). The first author conjectured in [2] that spheres are the only compact finite type surfaces in E^3. Since then, it was proved step by step and separately that finite type tubes, finite type ruled surfaces, finite type quadrics and finite type cones are surfaces of the only known examples (cf. [2, 6, 7, 10].) Our next natural target for this classification problem is the class of surfaces of revolution. However, this case seems to be much difficult than the other cases mentioned above. We therefore investigate this classification problem for this class and obtain classification theorems for surfaces of revolution which are either of rational or of polynomial kinds (cf. §1 for the definitions). As consequence, further supports for the conjecture cited above are obtained.

1. Introduction.

Let M be a (connected) smooth surface in a Euclidean 3-space E^3. Denote by Δ the Laplacian of M associated with the induced metric. Let x and H denote the position vector field and the mean curvature vector field of M in E^3,

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respectively. Then we have

$$\Delta x = -2H. \tag{1.1}$$

Formula (1.1) yields the following two well-known facts: (a) surface M is minimal if and only if all coordinate functions of E^3, restricted to M, are harmonic functions, *i.e.*, they are eigenfunctions of Δ with eigenvalue 0; and (b) M is an open portion of an ordinary sphere S^2 if and only if all coordinate functions of E^3, restricted to M are eigenfunctions of Δ with a fixed nonzero eigenvalue.

For a given surface M in E^3, a smooth function on M is said to be *of finite type* if it can be expressed as a finite sum of eigenfunctions of Δ. If all coordinate functions of E^3, restricted to M, are of finite type, then M is said to be *of finite type*. Otherwise, M is said to be *of infinite type*. (See [1, 2] for details). In terms of finite type surfaces, a well-known result of Takahashi [11] says that a surface in E^3 is of 1-type if and only if either it is a minimal surface of E^3 or it is an open portion of an ordinary sphere.

In [2], the first author proposed the following

PROBLEM. *Classify finite type surfaces in* E^3.

This is indeed an interesting but a very difficult problem. Because the problem involves a system of very complicated partical differential equations. It seems to the first author that the only surfaces of finite type in E^3 are open portions of planes, spheres, circular cylinders or minimal surfaces. For compact case, the first author made the following

CONJECTURE. [5] *The only compact surfaces of finite in* E^3 *are the ordinary spheres.*

The first result concerning the classification of surfaces of finite type in E^3 was obtained in [2]. In fact, it was proved in [2] that circular cylinders are the only tubes of finite type in E^3. In [10] it was shown that planes are the only finite type cones in E^3. It was proved in [7] that a ruled surface in E^3 is of finite type if and only if it is an open portion of a plane, of a circular cylinder or of a helicoid. Furthermore, the first author and F. Dillen proved in [6] that spheres and circular cylinders are the only quadrics of finite type in E^3 (even locally). Further results in this direction can be found in [5, 8, 9].

In this article, we consider this classification problem for surfaces of revolution. However, this case seems to be much complicated than the previous cases. We therefore investigate in this article the classification of surfaces of revolution which are either of rational kind or of polynomial kinds.

A surface in E^3 is called a surface of revolution if it is generated by a curve C on a plane π when π is rotated around a straight line L in π. By choosing π to be the xz-plane and line L to be the z-axis, a surface of revolution is parametrized by

$$x(u, v)=(f(u)\cos v,\ f(u)\sin v,\ g(u)) . \tag{1.2}$$

A surface of revolution given by (1.2) is said to be *of the polynomial kind* if $f(u)$ and $g(u)$ are polynomial functions in u; and it is said to be *of the rational kind* if g is a rational function in f, *i.e.*, g is the quotient of two polynomial functions in f.

In this article we prove the following classification theorems:

THEOREM 1. *Let M be a surface of revolution of the polynomial kind. Then M is a surface of finite type if and only if M is either an open portion of a plane or an open portion of a circular cylinder.*

THEOREM 2. *Let M be a surface of revolution of the rational kind. Then M is a surface of finite type if and only if M is an open portion of a plane.*

In fact, Theorem 2 follows from the following more general result:

THEOREM 3. *Let M be a finite type surface of revolution parametrized by*

$$x(t, v)=(t\cos v,\ t\sin v,\ g(t)) .$$

If $g'(t)^2=Q(t)/R(t)$ for some polynomial functions $Q(t)$ and $R(t)$ in t, then M is an open portion of a plane, or M is an open portion of a catenoid, or $\deg Q=\deg R=2+\deg(Q+R)$.

REMARK. Up to similarity transformations on E^3, a catenoid can be parametrized by

$$x(t, v)=(t\cos v,\ t\sin v,\ g(t)) ,$$

with $g(t)=\cosh^{-1}t$. Thus $g'(t)^2=1/(t^2-1)$ is the quotient of two polynomials $Q=1$ and $R=t^2-1$ with $\deg R=\deg(Q+R)$. On the other hand, it is easy to verify that spheres centered at the origin in E^3 are 1-type surfaces which do satisfy the condition on $g'(t)$ in Theorem 3 with $\deg Q=\deg R=2+\deg(Q+R)$.

And planes are null 1-type surfaces with the function $g(t)$ of Theorem 3 satisfying $g'(t)=0$.

2. Preliminaries.

Let $x: M \to E^3$ be an immersion from a 2-dimensional connected manifold M into a Euclidean 3-space E^3. Denote the Laplacian operator of M with the induced metric by Δ. Then the immersion x (or the surface M) is *of finite type* if each component of the position vector field x of M in E^m can be written as a finite sum of eigenfunctions of the Laplacian operator. Hence x is of finite type if and only if x has the following decomposition:

$$x=c+x_1+x_2+\cdots+x_k \tag{2.1}$$

where c is a constant vector and $x_1, x_2, \cdots, x_k$ are non-constant maps satisfying $\Delta x_i=\lambda_i x_i$, $i=1, 2, \cdots, k$. Moreover, if all eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ are mutually different, then the immersion x (or the surface M) is said to be of *k-type* and the decomposition (2.1) is called the *spectral decomposition* of the immersion x (or of the surface M.) In particular, if one of $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ is zero, then M is said to be of *null k-type*.

We need the following results for later use:

PROPOSITION 1. [1, 2] *Let M be a k-type surface whose spectral decomposition is given by* (2.1). *If we put*

$$P(T)=\prod_{i=1}^{k}(T-\lambda_i), \tag{2.2}$$

then $P(\Delta)(x-c)=0$.

PROPOSITION 2. [3, 5] *Let M be a surface in E^3. If there exists a constant b such that $\Delta H=bH$, then M is either of* 1*-type or of null* 2*-type.*

PROPOSITION 3. [4] *The only null* 2*-type surfaces in E^3 are open portions of circular cylinders.*

The monic polynomial P in Proposition 1 is called the *minimal polynomial* of the finite type surface M. To find out whether or nor a surface is of finite type, the minimal polynomial plays a very important role (cf. [5]).

Let M be a surface of revolution parametrized by

$$x(u, v)=(f(u)\cos v,\ f(u)\sin v,\ g(u)). \tag{2.3}$$

It is easy to verify by straight-forward computation that the Laplacian operator Δ of M is given by

$$(2.4)\qquad \Delta=\frac{-1}{(f')^2+(g')^2}\frac{\partial^2}{\partial u^2}-\left\{\frac{f'}{f((f')^2+(g')^2)}-\frac{f'f''+g'g''}{((f')^2+(g')^2)^2}\right\}\frac{\partial}{\partial u}-\frac{1}{f^2}\frac{\partial^2}{\partial v^2}.$$

3. Proof of Theorem 1.

Assume M is a surface of revolution of the polynomial kind in E^3. Then, without loss of generality, we may assume M is parametrized by

$$(3.1)\qquad x(u, v)=(Q(u)\cos v,\ Q(u)\sin v,\ R(u))$$

for some polynomials $Q(u)$ and $R(u)$ of degree q and r, respectively. In particular, we have $\max\{q, r\}\geqq 1$.

We need the following

LEMMA 1. *Let $F(u)$ and $G(u)$ be polynomial functions in u and M a surface of revolution of the polynomial kind which is parametrized by* (3.1). *Then $\Delta(F(u)/G(u))=F_1(u)/G_1(u)$ for some polynomial functions F_1, G_1 with*

$$\deg F_1-\deg G_1\leqq \deg F-\deg G-2\max\{\deg F, \deg G\}.$$

PROOF. By applying (2.4) and straight-fordward computation, we may obtain

$$\Delta\left(\frac{F(u)}{G(u)}\right)=\frac{F_1(u)}{G_1(u)}$$

where

$$(3.2)\qquad F_1=Q(Q'^2+R'^2)(G(F'G-FG')'-2G'(F'G-FG'))$$
$$+G(F'G-FG')\{Q'(Q'^2+R'^2)-Q(Q'Q''+R'R'')\}$$

and

$$(3.3)\qquad G_1=-G^3Q(Q'^2+R'^2)^2.$$

Let $\deg Q=q$ and $\deg R=r$. Then from (3.2) and (3.3) we may find

$$\deg F_1\leqq 2\deg G+q+2\max\{q, r\}+\deg F-4,$$

$$\deg G_1=3\deg G+q+4\max\{q, r\}-4,$$

which implies the lemma. □

Let

$$\Delta^i R=\frac{F_i}{G_i}, \qquad i=1, 2, 3, \cdots. \tag{3.4}$$

Then by Lemma 1 we have

$$\deg F_{i+1}-\deg G_{i+1}<\deg F_i-\deg G_i< \cdots <\deg F_1-\deg G_1<\deg R. \tag{3.5}$$

Assume M is of finite type, say of k-type. Let

$$P(T)=T^k+c_1T^{k-1}+\cdots+c_{k-1}T+c_k$$

be the minimal yolynomial of M given in Proposition 1. Then P has k distinct real roots. From (3.4) and Proposition 1, we have

$$\frac{F_k}{G_k}+c_1\frac{F_{k-1}}{G_{k-1}}+\cdots+c_{k-1}\frac{F_1}{G_1}+c_k(R-a)=0.$$

for some constant a. Let $K=G_1\cdots G_k$. Then we have

$$K\frac{F_k}{G_k}+c_1K\frac{F_{k-1}}{G_{k-1}}+\cdots+c_{k-1}K\frac{F_1}{G_1}+c_kK(R-a)=0. \tag{3.6}$$

If $\deg R=0$, M is an open portion of a plane. If $\deg R>0$, then by (3.5) we have

$$\deg K(R-a)>\deg K\frac{F_1}{G_1}>\cdots>\deg K\frac{F_{k-1}}{G_{k-1}}>\deg K\frac{F_k}{G_k}$$

which is impossible unless $P(T)=T^2+c_1T$ with $c_1\neq 0$ and $\Delta R=0$, since the minimal polynomial P has k distinct real roots. In this case, (1.1) and Proposition 1 imply

$$\Delta H=-c_1H. \tag{3.7}$$

Hence, by Proposition 2, we know that M is either of 1-type or of null 2-type. If M is of 1-type, then either M isa n open portion of a sphere or M is minimal. It is easy to check that spheres are not of the polynomial kind and the only minimal surfaces of revolution are open portions of planes and open portions of catenoids. Since as remarked in § 1, catenoids are also not of the polynomial kind, we conclude that M is of null 2-type. Consequently, by applying Proposition 3, we conclude that M should be an open portion of a circular cylinder. The converse of this is easy to verify. □

4. Proof of Theorem 2.

Let M be a surface of revolution of the rational kind. Then, without less of generality, we may assume M is parametrized by

$$x(t, v)=(t\cos v, t\sin v, g(t))$$

where $g(t)=G(t)/H(t)$ for some polynomial functions $G(t)$ and $H(t)$. We may assume $G(t)$ and $H(t)$ are relatively prime. We have

$$g'(t)^2=Q(t)/R(t), \qquad Q=(G'H-GH')^2, \qquad R=H^4. \tag{4.1}$$

It is easy to verify that $\deg Q=\deg R$ if and only if $\deg G=1+\deg H$. Moreover, in this case, we have $\deg Q=\deg R=\deg(Q+R)$. Assume M is of finite type. Then, by Theorem 3 proved in §5, M is either an open portion of a plane or an open portion of a catenoid. Since, up to similarity transformations on E^3, a catenoid can be parametrized by

$$x(t, v)=(t\cos v,\ t\sin v,\ \cosh^{-1}t), \tag{4.2}$$

it is not of the rational kind. Hence, M must be an open portion of a plane. The converse is trivial. □

5. Proof of Theorem 3.

Let M be a surface of revolution parametrized by

$$x(t, v)=(t\cos v,\ t\sin v,\ g(t)) \tag{5.1}$$

with $g'(t)^2=Q(t)/R(t)$ for some polynomial functions $Q(t)$, $R(t)$. Without loss of generality, we may assume that $Q(t)$ and $R(t)$ are relatively prime and $R(t)$ is a monic polynomial.

From (2.4) and (5.1) we find

$$\Delta=-\frac{R}{Q+R}\frac{\partial^2}{\partial t^2}-\frac{R}{t(Q+R)^2}\left\{Q+R-\frac{t(Q'R-R'Q)}{2R}\right\}\frac{\partial}{\partial t}-\frac{1}{t^2}\frac{\partial^2}{\partial v^2}. \tag{5.2}$$

From (5.2) we get

$$\Delta(t\cos v)=\frac{Q_1}{R_1}\cos v, \tag{5.3}$$

with

$$Q_1=-2R(Q+R)+t(Q'R-QR')+2(Q+R)^2, \qquad R_1=2t(Q+R)^2. \tag{5.4}$$

Moreover, by (5.2), (5.3) and (5.4), we may conclude inductively that

$$\Delta^i(t\cos v)=\frac{Q_i}{R_i}\cos v, \qquad i=1, 2, 3, \cdots \tag{5.5}$$

for some polynomial functions Q_i and R_i. Moreover, if we put

$$\bar{Q}_i=Q_i'R_i-Q_iR_i', \quad \bar{\bar{Q}}_i=(Q_i'R_i-Q_iR_i')'R_i^2-2(Q_i'R_i-Q_iR_i')R_iR_i', \tag{5.6}$$

then we have

$$Q_{i+1}=-2t^2R(Q+R)\bar{\bar{Q}}_i-2tRR_i^2(Q+R)\bar{Q}_i \tag{5.7}$$

$$+t^2(Q'R-QR')R_i^2\bar{Q}_i+2(Q+R)^2Q_iR_i^3,$$

$$R_{i+1}=2t^2(Q+R)^2R_i^4, \qquad i\geqq 1. \tag{5.8}$$

Assume M is of finite type, say of k-type. Let

$$P(T)=T^k+c_1T^{k-1}+\cdots+c_{k-1}T+c_k$$

be the minimal polynomial of M given in Proposition 1. Then P has k distinct real roots. From (5.5) and Proposition 1, we have

$$\frac{Q_k}{R_k}+c_1\frac{Q_{k-1}}{R_{k-1}}+\cdots+c_{k-1}\frac{Q_1}{R_1}+c_kt=0. \tag{5.9}$$

Let $K=R_1\cdots R_k$. Then we have

$$K\frac{Q_k}{R_k}+c_1K\frac{Q_{k-1}}{R_{k-1}}+\cdots+c_{k-1}K\frac{Q_1}{R_1}+c_ktK=0. \tag{5.10}$$

For simplicity, we put $q=\deg Q$, $r=\deg R$, $q_i=\deg Q_i$, and $r_i=\deg R_i$. We divide the proof of the theorem into three separate cases.

CASE (i): $q>r$.
In this case we have the following lemma.

LEMMA 2. *If $q>r$, then, for any $i\geqq 1$, we have*

$$r_i-q_i=2i-1. \tag{5.11}$$

PROOF OF LEMMA 2. From (5.4) we obtain $r_1=1+2q$ and $q_1=2q$. Thus (5.11) holds for $i=1$.

Assume (5.11) holds for some $i\geqq 1$. From (5.6) we have $\deg \bar{Q}_i=q_i+r_i-1$ and $\deg \bar{\bar{Q}}_i=q_i+3r_i-2$. Therefore, by (5.7) and (5.8), we may find

$$q_{i+1}=2q+q_i+3r_i, \qquad r_{i+1}=2q+4r_i+2.$$

This implies $r_{i+1}-q_{i+1}=r_i-q_i+2=2i+1$. This proves the lemma. □

By using Lemma 2 and (5.10) we can conclude that case (i) is impossible.

CASE (ii): $q<r$.
In this case we have the following lemma.

LEMMA 3. *If $q<r$, then, for any $i\geqq 1$, we have*

$$r_i-q_i=2i-1+s, \tag{5.12}$$

where s is a positive integer.

PROOF OF LEMMA 3. From (5.4) we obtain $r_1=1+2r$ and $q_1\leqq q+r$. Let $s=r_1-q_1-1\geqq r-q>0$. Then $r_1-q_1=1+s$. This proves (5.12) for $i=1$.

Assume (5.12) holds for some $i\geqq 1$. Then, by (5.6), we have

$$\deg \bar{Q}_i=q_i+r_i-1\,, \qquad \deg \bar{\bar{Q}}_i=q_i+3r_i-2\,.$$

Therefore, by (5.7) and (5.8), we get

$$q_{i+1}=2r+q_i+3r_i\,, \qquad r_{i+1}=2+2r+4r_i\,.$$

Consequently, we find $r_{i+1}-q_{i+1}=2+r_i-q_i=2(i+1)-1+s$. This proves the lemma. □

By using Lemma 3 and (5.10) we can conclude that Case (ii) is also impossible.

CASE (iii): $q=r$.

Let $m=\deg(Q+R)$. We divide this case into three subcases:

CASE (iii-a): $q=r=m$.

Without loss of generallty, we may assume the leading coefficients of Q and R are given by a and 1, respectively.

We need the following lemma.

LEMMA 4. *Assume $q=r=m$. Then we have*

(1) *if $a\neq 4n(n+1)$ for every natural number n, then*

$$r_i-q_i=2i-1\,, \qquad i=1,2,\cdots\,. \tag{5.13}$$

(2) *if $a=4n(n-1)$ for some natural number $n\geqq 2$, then*

$$r_i-q_i=2i-1\,, \qquad for\ i\leqq n \tag{5.14}$$

$$r_i-q_i=2i-1+s\,, \qquad for\ i>n\,, \tag{5.15}$$

where s is a positive integer.

PROOF OF LEMMA 4. Since $q=r=m$, we have

$$\deg R(Q+R)=2r,\quad \deg(Q'R-QR')\leqq 2r-2,\quad \deg(Q+R)^2=2r. \tag{5.16}$$

It is easy to see that the leading coefficient of Q_1 is $2a(a+1)$ which is not zero. Therefore, by (5.4), we have $q_1=2r$. Since $r_1=1+2r$, formulas (5.13) and

(5.14) hold for $i=1$.

Assume $a\neq 4n(n+1)$ for any natural number n and assume (5.13) holds for $i\geqq 2$. From (5.6) and (5.7), we see that $q_{i+1}\leqq 2r+q_i+3r_i$. It is not difficult to verify that the leading coefficient of Q_{i+1} is $2\{1+a-(r_i-q_i)^2\}a_i(1+a)$, where a_i is the leading coefficient of Q_i. Since $r_i-q_i=2i-1$ and $a\neq 4i(i-1)$ by assumption, we obtain $q_{i+1}=2r+q_i+3r_i$. Because $r_{i+1}=2+2r+4r_i$, we find $r_{i+1}-q_{i+1}=2+r_i-q_i=2(i+1)-1$. This proves Statement (1).

For Statement (2), let us assume that $a=4n(n-1)$ for some natural number $n\geqq 2$. By the same argument as the proof of Statement (1), we obtain (5.14). Since $a=4n(n-1)$ and $r_n-q_n=2n-1$, we find $1+a-(r_n-q_n)^2=0$. Thus $\deg Q_{n+1}<2r+q_n+3r_n$. Let $s=2r+q_n+3r_n-\deg Q_{n+1}$. Then we obtain $r_{n+1}-q_{n+1}=(2+2r+4r_n)-(2r+q_n+3r_n-s)=2+s+r_n-q_n=2n+1+s$. This proves (5.15) for $i=n+1$. Formula (5.15) for $i>n+1$ can be proceeded in the same way as Statement (1), now. □

From Lemma 4, we see that (5.10) is impossible unless $c_k=0$ and $Q_1=0$. Since the minimal polynomial P has exactly k distinct real roots, this implies $P(T)=T$. Thus, M is a minimal surface in E^3.

CASE (iii-b): $q=r=m+1$.

In this case, we may put

$$Q(t)=-t^r+A(t), \qquad R(t)=t^r+B(t), \tag{5.17}$$

where $A(t)$ and $B(t)$ are polynomials of degree $\leqq r-1=m$ and

$$\deg(Q+R)=\deg(A+B)=r-1, \qquad \deg(Q'R-QR')=2r-2. \tag{5.18}$$

We need the following lemma.

LEMMA 5. *If $q=r=m+1$, then we have $r_1=q_1$ and there is a non-negative integer s such that $r_i-q_i=i+s$ for any $i\geqq 2$.*

PROOF OF LEMMA 5. From (5.4) and (5.18) we have $q_1=r_1=2r-1$. Thus, by using (5.6) we may obtain

$$\deg \bar{Q}_1\leqq q_1+r_1-2=4r-4, \qquad \deg \bar{\bar{Q}}_1\leqq 8r-7. \tag{5.19}$$

From (5.8) we find $r_2=10r-4$. By applying (5.7) and (5.19), we get $q_2\leqq 10r-6$. Let $s=10r-6-q_2\geqq 0$. Then we have $r_2-q_2=2+s$.

Now, assume $r_i-q_i=i+s$ for some $i\geqq 2$. The, by direct computation, we obtain from (5.7) that the degree of each term in the right-hand-side of (5,7) is $\leqq 2r+q_i+3r_i-1$. Moreover, by using $r_i-q_i>0$, and by direct computation of the coefficient of Q_{i+1} in (5.7), we may conclude that $\deg Q_{i+1}=2r+q_i+3r_i-1$. Since $r_{i+1}=2r+4r_i$, we obtain $r_{i+1}-q_{i+1}=r_i-q_i+1=i+1+s$ by (5.8). This prove the lemma. □

From Lemma 5 we see that (5.10) is impossible unless $c_k=0$ and $Q_1=0$. Because the minimal polynomial P of M has exactly k distinct real roots, this implies $P(T)=T$. Thus, M is a minimal surface in E^3.

CASE (iii-c): $q=r>m+1$.

In this case we may put

$$Q(t)=-t^{m+1}D(t)+A(t)\,, \qquad R(t)=t^{m+1}D(t)+B(t)\,, \tag{5.20}$$

where $D(t)$ is a polynomial of degree $r-m-1$ and $A(t)$, $B(t)$ are polynomials of degree $\leqq m$ such that $\deg(A+B)=m$. It is easy to see from (5.20) that

$$\deg(Q'R-QR')=r+m-1\,. \tag{5.21}$$

We need the following lemma.

LEMMA 6. *If $q=r>m+1$, then*

$$r_i-q_i=i(2+m-r)-1\,. \tag{5.22}$$

PROOF OF LEMMA 6. From (5.4) and the assumption, we have $q_1=r+m$ and $r_1=2m+1$. Thus, $r_1-q_1=m+1-r$, which shows that (5.22) is true for $i=1$.

Assume (5.22) holds for some $i\geqq 1$. Then, from (5.6) and (5.21), we find

$$\deg \bar{Q}_i=q_i+r_i-1\,, \qquad \deg \bar{\bar{Q}}_i\leqq q_i+3r_i-2\,. \tag{5.23}$$

From (5.7), (5.8), (5.21) and (5.23), we may obtain $r_{i+1}=4r_i+2m+2$ and $q_{i+1}\leqq q_i+3r_i+r+m$. Furthermore, by direct computation, we may see that the coefficient of $t^{q_i+3r_i+r+m}$ in Q_{i+1} is a nonzero multiple of $(2r_i-2q_i+m-r)(q_i-r_i)$. Since $q_i-r_i\neq 0$ and $2r_i-2q_i+m-r=i(2+m-r)-1+m-r<0$, we get $q_{i+1}=q_i+3r_i+r+m$. Hence, $r_{i+1}-q_{i+1}=r_i-q_i+2+m-r=(i+1)(2+m-r)-1$. This proves the lemma. □

As we did before, let $K=R_1\cdots R_k$. Then we have

$$K\frac{Q_k}{R_k}+c_1K\frac{Q_{k-1}}{R_{k-1}}+\cdots+c_{k-1}K\frac{Q_1}{R_1}+c_ktK=0\,. \tag{5.24}$$

From Lemma 6, we have

$$\deg K = \sum_{i=1}^{k} q_i + \frac{1}{2} k(k+1)(2+m-r) - k\,. \tag{5.25}$$

By applying Lemma 6 and (5.25) we obtain

$$\deg K \frac{Q_i}{R_i} = \delta - i(2+m-r), \qquad i=1, \cdots, k\,, \tag{5.26}$$

$$\deg tK = \delta\,, \qquad \delta = \sum_{i=1}^{k} q_i + \frac{1}{2} k(k+1)(2+m-r) - k + 1\,. \tag{5.27}$$

(5.26) and (5.27) imply that if $r>m+2$, (5.24) is impossible unless $k=1$ and $c_k=0$. Hence, in this case either $r=q=m+2$ or M is minimal.

Consequently, by combining cases (i), (ii) and (iii), we conclude that either $\deg Q = \deg R = 2 + \deg(Q+R)$ or M is a minimal surface of revolution in E^3. The latter case occurs only when the surface is either an open portion of a plane or an open portion of a catenoid. This completes the proof of the theorem. □

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