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AUTOMORPHISMS WITH FIXED POINTS AND WEIERSTRASS POINTS OF COMPACT RIEMANN SURFACES

By

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§0. Introduction.

Let M be a compact Riemann surface of genus $g\geqq 2$ and T be a conformal automorphism of order N with t fixed points. We denote $\langle T\rangle$ the cyclic group generated by T and $M/\langle T\rangle$ the surface by identifying the equivalent points on M under the elements of $\langle T\rangle$. M is considered as a covering surface of $M/\langle T\rangle$ and the behavior of ramifications depends on the gap sequences of the fixed points.

Lewittes [7] proved that if $t\geqq 5$, then every fixed point of T is 1-Weierstrass point, and Guerrero [4] proved that if $t=1$ and the fixed point is not a 1-Weierstrass point, then T has order 6, $g\equiv 1 \pmod 6$ and the fixed point is a q-Weierstrass point for all $q\geqq 2$. Guerrero also gave examples of Riemann surfaces with automorphisms of prime order N whose two fixed points are not q-Weierstrass points. Furthermore several authors considered some cases for the relation of the fixed points and q-Weierstrass points.

Duma [2] proved that if $N=2$ and $t\geqq 3$, then every fixed point of T is a q-Weierstrass point for all $q\geqq 2$, and that if $N=3$ and $t\geqq 3$, then every fixed point of T is a q-Weierstrass point for $q\geqq 2$ ($q\neq 2 \pmod 3$). Farkas and Kra [3] proved that if T is of prime order N and $t\geqq 3$, then every fixed point is a q-Weierstrass point for $q\geqq 2$ ($q\equiv 1 \pmod N$). Accola [1] proved that if T is of prime order N and $t\geqq 3$, then every fixed point is a N-Weierstrass point. Recently Horiuchi and Tanimoto [5] gave a sufficient condition for fixed points to be q-Weierstrass point ($q\geqq 2$) and showed that the results mentioned above are obtained by using the condition and studied the case where $t\geqq 3$ and T is of order 5.

Almost all of the results mentioned above, however, are obtained under the condition that T is of prime order. In this paper we investigate the properties of automorphisms without the condition that T is of prime order. In the first

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section we collect for reference the terms and theorems useful for our purposes. In the second section we discuss the cases where automorphism T has a fixed point of which gap sequence is specified. We classify automorphisms according to the number of fixed points, show the relations of genera of M and $M/\langle T\rangle$, and determine the rotation constants of the fixed points. In the third section, we investigate a condition for fixed points to be q-Weierstrass point ($q\geqq 2$) in some cases classified in § 2. Furthermore we give some typical examples for the theorems. In the fourth section, we compute the dimension n_k^q of H_k^q, where H_k^q is the vector space of all holomorphic q-differentials θ such that $T(\theta)=\varepsilon^k\theta$, $\varepsilon=\exp(2\pi i/N)$. In the fifth section we give some miscellaneous results. In the sixth section we give some examples of certain construction of Riemann surfaces. Finally we compute the integral solution of some trigonometric "diophantine" equations in § 2 and § 3 as appendix.

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§ 1. Notations, preliminary and key lemmas.

Let M be a compact Riemann surface of genus $g\geqq 2$ and $\mathrm{Aut}(M)$ be the group of conformal automorphisms of M. Let $T\in\mathrm{Aut}(M)$ be of order N with t fixed points. Let $M\to M/\langle T\rangle$ be an analytic N-sheeted branched covering, $M/\langle T\rangle$ having genus g', then the Riemann-Hurwitz relation states that $2g-2=N(2g'-2)+\mathfrak{B}$ where $\mathfrak{B}$ is the sum of the orders of all branch points on M.

We denote H^q the complex vector space of holomorphic q-differentials. It is known that $\dim H^1=n^1=g$ and any $\theta\in H^1$, not identically zero, has $2g-2$ zeros, while for $q>1$, $\dim H^q=(2q-1)(g-1)$ and any $\theta\in H^q$, not identically zero, has $q(2g-2)$ zeros. $\mathrm{div}(\theta)$ denotes the divisor of zeros and poles of $\theta\in H^q$. Concerning divisors on M the Riemann-Roch theorem states $r(b^{-1})=\deg(b)+i(b)+1-g$ where $r(b^{-1})$ is the dimension of the space of functions whose divisors are multiples of b^{-1}, $\deg(b)$ is the degree of b, and $i(b)$ is the dimension of the subspace of meromorphic differentials consisting of multiples of b.

For each point $P\in M$, there are $d=d(q)$ integers $1=\gamma_1<\gamma_2<\cdots<\gamma_{d(q)}<2q(g-1)+2$ such that there exist a q-differential $\theta\in H^q(M)$ with a zero of order γ_j-1 at P, where $d=d(q)=\dim H^q$, i. e., $d(1)=g$, $d(q)=(2q-1)(g-1)$ if $q\geqq 2$. The sequence $\gamma_1<\cdots\gamma_d$ is called the q-gap sequence at P, and the non-negative integer $w_q(P)=\sum_{j=1}^{d(q)}(\gamma_j-j)$ is called the q-th weight at P. $P\in M$ is called a q-

Weierstrass point on M if $w_q(P)>0$. Then $W_1(g)=\sum_{P\in M} w_1(P)=(g-1)g(g+1)$ or $W_q(g)=\sum_{P\in M} w_q(P)=(2q-1)^2 g(g-1)^2$ if $q\geqq 2$.

$T\in \mathrm{Aut}(M)$ acts on H^q and has a matrix representation which is diagonal form for suitable choice of basis in H^q. Then a diagonal element is a power of N-th root of unity $\varepsilon=\exp(2\pi i/N)$.

LEMMA A. (*Lewittes*) [3] [7] *The representation of* T *on the space* H^q *is the* $d(q)\times d(q)$ *diagonal matrix*

$$\mathrm{diag}\,(\varepsilon^{\gamma_1-1+q},\ \varepsilon^{\gamma_2-1+q},\ \cdots,\ \varepsilon^{\gamma_{d(q)}-1+q})\,.$$

For $0\leqq k\leqq N-1$, $n^q{}_k$ denotes the number of ε^k in the diagonal elements, naturally we have $\sum_{k=0}^{N-1} n_k^q=\dim H^q=(2q-1)(g-1)$ for $q>1$ we have $\sum_{k=0}^{N-1} n_k^q=\dim H^1=g$ for $q=1$. Furthermore, denoting $H_k^q=\{\theta\in H^q \mid T(\theta)=\varepsilon^k\theta\}$, we have $n_k^q=\dim H_k^q$, particularly, $n_0^1=\dim H_0^1=g'$.

For each $m=1, 2, \cdots, t$, we choose a local coordinate z at P_m and an integer ν_m such that T^{-1} is given by $T^{-1}: z\to\varepsilon^{\nu_m}z$ near P_m (note that ν_m must be relatively prime to N). The integer ν_m is called the rotation constant of T at P_m. Then we have the following important lemmas.

LEMMA B. (*Eichler trace formula*) [3] *As above notations, we have*

$$tr\,(T)=1+\sum_{m=1}^{t}\varepsilon^{\nu_m}/(1-\varepsilon^{\nu_m}) \qquad \textit{for } q=1$$

$$tr\,(T)=\sum_{m=1}^{t}\varepsilon^{\nu_m r}/(1-\varepsilon^{\nu_m}) \qquad \textit{for } q>1$$

where $0\leqq r\leqq N-1$ *is chosen as the uniqe integer such that* $q=dN+r$ *with* $d\in N$. *In each case the sum is taken to be zero whenever* $t=0$.

As a corollary we have

LEMMA C. (*Lefschetz fixed point formula*) [3] *For* $q=1$,

$$tr\,(T)+\overline{tr\,(T)}=2-t\,.$$

§2.

1. Automorphisms with a fixed point which is not 1-Weierstrass point.

Let $P_1, \cdots, P_t$ be the fixed points of $T\in\mathrm{Aut}(M)$ and ν_m be the rotation constant of P_m. First we assume that $P=P_1$ is not 1-Weierstrass point and we may assume that $\nu_1=1$. Put $g=lN+s$ $(0\leqq s\leqq N-1)$. Then, from Lemma A, we have $l=g'$ and

$$tr(T)=\varepsilon+\varepsilon^2+\cdots+\varepsilon^g=\varepsilon+\varepsilon^2+\cdots+\varepsilon^s=(\varepsilon-\varepsilon^{s+1})/(1-\varepsilon).$$

On the other hand, from Lemma C, we have

$$(\varepsilon-\varepsilon^{s+1})/(1-\varepsilon)+\overline{(\varepsilon-\varepsilon^{s+1})/(1-\varepsilon)}=2-t \quad \text{or} \quad 1+\varepsilon+\varepsilon^2+\cdots+\varepsilon^{2s}=(3-t)\varepsilon^s.$$

From $\varepsilon=\cos(2\pi/N)+i\sin(2\pi/N)$ and the above equation, we have

$$\sin((s+1)\pi/N)\cos(s\pi/N)=((4-t)/2)\sin(\pi/N).$$

Namely, $\sin((2s+1)\pi/N)=(3-t)\sin(\pi/N)$.

For $t=1$, we have $s=1$, $N=6$.

For $t=2$, we have $s=0$ or $s+1=N/2$.

For $t=3$, we have $s=(N-1)/2$.

For $t=4$, we have $s=N-1$ or $s=N/2$.

But we can exclude the case $t=4$, $s=N/2$. Because from the Riemann-Hurwitz relation (hereafter we say the R-H relation), $s\geqq N-1$, consequently we have $N=2$ and this case is contained in the case $s=N-1$. The case $t\geqq 5$ does not occur [7]. Indeed from the R-H relation, $\mathfrak{B}=2g-2-N(2g'-2)=2N+2s$. On the other hand $\mathfrak{B}\geqq 5(N-1)$. Hence we have $2s\geqq 3(N-1)$, which contradicts $s\leqq N-1$.

We will investigate the above cases in detail.

(a) Case $t=1$ [4]. In this case, $N=6$, $g=6g'+1$.

From the R-H relation, we have $\mathfrak{B}=12$. On the other hand the branch number must be of the form $5+3x+4y$, where x (resp. y) denotes the number of points in $M/\langle T\rangle$ whose fibre consists of fixed of T^3 (resp. T^2). The only posibility is that $x=y=1$.

(b) Case $t=2$. $g=Ng'$.

From Lemma B, $tr(T)=1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})=(1-\varepsilon^{\nu_2})/(1-\varepsilon)(1-\varepsilon^{\nu_2})$, while from Lemma A, $tr(T)=0$. Cosequently we have $\nu_2=N-1$. From the R-H relation we have $\mathfrak{B}=2(N-1)$, so there is no branch point except P_1 and P_2.

(c) Case $t=2$. $g=Ng'+(N/2-1)$ (N is even).

From Lemma B, $tr(T)=(1-\varepsilon^{\nu_2+1})/(1-\varepsilon)(1-\varepsilon^{\nu_2})$, while from Lemma A, $tr(T)=(1+\varepsilon)/(1-\varepsilon)$. Consequently we have $\nu_2=1$. From the R-H relation $\mathfrak{B}=2(N-1)+N-2$, so there is one point in $M/\langle T\rangle$ whose fibre consists of fixed points Q_1, Q_2 of T^2.

(d) Case $t=3$. $g=Ng'+(N-1)/2$ (N is odd).

From Lemma B, $tr(T)=1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})$, while from Lemma

A, $tr(T)=\varepsilon(1-\varepsilon^{(N-1)/2})/(1-\varepsilon)$. By simple and concrete calculations we can show that $\nu_2=1$, $\nu_3=(N-1)/2$ (assuming $\nu_2\leqq\nu_3$) (c. f., Appendix 1). From the R-H relation $\mathfrak{B}=3(N-1)$, so there is no branch point except P_1, P_2, P_3.

(e) Case $t=4$. $g=Ng'+(N-1)$.

From Lemma B, $tr(T)=1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})+\varepsilon^{\nu_4}/(1-\varepsilon^{\nu_4})$, while from Lemma A, $tr(T)=(\varepsilon-\varepsilon^N)/(1-\varepsilon)=-1$. Then by simple and concrete calculations we can show that $\nu_2+\nu_3=N$, $\nu_4=N-1$ (assuming $\nu_2\leqq\nu_3\leqq\nu_4$) (c. f., Appendix 2). From the R-H relation, $\mathfrak{B}=4(N-1)$, so there is nn branch point except P_1, P_2, P_3, P_4.

We summarize the above results in the following theorem:

THEOREM 1. *If $T\in Aut(M)$ has a fixed point which is not* 1*-Weierstrass point, then there are only five possible cases*;

(a) $t=1$, $g=6g'+1$ $(N=6)$

(b) $t=2$, $g=Ng'$ $(\nu_1, \nu_2)=(1, N-1)$

(c) $t=2$, $g=Ng'+(N/2-1)$ $(\nu_1, \nu_2)=(1, 1)$

(d) $t=3$, $g=Ng'+(N-1)/2$ $(\nu_1, \nu_2, \nu_3)=(1, 1, (N-1)/2)$

(e) $t=4$, $g=Ng'+(N-1)$ $(\nu_1, \nu_2, \nu_3, \nu_4)=(1, \nu, N-\nu, N-1)$

REMARK 1. In the case (c), we consider T^2 instead of T. Then the analytic $N/2$-sheeted covering $M\to M/\langle T^2\rangle$ has four branch points P_1, P_2, Q_1, Q_2 and $g=(N/2)\hat{g}+(N/2-1)$ where $\hat{g}=$the genus of $M/\langle T^2\rangle$. Therefore this case can be reduced in the case (e) and we know that the rotation constants of Q_1, Q_2 (which are the fixed points of T^2) are $2(N/2-1)=N-2$.

REMARK 2. We say that a N-sheeted covering $M\to\tilde{M}$ is totally ramified provided all branch points have order $N-1$. Briefly M is totally ramified (over $\tilde{M}$). Now in cases (a), (b), (d) and (e) M is totally ramified.

REMARK 3. Lewittes [7] obtained a theorem of the same type under the condition that N is prime number.

2. Automporphisms with a fixed point which is normal Weierstrass point. (i. e., gas sequence 1, 2, 3, $\cdots$, $g-1$, $g+1$)

We put $g=lN+s$ $(0\leqq s\leqq N-1)$. If $s=0$, then $g'=l-1$. If $1\leqq s\leqq N-2$ $(N\geqq3)$, then $g'=l$. If $s=N-1$ $(N\geqq3)$, theu $g'=l+1$.

(1) If $s=0$, $g=Ng'+N$. From the R-H relation $\mathfrak{B}=4N-2$ and $\mathfrak{B}>(N-1)$, we have $t\leqq4+2/(N-1)$. So if $N=2$, then $t=6$ and if $N=3$, then $t=5$ and if

$N \geqq 4$, then $t \leqq 4$.

(2) If $1 \leqq s \leqq N-2$, $g=Ng'+s$. From the R-H relation $\mathfrak{B}=2N+2s-2$ we have $t \leqq 2+2(N-2)/(N-1)$, hence $t \leqq 3$.

(3) If $s=N-1$, $g=Ng'-1$. From the R-H relation $\mathfrak{B}=2N-4$ we have $t \leqq 2-2/(N-1)$, hence $t=1$.

We investigate more detail; Put $g=lN+s$ $(0 \leqq s \leqq N-1)$, then from Lemma A, $tr(T)=\varepsilon+\varepsilon^2+\cdots+\varepsilon^{s-1}+\varepsilon^{s+1}=(\varepsilon-\varepsilon^s+\varepsilon^{s+1}-\varepsilon^{s+2})/(1-\varepsilon)$. Therefore Lemma C,

$$tr(T)+tr(T)=(\varepsilon-\varepsilon^s+\varepsilon^{s+1}-\varepsilon^{s+2})/(1-\varepsilon)+(\varepsilon^{-1}-\varepsilon^{-s}+\varepsilon^{-s-2})/(1-\varepsilon^{-1})$$
$$=2-t .$$

Hence

$$(1-\varepsilon^{2s+1})(1-\varepsilon+\varepsilon^2)=(3-t)\varepsilon^{s+1}(1-\varepsilon) \qquad (*)$$

(a) Case $t=1$. From Lemma B, $tr(T)=1/(1-\varepsilon)$, so we have

$$(\varepsilon-\varepsilon^s+\varepsilon^{s+1}-\varepsilon^{s+2})/(1-\varepsilon)=1/(1-\varepsilon) \quad \text{or} \quad \varepsilon^{s+2}-\varepsilon^{s+1}+\varepsilon^s-\varepsilon+1=0 .$$

Then we have $N=10$ and $g=10g'+2$.

(b) Case $t=2$. From $(*)$, $1+\varepsilon^2+\varepsilon^{s+2}+\varepsilon^{2s+2}=\varepsilon+\varepsilon^{s+1}+\varepsilon^{2s+1}+\varepsilon^{2s+3}$.

By the same way in Appendix, we have the following solutions $(N, s)=(4, 2), (4, 3), (8, 1), (8, 2), (9, 3)$.

But we must exclude the two cases $(4, 3)$, $(8, 1)$, for the contradictions occur according to the R-H relation.

If $(N, s)=(4, 2)$, then $g=4g'+2$. From Lemma B, $tr(T)=1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}$, while $tr(T)=(\varepsilon-\varepsilon^2+\varepsilon^3-\varepsilon^4)/(1-\varepsilon)=0$. Hence $\nu_2=3$. From the R-H relation $\mathfrak{B}=10$, so there are two points on $M/\langle T\rangle$ whose fibres consist of fixed points Q_1, Q_2 and R_1, R_2 of T^2 respectively.

If $(N, s)=(8, 2)$, then $g=8g'+2$. From Lemma B and $tr(T)=(\varepsilon-\varepsilon^2+\varepsilon^3-\varepsilon^4)/(1-\varepsilon)$, $\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})=(\varepsilon-\varepsilon^2+\varepsilon^3)/(1-\varepsilon)$. Hence $\nu_2=3$. From the R-H relation $\mathfrak{B}=18$, so there is one point on $M/\langle T\rangle$ whose fibre consists of the fixed points $Q_1, \cdots, Q_4$ of T^4.

If $(N, s)=(9, 3)$, then $g=9g'+3$. From Lemma B and $tr(T)=(\varepsilon-\varepsilon^3+\varepsilon^4-\varepsilon^5)/(1-\varepsilon)$, we have $\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})=(-1+\varepsilon-\varepsilon^3+\varepsilon^4-\varepsilon^5)/(1-\varepsilon)$. Hence $\nu_2=2$. From the R-H relation $\mathfrak{B}=22$, so there is one point on $M/\langle T\rangle$ whose fibre consists of the fixed points Q_1, Q_2, Q_3 of T^3.

(c) Case $t=3$. From $(*)$, $(1-\varepsilon+\varepsilon^2)(1-\varepsilon^{2s+1})=0$.

(i) If $1-\varepsilon+\varepsilon^2=0$, then $N=6$.

Assume $s=0$, then $g=6g'+6$. From Lemma B and $tr(T)=\varepsilon-1$, we have $1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})=\varepsilon-1$. Hence we have $\nu_2+\nu_3=6$. Therefore we may assume $\nu_2=1$, $\nu_3=5$. From the R-H relation $\mathfrak{B}=22$, so there are two points on $M/\langle T\rangle$ whose fibres consist of fixed points Q_1, Q_2 and R_1, R_2, R_3 of T^2 and T^3 respectively.

Assume $1\leqq s\leqq 4$, then $g=6g'+s$. From the R-H relation $\mathfrak{B}=10+s$, so all of the branch number is $2s-5$ except for the fixed points P_1, P_2, P_3. Then the possible value of s is 4, namely $g=6g'+4$. There is one point of $M/\langle T\rangle$ whose fibre consists of the fixed points Q_1, Q_2, Q_3 of T^3. We may assume $\nu_2=1$, $\nu_3=5$ as the above case.

(ii) If $\varepsilon^{2s+1}=1$, then $s=(N-1)/2$ and $g=Ng'+(N-1)/2$. In this case there is no branch point except P_1, P_2, P_3. From Lemma B and $tr(T)=(\varepsilon-\varepsilon^s+\varepsilon^{s+1}-\varepsilon^{s+2})/(1-\varepsilon)$, we obtain

$$1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})=(\varepsilon-\varepsilon^s+\varepsilon^{s+1}-\varepsilon^{s+2})/(1-\varepsilon)\,.$$

By the same way in Appendix, we have only the following four solutions under the assumption $\nu_2\leqq\nu_3$.

$$(N,\ \nu_2,\ \nu_3)=(5,\ 3,\ 3),\ (7,\ 2,\ 4),\ (9,\ 1,\ 7),\ (11,\ 2,\ 3)$$

Namely

$g=5g'+2$ $(\nu_1, \nu_2, \nu_3)=(1, 3, 3)$, $g=7g'+3$ $(\nu_1, \nu_2, \nu_3)=(1, 2, 4)$,

$g=9g'+4$ $(\nu_1, \nu_2, \nu_3)=(1, 1, 7)$, $g=11g'+5$ $(\nu_1, \nu_2, \nu_3)=(1, 2, 3)$.

(d) Case $t=4$. According to the consideration in the first step of this subsection, we have $s=0$. From (*), $(1-\varepsilon)(1-\varepsilon+\varepsilon^2)=-\varepsilon(1-\varepsilon)$, so we have $\varepsilon^2=-1$. Hence $N=4$ and $g=4g'+4$. In this case there is one point of $M/\langle T\rangle$ whose fibre consists of the fixed points Q_1, Q_2 of T^2. From Lemma B and $tr(T)=\varepsilon-1$, we obtain

$$1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})+\varepsilon^{\nu_4}/(1-\varepsilon^{\nu_4})=\varepsilon-1\,.$$

Then we obtain $(\nu_2, \nu_3, \nu_4)=(1, 1, 3)$ under the assumption $\nu_2\leqq\nu_3\leqq\nu_4$.

(e) Case $t=5$. According to the consideration in the first step of this subsection, we have $N=3$, $s=0$ and $g=3g'+3$. In this case there is no branch point except $P_1, \cdots, P_5$. From Lemma B and $tr(T)=\varepsilon-1$, we have

$$1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})+\varepsilon^{\nu_4}/(1-\varepsilon^{\nu_4})+\varepsilon^{\nu_5}/(1-\varepsilon^{\nu_5})=\varepsilon-1\,.$$

Then we obtain $(\nu_2, \nu_3, \nu_4, \nu_5)=(1, 1, 1, 2)$ under the assumption $\nu_2\leqq\nu_3\leqq\nu_4\leqq\nu_5$.

(f) Case $t=6$. According to the first step of this subsection, we have $N=2$, $s=0$ and $g=2g'+2$. In this case it is trivial that there is no branch point except $P_1, \cdots, P_6$ and $(\nu_1, \cdots, \nu_6)=(1, \cdots, 1)$.

REMARK 4. As a matter of fact, we are able to know that the cases $g=$

$6g'+4$ and $g=9g'+4$ in the above do not happen on account of the results of the computation n_k^1. Indeed n_k^1 can not be obtained as integer in the two cases.

We summalize the above results in the following theorem;

THEOREM 2. *If $T\in Aut(M)$ has a fixed point which is normal Weierstrass point, then there are only* 11 *possible cases*:

(a) $t=1$, $g=10g'+2$ $(N=10)$

(b-1) $t=2$, $g=4g'+2$ $(N=4)$ $(\nu_1, \nu_2)=(1, 3)$

(b-2) $t=2$, $g=8g'+2$ $(N=8)$ $(\nu_1, \nu_2)=(1, 3)$

(b-3) $t=2$, $g=9g'+3$ $(N=9)$ $(\nu_1, \nu_2)=(1, 2)$

(c-1) $t=3$, $g=5g'+2$ $(N=5)$ $(\nu_1, \nu_2, \nu_3)=(1, 3, 3)$

(c-2) $t=3$, $g=6g'+6$ $(N=6)$ $(\nu_1, \nu_2, \nu_3)=(1, 1, 5)$

(c-3) $t=3$, $g=7g'+3$ $(N=7)$ $(\nu_1, \nu_2, \nu_3)=(1, 2, 4)$

(c-4) $t=3$, $g=11g'+5$ $(N=11)$ $(\nu_1, \nu_2, \nu_3)=(1, 2, 3)$

(d) $t=4$, $g=4g'+4$ $(N=4)$ $(\nu_1, \nu_2, \nu_3, \nu_4)=(1, 1, 1, 3)$

(e) $t=5$, $g=3g'+3$ $(N=3)$ $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)=(1, 1, 1, 1, 2)$

(f) $t=6$, $g=2g'+2$ $(N=2)$ $(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6)=(1, 1, 1, 1, 1, 1)$.

We can say briefly;

COROLLARY 1. *If T has a fixed point which is normal Weierstrass point, then we have $t\leqq 6$ and $N\leqq 11$.*

In other words;

COROLLARY 2. *If T has $t\geqq 7$ fixed points, every fixed point of T must be a Weierstrass point which have weight $\geqq 2$. If T has of order $N\geqq 12$, every fixed point of T is not normal Weierstrass point.*

REMARK 5. In cases (c-1), (c-3), (c-4), (e) and (f), M is totally ramified.

3. Automorphisms with a fixed point which is hyperelliptic Weierstrass point. (gap sequence 1, 3, 5, $\cdots$, $2g-1$)

We assume that M is hyperelliptic and that a fixed point $P=P_1$ of $T\in Aut(M)$ is a hyperelliptic Weierstrass point. As well known, if M is hyperelliptic, $T\in Aut(M)$ has strictry $2g+2$ or at most 4 fixed points. Indeed, there is a meromorphic function f which takes any value twice and so we put $F=f-f\circ T$. If $F\neq 0$, then F has at most 4 poles, and so T has at most 4 fixed points. If $F\equiv 0$, then for any $P\in M$, $f(P)=f(Q)$ $(Q=T(P))$. That is T is an involution of M if M is considered as 2-sheeted covering of $\mathbf{P}^1$ by f, and the $2g+2$ ramification points are fixed points of T clearly.

Now, from Lemma A, $tr(T)=\varepsilon+\varepsilon^3+\varepsilon^5+\cdots+\varepsilon^{2g-1}$.

We put

$$g=lN+s \quad (0\leqq s\leqq N-1).$$

If N is even, then there is no multiple of N in $\{1, 3, \cdots, 2g-1\}$ and so

the genus g' of $M/\langle T\rangle$ is 0.

If N is odd, there are $l+1$ (resp. l) multiples of N in $\{1, 3, \cdots, 2g-1\}$ if $s\geqq(N+1)/2$ (resp. $s\leqq(N-1)/2$), and so

$$g'=\begin{cases} l+1 & (s\geqq(N+1)/2) \\ l & (s\leqq(N-1)/2) \end{cases}.$$

If $N=2$, $g=2l+s$ $(g'=0)$, then from Lemma C,

$$tr(T)+\overline{tr(T)}=-g+(-g)=2-t \quad \text{or} \quad t=2g+2,$$

namely T is hyperelliptic involution.

If N is odd and $s\geqq(N+1)/2$, then from the R-H relation,

$$2g-2=2(Ng'-N+s)-2=N(2g'-2)+\mathfrak{B},$$

therefore $\mathfrak{B}=2(s-1)\geqq(N-1)t$. Hence $t=1$. If N is odd and $s\leqq(N-1)/2$, then from the R-H relation,

$$2g-2=2(Ng'+s)-2=N(2g'-2)+\mathfrak{B},$$

therefore $\mathfrak{B}=2N+2s-2\geqq(N-1)t$. Hence $t\leqq 3$.

We will investigate the above cases in detail. $tr(T)=\varepsilon+\varepsilon^3+\cdots+\varepsilon^{2g-1}=\varepsilon(1-\varepsilon^{2s})/(1-\varepsilon^2)$ and from Lemma C, we have $1-\varepsilon^{4s}=(2-t)\varepsilon^{2s-1}(1-\varepsilon^2)$.

Case $t=1$. $1-\varepsilon^{4s}=\varepsilon^{2s-1}(1-\varepsilon^2)$ so $(1-\varepsilon^{2s-1})(1+\varepsilon^{2s+1})=0$.

If N is even, $\varepsilon^{2s+1}=-1$. Therefore $s=N/4-1/2$ or $s=3N/4-1/2$.

If N is odd, $\varepsilon^{2s-1}=1$ namely $2s-1=N$, but this case does not happen, for the branch point is the only one point P_1.

Case $t=2$. $1-\varepsilon^{4s}=0$.

From $0\leqq 4s\leqq 4(N-1)$, we have the following

$$s=0 \ (N\text{: arbitrary}) \quad \text{and} \quad s=N/4,\ N/2,\ 3N/4 \ (N\text{: even}).$$

Case $t=3$. $1-\varepsilon^{4s}=-\varepsilon^{2s-1}(1-\varepsilon^2)$ so $(1-\varepsilon^{2s+1})(1+\varepsilon^{2s-1})=0$,

If N is odd, $\varepsilon^{2s+1}=1$, hence $s=(N-1)/2$.

If N is even, $\varepsilon^{2s-1}=-1$, hence $s=N/4+1/2$ or $s=3N/4+1/2$.

Case $t=4$. $1-\varepsilon^{4s}=-2\varepsilon^{2s-1}(1-\varepsilon^2)$.

From $\varepsilon=\cos(2\pi/N)+i\sin(2\pi/N)$, we have $\sin(4\pi s/N)+2\sin(2\pi/N)=0$. But this equation has no solution when $N\geqq 3$ by simple calculation.

Now, we will determine the rotation constants and investigate the behavior of ramification in the above cases.

(a) case $t=1$.

We do not need to determine the rotation constant in this case.

If $s=N/4-1/2$, the branch number is $(2l+3/2)N-2$ excepting the number of P_1 from the R-H relation $\mathfrak{B}=2g-2+2N$. If $s=3N/4-1/2$, the branch number is $(2l+5/2)N-2$ excepting the number of P_1 from $\mathfrak{B}=2g-2+2N$.

(b) case $t=2$.

From Lemma B, $tr(T)=1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})=(1-\varepsilon^{\nu_2+1})/(1-\varepsilon)(1-\varepsilon^{\nu_2})$.

If $s=0$, then $tr(T)=0$. So we have $\nu_2=N-1$. From R-H relation $\mathfrak{B}=2lN+2N-2$, the branch number is $2lN$ except for P_1, P_2. If $s=N/2$, then $tr(T)=0$. So we have $\nu_2=N-1$. From the R-H relation $\mathfrak{B}=2lN+3N-2$, the branch number is $2lN+N$ except for P_1 and P_2. If $s=N/4$, then $tr(T)=\varepsilon(1-\varepsilon^{N/2})/(1-\varepsilon^2)=2\varepsilon/(1-\varepsilon^2)$. So we have $(1-\varepsilon^{\nu_2+1})/(1-\varepsilon)(1-\varepsilon^{\nu_2})=2\varepsilon/(1-\varepsilon^2)$. Therefore $\varepsilon^{\nu_2+1}=-1$, hence $\nu_2=N/2-1$. From the R-H relation, $\mathfrak{B}=2lN+5N/2-3$, so the branch number is $2lN+N/2$ except for P_1 and P_2.

If $s=3N/4$, then $tr(T)=\varepsilon(1-\varepsilon^{3N/2})/(1-\varepsilon^2)=2\varepsilon/(1-\varepsilon^2)$. So we have $\nu_2=N/2-1$. From the R-H relation, $\mathfrak{B}=2lN+7N/2-2$, so the branch number is $2lN+3N/2$ except for P_1 and P_2.

(c) case $t=3$.

From Lemma B, $tr(T)=1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})$. If N is odd and $s=(N-1)/2$, then $tr(T)=\varepsilon(1-\varepsilon^{2s})/(1-\varepsilon^2)=-1/(1+\varepsilon)$. Hence $1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})=-1/(1+\varepsilon)$. By the same way in Appendix, we have the following solution $(\nu_1, \nu_2)=(N-2, N-2)$. From the R-H relation, there is no branch point except P_1, P_2, P_3.

If N is even and $s=N/4+1/2$, then $N\equiv 2 \pmod 4$. We consider $M/\langle T^2\rangle$ and denote $\hat{g}$ the genus of $M/\langle T^2\rangle$. Since $N/2$ (=the order of T^2) is odd, we have $g=(N/2)\hat{g}$ or $g=(N/2)\hat{g}+(N/2-1)/2$. These, however, do not occur, because N must be 2 and T is hyperelliptic involution in these cases. Indeed, from $g=(N/2)\hat{g}$ for example, we have $\hat{g}=2l+1/2+1/N$, so N must be 2. Also from $g=(N/2)\hat{g}+(N/2-1)/2$, we have the same. The case that N is even and $s=3N/4+1/2$ is the same as above.

We summarize the above results in the following theorem.

THEOREM 3. *If $T \in Aut(M)$ has a fixed point which is hpperelliptic Weierstrass point, then there are only 8 possible cases except that T is the hyperelliptic involution*:

If N is odd, then

(a) $t=2,\ g=Ng'$ $(\nu_1, \nu_2)=(1, N-1)$

(b) $t=3,\ g=Ng'+(N-1)/2$ $(\nu_1, \nu_2, \nu_3)=(1, N-2, N-2)$

If N is even, then $g'=0$ and

(c-1) $t=1,\ g=lN+N/4-1/2$

(c-2) $t=1,\ g=lN+3N/4-1/2$

(d-1) $t=2,\ g=lN$ $(\nu_1, \nu_2)=(1, N-1)$

(d-2) $t=2,\ g=lN+N/2$ $(\nu_1, \nu_2)=(1, N-1)$

(d-3) $t=2,\ g=lN+N/4$ $(\nu_1, \nu_2)=(1, N/2-1)$

(d-4) $t=2,\ g=lN+3N/4$ $(\nu_1, \nu_2)=(1, N/2-1)$.

REMARK 6. In cases (a) and (b), M is totally ramified.

REMARK 7. Horiuchi [6] investigated normal coverings of hyperelliptic surfaces which are also hyperelliptic and obtained most part of the above in other aspects.

§ 3. Higher order Weierstrass points for some cases

1. Now we give sufficient conditions for fixed points to be q-Weierstrass point ($q \geqq 2$) in some cases.

THEOREM 4. *If $T \in Aut(M)$ is of odd order N and has three fixed points P_1, P_2, P_3 whose rotation constants (ν_1, ν_2, ν_3) are $(1, 1, N-1/2)$, then we have the following*;

(i) *P_1, P_2 are q-Weierstrass points ($q \geqq 2$) except possibly for the following cases*:

(1) $N \equiv 1 \pmod 6$ *and either* $q \equiv \frac{N+2}{3} \pmod N$ *or* $q \equiv \frac{2N+1}{3} \pmod N$

(2) $N \equiv 5 \pmod 6$ *and either* $q \equiv \frac{N+1}{3} \pmod N$ *or* $q \equiv \frac{2(N+1)}{3} \pmod N$

(ii) *P_3 is q-Weierstrass point for all $q \geqq 2$.*

PROOF. (i) Assume that P_i $(i=1, 2)$ is not q-Weierstrass point, then from Lemma A, $tr(T)=\varepsilon^q(1+\varepsilon+\cdots \varepsilon^{(2q-1)(g-1)-1})=(\varepsilon^q-\varepsilon^{-2q-(N-3)/2})/(1-\varepsilon)$. While, from Lemma B, $tr(T)=2\varepsilon^q/(1-\varepsilon)+\varepsilon^{(N-1)q/2}/(1-\varepsilon^{(N-1)/2})$.

Hence, $\varepsilon^q-\varepsilon^{q+(N-1)/2}+\varepsilon^{-2q-(N-3)/2}-\varepsilon^{-2q+1}+\varepsilon^{(N-1)q/2}-\varepsilon^{(N-1)q/2+1}=0$. Then we can obtain the desired condition (cf., Appendix 3).

(ii) Assume that P_3 is not q-Weierstrass point, then from Lemma B, $tr(T)=(\varepsilon^q-\varepsilon^{-2q-(N-3)/2})/(1-\varepsilon)$, while from Lemma B, $tr(T)=\varepsilon^q/(1-\varepsilon)+2\varepsilon^{(N-2)q}/(1-\varepsilon^{N-2})$. Hence $2\varepsilon^{(N+1)/2}=1+\varepsilon$. This equation has no integer solution N.

⟨q. e. d.⟩

THEOREM 5. *If $T \in Aut(M)$ has four fixed points P_1, P_2, P_3, P_4 whose rotation constants $(\nu_1, \nu_2, \nu_3, \nu_4)$ are $(1, \nu, N-\nu, N-1)$, then P_i $(i=1, 2, 3, 4)$ is q-Weierstrass point $(q \geqq 2)$ except possibly for the following case; N is odd and $q \equiv N+1/2 \pmod N$.*

PROOF. Assume that P_i $(i=1, 2, 3, 4)$ is not q-Weierstrass point, then from Lemma A, $tr(T)=(\varepsilon^q-\varepsilon^{-3q+2})/(1-\varepsilon)$, while from Lemma B, $tr(T)=\varepsilon^q/(1-\varepsilon)+\varepsilon^{\nu q}/(1-\varepsilon^\nu)+\varepsilon^{(N-\nu)q}/(1-\varepsilon^{(N-\nu)})+\varepsilon^{(N-1)q}/(1-\varepsilon^{(N-1)})$. Hence we have

$$\varepsilon^{-3q+2}/(1-\varepsilon)+\varepsilon^{\nu q}/(1-\varepsilon^\nu)+\varepsilon^{(N-\nu)q}/(1-\varepsilon^{(N-\nu)})+{}^{(N-1)q}/(1-\varepsilon^{(N-1)})=0\,.$$

Then we can obtain the desired conclusion (c. f., Appendix 4). ⟨q. e. d.⟩

THEOREM 6. *If $T \in Aut(M)$ is of even order and has two fixed points P_1, P_2 where the rotation constants (ν_1, ν_2) are $(1, 1)$, then P_1, P_2 are q-Weierstrass points except possibly for*

$$2q \equiv 1 \pmod{N/2}\,.$$

PROOF. Assume P_1 (resp. P_2) is not q-Weierstrass point, then from Lemma A and Lemma B, we have $\varepsilon^{4q}=\varepsilon^2$. Hence $2q \equiv 1 \pmod{N/2}$. ⟨q. e. d.⟩

2. We give the examples of the above theorems.

EXAMPLE OF THEOREM 4.

We will now consider the Riemann surfaces of the algebraic function

$$y^N=x(x-1) \qquad (N\text{: odd})\,.$$

This is the surface of genus $g=N-1/2$ and has a representation as N-sheeted cover of the sphere with 3 ramification points P_0, P_1, P_∞ over 0, 1, ∞ respectively. Also P_0, P_1, P_∞ are the fixed points of the automorphism $T:(x, y)\rightarrow(x, \varepsilon y)$.

A basis for the complex vector space H^1 of holomorphic differentials is as follows:

$$\frac{dx}{y^{N-1}},\ \frac{dx}{y^{N-2}},\ \cdots,\ \frac{dx}{y^{(N-1)/2}}.$$

The 1-gap sequence at P_1, P_2 is 1, 2, $\cdots$, g and 1-gap sequence at P_∞ is 1, 3, $\cdots$, $2g-1$, so the surface is hyperelliptic.

Now we can compute the q-gap sequences and q-weights at P_1, P_2, P_∞ for all $q\geqq 2$ constructing a basis for the complex vector space H^q of holomorphic q-differentials inductively. We mention the results as follows; Let $[\![a, b]\!]$ be the set of integers n such that $a\leqq n\leqq b$.

<q-gap sequences and q-weights of P_1, P_2*>*

(I) $N=6n+1$ (*i.e.*, $N\equiv 1\ (\mathrm{mod}\ 6)$)

i) If $q\equiv 2k-1\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $(3k-2)(3n-3k+1)$ and $[\![1, 2q(3n-1)-6n+3k-2]\!][\![2q(3n-1)-6n+6k-3, 2q(3n-1)-3n+3k-1]\!]$.

ii) If $q\equiv 2k\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $(3k-1)(3n-3k+1)$ and $[\![1, 2q(3n-1)-6n+3k]\!][\![2q(3n-1)-6n+6k, 2q(3n-1)-3n+3k]\!]$.

iii) If $q\equiv 2n+2k-1\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $9(k-1)(n-k+1)$ and $[\![1, 2q(3n-1)-6n+3k-2]\!][\![2q(3n-1)-6n+6k-4, 2q(3n-1)-3n+3k-2]\!]$.

iv) If $q\equiv 2n+2k\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $(3k-1)(3n-3k+2)$ and $[\![1, 2q(3n-1)-6n+3k-2]\!][\![2q(3n-1)-6n+6k-1, 2q(3n-1)-3n+3k]\!]$.

v) If $q\equiv 4n+2k-1\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $3(k-1)(3n-3k+4)$ and $[\![1, 2q(3n-1)-6n+3k-3]\!][\![2q(3n-1)-6n+6k-5, 2q(3n-1)-3n+3k-2]\!]$.

vi) If $q\equiv 4n+2k\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $(3k-2)(3n-3k+2)$ and $[\![1, 2q(3n-1)-6n+3k-1]\!][\![2q(3n-1)-6n+6k-2, 2q(3n-1)-3n+3k-1]\!]$.

(II) $N=6n-1$ (*i.e.*, $N\equiv 5\ (\mathrm{mod}\ 6)$)

i) If $q\equiv 2k-1\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $(3k-2)(3n-3k+2)$ and $[\![1, 2q(3n-2)-6n+3k]\!][\![2q(3n-2)-6n+6k-1, 2q(3n-2)-3n+3k]\!]$.

ii) If $q\equiv 2k\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $3(3k-1)(n-k)$ and $[\![1, 2q(3n-2)-6n+3k+2]\!][\![2q(3n-2)-6n+6k+2, 2q(3n-2)-3n+3k+1]\!]$.

iii) If $q\equiv 2n+2k-1\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $(3k-2)(3n-3k+1)$ and $[\![1, 2q(3n-2)-6n+3k+1]\!][\![2q(3n-2)-6n+6k, 2q(3n-2)-3n+3k]\!]$.

iv) If $q\equiv 2n+2k\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $9k(n-k)$ and $[\![1, 2q(3n-2)-6n+3k+2]\!][\![2q(3n-2)-6n+6k+3, 2q(3n-2)-3n+3k+2]\!]$.

v) If $q\equiv 4n+2k-1\ (\mathrm{mod}\ N)$ $(1\leqq k\leqq n)$, then q-weight $(3k-1)(3n-3k+1)$

and $[\![1,\ 2q(3n-2)-6n+3k+1]\!][\![2q(3n-2)-6n+6k+1,\ 2q(3n-2)-3n+3k+1]\!]$.

vi) If $q\equiv 4n+2k \pmod N$ $(1\leqq k\leqq n)$, then q-weight $3k(3n-3k-1)$ and $[\![1,\ 2q(3n-2)-6n+3k+1]\!][\![2q(3n-2)-6n+6k+1,\ 2q(3n-2)-3n+3k+1]\!]$.

(III) $N=6n+3$ (*i.e.*, $N\equiv 3 \pmod 6$)

i) If $q\equiv 2k-1 \pmod{N/3}$ $(1\leqq k\leqq n+1)$, then q-weight $(3k-2)(3n-3k+4)$ and $[\![1,\ 6qn-6n+3k-4]\!][\![6qn-6n+6k-5,\ 6qn-3n+3k-2]\!]$.

ii) If $q\equiv 2k \pmod{N/3}$ $(1\leqq k\leqq n)$, then q-weight $(3k-1)(3n-3k+2)$ and $[\![1,\ 6qn-6n+3k-2]\!][\![6qn-6n+6k-2,\ 6qn-3n+3k-1]\!]$.

<*q-gap sequences and q-weight of* P_∞>

$$q\text{-weight } \frac{g(g+1)}{2}$$

and $[\![1,\ 2q(g-1)-2g+1]\!]$, $2q(g-1)-2g+3$, $2q(g-1)-2g+5$, $\cdots$, $2q(g-1)+1$.

EXAMPLE OF THEOREM 5.

We will now consider the Riemann surfaces of the algebraic function

$$y^N=x(x-1)(x-\alpha)^{N-1}.$$

This is the surface of genus $g=N-1$ and has a representation as an N-sheeted cover of the sphere with the four ramification points P_0, P_1, P_α, P_∞ over 0, 1, α, ∞ respectively. Also these points are fixed points of the automorphism $T:(x,\ y)\to(x,\ \varepsilon y)$.

A basis for the complex vector space H^1 of holomorphic differentials is as follows:

$$\frac{(x-\alpha)^{N-2}}{y^{N-1}}dx,\ \frac{(x-\alpha)^{N-3}}{y^{N-2}}dx,\ \cdots,\ \frac{(x-\alpha)dx}{y^2},\ \frac{dx}{y}.$$

The 1-gap sequence at P_0, P_1, P_α, P_∞ is $1, 2, \cdots, g$ and we can easily show that this surface is hyperelliptic. We can compute the q-gap sequences and q-weights at these points by the same way as Example of Theorem 4. We mention the results as follows;

<*q-gap sequences and q-weights of* P_0, P_1, P_α, P_∞>

(I) N: odd

i) If $q\equiv k \pmod N$ $(1\leqq k\leqq (N-1)/2)$, then q-weight $(g-2k+2)(2k-1)$ and $[\![1,\ (2q-1)(g-1)-N+2k-1]\!][\![(2q-1)(g-1)-N+4k-1,\ (2q-1)(g-1)+2k-1]\!]$.

ii) If $q\equiv (N+1)/2 \pmod N$, then q-weight 0 and $[\![1,\ (2q-1)(g-1)]\!]$.

iii) If $q\equiv (N+1)/2+k \pmod N$ $(1\leqq k\leqq (N-1)/2)$, then q-weight $2(g-2k+1)k$ and $[\![1,\ (2q-1)(g-1)-N+2k]\!][\![(2q-1)(g-1)-N+4k+1,\ (2q-1)(g-1)+2k]\!]$.

(II) N: even

If $q \equiv k \pmod{N/2}$ $(1 \leqq k \leqq N/2)$, then q-weight $(g-2k+2)(2k-1)$ and $[\![1, (2q-1)(g-1)-N+2k-1]\!][\![(2q-1)(g-1)-N+4k-1, (2q-1)(g-1)+2k-1]\!]$.

EXAMPLE OF THEOREM 6.

We consider the Riemann surface of the algebraic function

$$y^N = x(x-1) \qquad (N\text{: even})$$

This is the same equation of Example of Theorem 4 except for the difference between even or odd. This surface is of genus $N/2-1$ and is hyperelliptic. But this case can be reduced in the case of example of theorem 5. Indeed, as we noticed in Remark 1, we can see that the analytic $N/2$ cover $M \to M/\langle T^2 \rangle$ has four ramification points P_0, P_1, P_∞^1, P_∞^2 whose rotation constants are $(\nu_1, \nu_2, \nu_3, \nu_4) = (1, 1, N/2-1, N/2-1)$. So we can see that the q-gap sequences and q-weights at P_0, P_1, P_∞^1, P_∞^2 is as follows;

< q-gap sequences and q-weights of P_0, P_1, P_∞^1, P_∞^2>

(I) $N/2$: odd

i) If $q \equiv k \pmod{N/2}$ $(1 \leqq k \leqq (N/2-1)/2)$, then q-weight $(q-2k+2)(2k-1)$ and $[\![1, (2q-1)(g-1)-N/2+2k-1]\!][\![(2q-1)(g-1)-N/2+4k-1, (2q-1)(g-1)+2k-1]\!]$.

ii) If $q \equiv (N/2+1)/2 \pmod{N/2}$, then q-weight 0 and $[\![1, (2q-1)(g-1)]\!]$.

iii) If $q \equiv (N/2+1)/2+k \pmod{N/2}$ $(1 \leqq k \leqq (N/2-1)/2)$, then q-weight $2(g-2k+1)k$ and $[\![1, (2q-1)(g-1)-N/2+2k]\!][\![(2q-1)(g-1)-N/2+4k+1, (2q-1)(g-1)+2k]\!]$.

(II) $N/2$: even

If $q \equiv k \pmod{N/4}$ $(1 \leqq k \leqq N/4)$, then q-weight $(g-2k+2)(2k-1)$ and $[\![1, (2q-1)(g-1)-N/2+2k-1]\!][\![(2q-1)(g-1)-N/2+4k-1, (2q-1)(g-1)+2k-1]\!]$.

§ 4. **Differentials.** (Dimension n_k^q of H_k^q)

We can obtain $n_k^q = \dim H_k^q$ in each case of Theorem 1, 2 and 3. Methods of computation are owed to J. Lewittes [7], [8] essentially and Lewittes and Duma obtained inequalities on n_k^q. By the same way we can compute n_k^q explicitly, because in our special cases we have known the rotation constants exactly.

We mention the tables of n_k^q in each case of Theorem 1.

Here we put $q = Nd + r$ $(0 \leqq r \leqq N-1)$ and we denote $n_k^q = n_k$ for simplicity's sake.

TABLE 1. (Case $q=1$)

I) $t=1$, $g=6g'+1$, $N=6$. Then, $n_k=g'$ $(k\neq 1)$, $n_1=g'+1$

II) $t=2$, $g=Ng'$. Then, $n_k=g'$ for all k $(0\leqq k\leqq N-1)$

III) $t=2$, $g=Ng'+(N/2-1)$. Then,

$n_0=g'$, $n_k=g'+1$ $(1\leqq k\leqq N/2-1)$, $n_k=g'$ $(N/2\leqq k\leqq N-1)$.

IV) $t=3$, $g=Ng'+(N-1)/2$. Then,

$n_0=g'$, $n_k=g'+1$ $(1\leqq k\leqq (N-1)/2)$, $n_k=g'$ $((N+1)/2\leqq k\leqq N-1)$.

V) $t=4$, $g=Ng'+(N-1)$. Then, $n_0=g'$, $n_k=g'+1$ $(1\leqq k\leqq N-1)$.

TABLE 2. (Case $q\geqq 2$)

I) $t=1$, $g=6g'+1$, $N=6$. Then,

If $r=0$, $n_0=(2q-1)g'+1$, $n_k=(2q-1)g'$ $(1\leqq k\leqq 4)$, $n_5=(2q-1)g'-1$.

If $r\geqq 1$, $n_r=(2q-1)g'+1$, $n_{r-1}=(2q-1)g'-1$, $n_k=(2q-1)g'$ $(k\neq r,\ r-1)$.

II) $t=2$, $g=Ng'$, rotation constants $(1,\ N-1)$. Then,

If $r=0$, $n_0=(2q-1)g'-2d+1$, $n_k=(2q-1)g'-2d$ $(1\leqq k\leqq N-1)$.

If $r\geqq 1$, $n_k=(2q-1)g'-2d-2$ $(k<r,\ N-k<r)$,

$n_k=(2q-1)g'-2d$ $(k\geqq r,\ N-k\geqq r)$

$n_k=(2q-1)g'-2d-1$ $(k<r,\ N-k\geqq r$ or $k\geqq r,\ N-k<r)$.

III) $t=2$, $g=Ng'+(N/2-1)$, rotation constants $(1,\ 1)$. Then,

If $r=0$, $n_0=(2q-1)g'+q-4d+1$, $n_k=(2q-1)g'+q-4d$ $(1\leqq k\leqq N/2)$,

$n_k=(2q-1)g'+q-4d-1$ $(N/2+1\leqq k\leqq N-1)$.

If $1\leqq r\leqq N/2$,

$n_k=(2q-1)g'+q-4d$ $(k\geqq r,\ k+r\leqq N/2)$,

$n_k=(2q-1)g'+q-4d-1$ $(k\geqq r,\ N/2<k+r\leqq N)$,

$n_k=(2q-1)g'+q-4d-2$ $(k<r,\ k+r\leqq N/2$ or $k\geqq r,\ N<k+r)$,

$n_k=(2q-1)g'+q-4d-3$ $(k<r,\ N/2<k+r\leqq N)$.

If $N/2<r\leqq N-1$,

$n_k=(2q-1)g'+q-4d-2$ $(k\geqq r,\ k+r\leqq 3N/2)$,

$n_k=(2q-1)g'+q-4d-3$ $(k<r,\ k+r\leqq N$ or $k\geqq r,\ 3N/2<k+r)$,

$n_k=(2q-1)g'+q-4d-4$ $(k<r,\ N<k+r\leqq 3N/2)$,

$n_k=(2q-1)g'+q-4d-5$ $(k<r,\ 3N/2<k+r)$.

IV) $t=3$, $g=Ng'+(N-1)/2$, rotation constants $(1,\ 1,\ (N-1)/2)$. Then,

If $r=0$, $n_0=(2q-1)g'+q-3d+1$, $n_k=(2q-1)g'+q-3d$ $(1\leqq k\leqq (N-1)/2)$,

$n_k=(2q-1)g'+q-3d-1$ $((N+1)/2\leqq k\leqq N-1)$.

If $r\geqq 1$, $n_k=(2q-1)g'+q-3d$ $(k\leqq (N-1)/2,\ k\geqq r,\ N-2k\geqq r)$,

$n_k=(2q-1)g'+q-3d-1$ $(k\leqq (N-1)/2,\ N-2k<r$ or

$k\geqq (N+1)/2,\ k\geqq r,\ 2N-2k\geqq r)$,

$n_k=(2q-1)g'+q-3d-2$ $(k\leqq(N-1)/2,\ k<r,\ N-2k\geqq r$ or
$k\geqq(N+1)/2,\ k\geqq r,\ 2N-2k<r)$,
$n_k=(2q-1)g'+q-3d-3$ $(k\leqq(N-1)/2,\ k<r,\ N-2k<r$ or
$k\geqq(N+1)/2,\ k<r,\ 2N-2k\geqq r)$,
$n_k=(2q-1)g'+q-3d-4$ $(k\geqq(N+1)/2,\ k<r,\ 2N-2k<r)$.

V) $t=4$, $g=Ng'+(N-1)$, rotation constants $(1, \nu, N-\nu, N-1)$.

Let λ_k, μ_k be integers such that $\lambda_k\cdot\nu\equiv k \pmod N$, $\lambda_k+\mu_k=N$, $1\leqq\lambda_k$, $\mu_k\leqq N-1$. Then,

If $r=0$, $n_0=(2q-1)g'+2q-4d+1$, $n_k=(2q-1)g'+2q-4d-1$ $(1\leqq k\leqq N-1)$.

If $r\geqq 0$, $n_0=(2q-1)g'+2q-4d-3$, $n_k=(2q-1)g'+2q-4d-1$ (1),
$n_k=(2q-1)g'+2q-4d-2$ (2), $n_k=(2q-1)g'+2q-4d-3$ (3),
$n_k=(2q-1)g'+2q-4d-4$ (4), $n_k=(2q-1)g'+2q-4d-5$ (5),

where, case (1) means that all of four integers k, $N-k$, λ_k, μ_k are smaller than r, case (2) means that only one of them is lager than or equal to r, case (3) means that just two of them are smaller than r, case (4) means that only one of them is smaller than r, and case (5) means that uone of them is smaller than r.

We have the tables in other cases of Theorem 2 and 3 but we will omit those on account of their length.

EXAMPLE OF CALCULATION OF n_k^q.

We give an example of calculation of n_k^q in the case Table 2 I), namely, $g=6g'+1$. There are five branch points except P_1 which are the fixed points of T^2 and T^3. We denote the fixed points of T^2 by Q_1, Q_2 and the fixed points of T^3 by R_1, R_2, R_3. We denote $\pi(P_1)=\tilde{P}$, $\pi(Q_1)=\pi(Q_2)=\tilde{Q}$, $\pi(R_1)=\pi(R_2)=\pi(R_3)=\tilde{R}$. Considering the projection $M\to M/\langle T^2\rangle$, we see that M is covering surface over $M/\langle T^2\rangle$ with three fixed points P_1, Q_1, Q_2 and M has no branch points except P_1, Q_1, Q_2. So the rotation constants with respect to T^2 are (1, 1, 1) from Theorem 1 (d), i.e., $(T^2)^{-1}: Z\to\varepsilon^2 Z$ at P_1, Q_1, Q_2. Now we may assume $n_k\neq 0$ for some k, namely there is a holomorphic q-differential θ such that $T(\theta)=\varepsilon^k\theta$. If $U\in M$, not a branch point, is a zero of θ of order u, then each of the points $T^\alpha(U)$ $(1\leqq\alpha\leqq 5)$ is a zero of order u. Thus the divisor of θ has the form

$$\operatorname{div}(\theta)=P_1^{m_1}(Q_1Q_2)^{m_2}(R_1R_2R_3)^{m_3}\prod_{j=1}^{s}\Big(\prod_{\alpha=0}^{5}T^\alpha(U_j)\Big)^{u_j},\qquad m_j\geqq 0,\ u_j\geqq 0,$$

where

$$m_1+2m_2+3m_3+6(\sum_{j=1}^{s} u_j)=2q(g-1).$$

Note that u_j may be arbitrary but not m_j. In fact $T^{-1}: z\rightarrow \varepsilon z$ at P_1 locally and θ is represented by $\theta=(a_0+a_1z+\cdots+a_nz^n+\cdots)dz^q$ locally, so the condition $T(\theta)=\varepsilon^k\theta$ says that $\varepsilon^{n+q}a_n=\varepsilon^k a_n$, therefore $a_n=0$ for n not satisfying $n+q\equiv k$ (mod 6). In particular the first non zero coefficient has index of the form $6h_1+k-q$, $h_1\geqq 0$ an integer. Similarly $(T^2)^{-1}: z\rightarrow\varepsilon^2 z$ at Q_1 (resp. Q_2) and $\theta=(a_0+a_1z+\cdots+a_nz^n+\cdots)dz^q$ locally so the condition $T^2(\theta)=\varepsilon^{2k}\theta$ says that $\varepsilon^{2(n+q)}a_n=\varepsilon^{2k}a_n$, therefore $a_n=0$ for n not satisfying $n+q\equiv k$ (mod 3). In particular the first non zero coefficient has index of the form $3h_2+k-q$, $h_2\geqq 0$ an integer. Similarly $(T^3)^{-1}: z\rightarrow\varepsilon^3 z$ at R_1 (resp. R_2 and R_3) locally so the condition $T^3(\theta)=\varepsilon^{3k}\theta$ says that the first non zero coefficient has index of the form $2h_3+k-q$, $h_3\geqq 0$ an integer. Therefore the divisor of θ is denoted by

$$\operatorname{div}(\theta)=P_1^{6h_1+k-q}(Q_1Q_2)^{3h_2+k-q}(R_1R_2R_3)^{2h_3+k-q}\prod_{j=1}^{s}(\prod_{\alpha=0}^{5}T^\alpha(U_j))^{u_j}.$$

If $\theta^*\in H_k^q$ is another q-differential of which divisor is

$$\operatorname{div}(\theta^*)=P_1^{6h_1^*+k-q}(Q_1Q_2)^{qh_2^*+k-q}(R_1R_2R_3)^{2h_3^*+k-q}\prod_{j=1}^{s^*}(\prod_{\alpha=0}^{5}T^\alpha(U_j^*))^{u_j^*},$$

then $\theta^*/\theta=f$ is an T-invariant meromorphic function on M, for $T(f)=T(\theta^*)/T(\theta)=\varepsilon^k\theta^*/\varepsilon^k\theta=f$. The divisor of f is

$$\operatorname{div}(f)=P_1^{6(h_1^*-h_2)}(Q_1Q_2)^{3(h_2^*-h_2)}(R_1R_2R_3)^{2(h_3^*-h_3)}$$
$$\times\prod_{j=1}^{s^*}(\prod_{\alpha=0}^{5}T^\alpha(U_j^*))^{u_j^*}\prod_{j=1}^{s}(\prod_{\alpha=0}^{5}T^\alpha(U_j))^{-u_j}.$$

Then $\tilde{f}=\pi(f)$ becomes a meromorphic function on $M/\langle T\rangle$ of which divisor is

$$\operatorname{div}(\tilde{f})=\tilde{P}_1^{h_1^*-h_1}\tilde{Q}^{h_2^*-h_2}\tilde{R}^{h_3^*-h_3}\prod_{j=1}^{s^*}\tilde{U}^{*u_j^*}\prod_{j=1}^{s}\tilde{U}^{-u_j}.$$

Here we consider the following 36 cases; $(r, k)=(0, 0), \cdots, (5, 5)$: If $r=0$, $k=0$, then $m_1=6h_1-q=6(h_1-d)$, $m_2=3(h_2-2d)$, $m_2=2(h_3-3d)$ and $h_1\geqq d$, $h_2\geqq 2d$, $h_3\geqq 3d$ because $m_j\geqq 0$. Let $\tilde{b}_0$ be negative divisor on $M/\langle T\rangle$ as follows, $\tilde{b}_0=\tilde{P}_1^{d-h_1}\tilde{Q}^{2d-h_2}\tilde{R}^{3d-h_3}\prod_{j=1}^{s}\tilde{U}^{-u_j}$. Then any $\theta^*\in H_0^q$ induces a meromorphic function $\tilde{f}=\pi(\theta^*/\theta)$ and $\operatorname{div}(f)$ is a multiple of $\tilde{b}_0$. On the otherhand any meromorphic function $\tilde{f}$ of which divisor is a multiple of $\tilde{b}_0$ induces a meromorphic function $f=\pi^{-1}(\tilde{f})$ such .hat $f\theta=\theta^*\in H_0^q$. Hence n_0^q is the dimension of the space of meromorphic function of which divisor is a multiple of $\tilde{b}_0$, namely $n_0^q=r(\tilde{b}_0)$. We are now in a position to apply the Riemann-Roch theorem. Now,

$$\deg(\tilde{b}_0^{-1}) = h_1 + h_2 + h_3 - 6d + \sum u_j,$$

while,

$$2q(g-1) = m_1 + 2m_2 + 3m_3 + 6(\sum u_j) = 6(h_1 - d) + 6(h_2 - 2d) + 6(h_3 - 3d) + 6(\sum u_j),$$

so $\deg(\tilde{b}_0^{-1}) = 2qg' > 2g' - 2$. Hence $i(\tilde{b}_0^{-1}) = 0$. Using the Riemann-Roch theorem,

$$n_0^q = r(\tilde{b}_0) = \deg(\tilde{b}_0^{-1}) + 1 - g' + i(\tilde{b}_0^{-1}) = (2q-1)g' + 1.$$

In the other (r, k) cases we can compute n_k^q similarly.

In all other cases in the above tables we can compute n_k^q by the same routine work.

§ 5. Miscellaneous results.

1. We consider the case $t=2$. Theorem 1 asserts that, if P_1 is not a 1-Weierstrass point, we have $g = Ng'$ and the rotation constants are $(1, N-1)$. Now conversely we start the following condition; $g = N$, $g' = 1$ and the rotation constants $(1, N-1)$, i.e., we don't assume that P_1 is not a 1-Weierstrass point. Then we have the following theorem.

THEOREM 7. *We assume that $t=2$, $g=N$, $g'=1$ and the rotation constants $(1, N-1)$. Then the two fixed points P_1 and P_2 are not 1-Weierstrass points or 1-Weierstrass points with the same gap sequence of which weight are a multiple of N.*

PROOF. From Table 1, we have $n_0^1 = n_1^1 = \cdots = n_{g-1}^1 = 1$. Therefore there is a basis $\{\theta_k\}$ of H^1 such that $T(\theta_k) = \varepsilon^k \theta_k$. The condition $T(\theta_0) = \theta_0$ says that the order of the zeros of θ_0 at P_j is $h_j g + g - 1$ $(j=1, 2)$. The condition $T(\theta_k) = \varepsilon^k \theta_k$ $(k \geqq 1)$ says that the order of the zeros at P_1 (resp. P_2) is $h_1 g + k - 1$ (resp. $h_2 g + g - k - 1$). Since total orders are $2g-2$, we observe that $\operatorname{div}(\theta_k) = P_1^{k-1} P_2^{g-k-1} \prod_{\alpha=0}^{g-1} T^\alpha(Q_k)$ or $\operatorname{div}(\theta_k) = P_1^{g+k-1} P_2^{g-k-1}$ or $\operatorname{div}(\theta_k) = P_1^{k-1} P_2^{2g-k-1}$.

If there is a θ_k such that $\operatorname{div}(\theta_k) = P_1^{g+k-1} P_2^{g-k-1}$, then we have $\operatorname{div}(\theta_k/\theta_0) = P_1^k P_2^{-k}$, therefore θ_k/θ_0 (resp. θ_0/θ_k) is a meromorphic function of which poles are P_2^k (resp. P_1^k) so k is a non gap of P_1 and P_2.

If there is a θ_k such that $\operatorname{div}(\theta_k) = P_1^{k-1} P_2^{2g-k-1}$, then we have $\operatorname{div}(\theta_k/\theta_0) = P_1^{-g+k} P_2^{g-k}$, and so $g-k$ is a non gap of P_1 and P_2.

If $\operatorname{div}(\theta_k) = P_1^{k-1} P_2^{g-k-1} \prod_{\alpha=0}^{g-1} T^\alpha(O_k)$ $(k=1, \cdots, g-1)$ and $Q_k = Q_m$ $(k \neq m)$ for cirtain k and m, then $\operatorname{div}(\theta_k/\theta_m)=P_1^{k-m}P_2^{m-k}$ so $k-m$ is a non gap of P_1 and P_2. We put $\theta'=(\theta_k/\theta_m)\theta_0$, then $\operatorname{div}(\theta')=P_1^{g+(k-m)-1}P_2^{g-(k-m)-1}$.

Therefore if P_1 is not 1-Weierstrass point, the divisor of θ_k must be $\operatorname{div}(\theta_k)=P_1^{k-1}P_2^{g-k-1}\prod_{\alpha=0}^{g-1}T^{\alpha}(Q_k)$ for any k and Q_k are different from each other. Also non gap k $(1\leqq k\leqq g-1)$ exists if and only if there exists θ_k such that $\operatorname{div}(\theta_k)=P_1^{g+k-1}P_2^{g-k-1}$ or $\operatorname{div}(\theta_k)=P_1^{k-1}P_2^{2g-k-1}$. Moreover it is clear that P_1 and P_2 have the same gap sequence and the weight is a multiple of $N=g$ from the form of the divisor of θ_k. ⟨q. e. d.⟩

For example, if $g=N=3$, the gap sequence of P_j is $\{1, 2, 3\}$ or $\{1, 3, 5\}$, if $g=N=4$, the gap sequence is $\{1, 2, 3, 4\}$ or $\{1, 2, 4, 7\}$ and if $g=N=5$, the gap sequence is $\{1 ‘2, 3, 4, 5\}$ or $\{1, 2, 3, 5, 9\}$ or $\{1, 2, 4, 5, 8\}$ or $\{1, 3, 5, 7, 9\}$ etc.

2. We consider the case $t=3$. Theorem 1 asserts that if P_1 is not a 1-Weierstrass point we have $g=Ng,+(N-1)/2$ and the rotation constants are $(1, 1, (N-1)/2)$. Conversely we obtain the following:

THEOREM 8. *We assume that* $t=3$, $g=Ng'+(N-1)/2$ *and the rotation constants are* $(1, 1, (N-1)/2)$.

(i) *If* $g'=0$, M *is hyperelliptic.*

(ii) *If* M *is hyperelliptic,* P_1, P_2 *are not* 1-*Weierstrass points and* P_3 *is a* 1-*Weierstrass point and* $J(P_1)=P_2$, *where* J *is the hyperelliptic involution.*

PROOF. (i) If $g'=0$, then $g=(N-1)/2$. From Table 1, we have $n_0^1=0$, $n_k^1=1$ $(1\leqq k\leqq(N-1)/2=g)$. Therefore there is a basis $\{\theta_k\}$ of H^1 such that $T(\theta_k)=\varepsilon^k\theta_k$. Considering the rotation constants, the condition $T(\theta_k)=\varepsilon^k\theta_k$ says that the divisor of θ_k is $\operatorname{div}(\theta_k)=P_1^{k-1}P_2^{k-1}P_3^{2g-2k}$. Hence the gap sequence of P_1 and P_2 is $1, 2, \cdots, g$ and the gap sequence of P_3 is $1, 3, \cdots, 2g-1$, so M is hyperelliptic.

(ii) If M is hyperelliptic, the number of 1-Weierstrass points is $2g+2=(2g'+1)/N+1$. Hence one of the points P_1, P_2, P_3 is 1-Weierstrass point. If P_1 (resp. P_2) is a 1-Weierstrass point, then according to Theorem 3, the rotation constants are $(1, N-2, N-2)$, so the rotation constants of P_2 (resp. P_1) and P_3 must be the the same. It contradicts. So P_3 must be a 1-Weierstrass point. Since J is in center of $Aut(M)$, $T(J(P_j))=J(T(P_j)=J(P_j)$ and $J(P_j)$ is also a fixed point, hence $J(P_1)=P_2$. ⟨q. e. d.⟩

Lewittes [7] showed above theorem in the case that N is prime.

3. We consider the case $t=4$. Theorem 1 asserts that if P_1 is not a 1-Weierstrass point we have $g=Ng'+N-1$ and the rotation constants are $(1, \nu, N-\nu, N-1)$. Conversely if the rotation constants are $(1, \nu, N-\nu, N-1)$, is P_j $(j=1, \cdots, 4)$ not a 1-Weierstrass point? It is not correct in general, however if $g'=0$ we have the following:

THEOREM 9. *We assume that* $t=4$, $g=N-1$, $g'=0$. *Then we have*

(i) P_1, P_2, P_3 *and* P_4 *are not* 1-*Weierstrass points if and only if the rotation constants are* $(1, \nu, N-\nu, N-1)$,

(ii) *moreover,* M *is hyperelliptic if and only if* $\nu=1$, *namley* $(1, 1, N-1, N-1)$.

PROOF. (i) If P_1 is not a 1-Weierstrass point, from Theorem 1 the rotation constants are $(1, \nu, N-\nu, N-1)$. Since $g'=0$, from Table 1, $n_0^1=0$, $n_k^1=1$ $(k=1, \cdots, g)$. Hence there is a basis $\{\theta_k\}$ of H^1 such that $T(\theta_k)=\varepsilon^k\theta_k$ $(k=1, \cdots, g)$. Considering the rotation constants, we have that the divisor of θ_k is $\mathrm{div}(\theta_k)=P_1^{k-1}P_2^{\lambda_k-1}P_3^{N-\lambda_k-1}P_4^{N-k-1}$, where $\lambda_k\cdot\nu\equiv 1 \pmod N$, $1\leqq\lambda_k\leqq N-1$. Hence the gap sequence of all P_j is $1, \cdots, g$ and so all of P_j are not 1-Weierstrass points. Conversely if the rotation constants are $(1, \nu, N-\nu, N-1)$, then the form of the divisor θ_k is as above, therefore it is clear that all of P_j are not 1-Weierstrass points.

(ii) If the rotation constants are $(1, 1, N-1, N-1)$, then there is a as is $\{\theta_k\}$ with $\mathrm{div}(\theta_k)=(P_1P_2)^{k-1}(P_3P_4)^{N-k-1}$. Hence $\mathrm{div}(\theta_k/\theta_{k-1})=(P_1P_2)(P_3P_4)^{-1}$, therefore θ_k/θ_{k-1} is a meromorphic function of order two, and so M is hyperelliptic.

If $2\leqq\nu<N-\nu\leqq N-2$, we consider $\theta_k\theta_l$ $(1\leqq k, l\leqq N-1)$ as above. In this case the divisor of $\theta_k\theta_l$ is

$$\mathrm{div}(\theta_k\theta_l)=P_1^{k+l-2}P_2^{\lambda_k+\lambda_l-2}P_3^{2N-\lambda_k-\lambda_l-2}P_4^{2N-k-l-2}.$$

Then there are k_1, k_2, l_1 and l_2 such that $k_1+l_1=k_2+l_2$, $\lambda_{k_1}+\lambda_{l_1}\neq\lambda_{k_2}+\lambda_{l_2}$ since $2\leqq\nu<N-\nu\leqq N-2$, $(\nu, N)=1$. Therefore the number of independent $\theta_k\theta_l$ is is larger than $2g-1$, and so M is not hyperelliptic. ⟨q.e.d⟩

§6. Examples of the construction of Riemann surfaces.

We consider the case $t=2$, $g=Ng'$, rotation constants $(1, N-1)$.

EXAMPLE 1. Assume $g'=1$, $N=3$, namely $g=3$.

From Table 1 of §5, there is a basis θ_0, θ_1, θ_2 of H^1 such that $T(\theta_k)=\varepsilon^k\theta_k$ ($k=0, 1, 2$), $\varepsilon=\exp(2\pi i/3)$. Then we obtain the following 15 differentials of H^4 as products of θ_k.

$$\theta_0^4,\quad \theta_1^3\theta_0,\quad \theta_2^3\theta_0,\quad \theta_0^2\theta_1\theta_2,\quad \theta_1^2\theta_2^2\in H_0^4$$

$$\theta_1^4,\quad \theta_0^3\theta_1,\quad \theta_2^3\theta_1,\quad \theta_1^2\theta_0\theta_2,\quad \theta_0^2\theta_2^2\in H_1^4$$

$$\theta_2^4,\quad \theta_0^3\theta_2,\quad \theta_1^3\theta_2,\quad \theta_2^2\theta_0\theta_1,\quad \theta_0^2\theta_1^2\in H_2^4 .$$

From Table 2, $n_0^4=4$, $n_1^4=5$, $n_2^4=5$, so there is a linear relation in the first group.

$$a_0\theta_0^4+a_1\theta_0^2\theta_1\theta_2+a_2\theta_0\theta_1^3+a_3\theta_0\theta_2^3+a_4\theta_1^2\theta_2^2=0 .$$

We put $\theta_1/\theta_0=x$, $\theta_2/\theta_0=y$. Then

$$a_0+a_1xy+a_2x^3+a_3y^3+a_4x^2y^2=0 .$$

If $a_2=0$. Then $a_0+a_1xy+a_3y^3+a_4x^2y^2=0$. We take birational transformation $y=Y$, $xy=X$, then $a_0+a_1X+a_3Y^3+a_4X^2=0$. This Rieman surface is of genus $g\leqq 1$. It contradicts. So we have $a_2\neq 0$. Similarly we have $a_3\neq 0$. Also, if $a_0=0$, then $a_1xy+a_2x^3+a_3y^3+a_4x^2y^2=0$. But this Riemann surface is of genus $g\leqq 2$. So we have $a_0\neq 0$. Consequentry we can normalize;

$$M:\ x^3+y^3+ax^2y^2+bxy+1=0 .$$

This Riemann has the group of automorphisms $Aut(M)$ whose order six in general. Its generators are

$$T:(x, y)\longrightarrow(\varepsilon x, \varepsilon^2 y)\quad\text{and}\quad S:(x, y)\longrightarrow(y, x).$$

The fixed points of T are P_1 and P_2 which lie over $x=\infty$. And we can easily see that these points are not 1-Weierstrass points, for P_1 and P_2 correspond to (0, 1, 0) and (1, 0, 0) provided that they are considered on homogeneous coordinate and the Hessian of $X^3Z+Y^3Z+aX^2Y^2+bXYZ^2+Z^4$ is not zero at (0, 1, 0) and (1, 0, 0).

EXAMPLE 2. Assume $g'=1$, $N=4$. Namely $g=4$. From Table 1, there is a basis $\{\theta_k\}$ of H^1 such that $T(\theta_k)=\varepsilon^k\theta_k$, ($k=0,1,2,3$) $\varepsilon=\exp(2\pi i/4)=i$. Then we obtain the following 10 differentials of H^2 as product θ_k.

$$\theta_0^2,\quad \theta_1\theta_3,\quad \theta_2^2\in H_0^2\qquad \theta_0\theta_1,\quad \theta_2\theta_3\in H_1^2$$

$$\theta_0\theta_2,\quad \theta_1^2,\quad \theta_3^2\in H_2^2\qquad \theta_0\theta_3,\quad \theta_1\theta_2\in H_3^2 .$$

From Table 2, $n_0^2=2$, $n_1^2=2$, $n_2^2=3$, $n_3^2=2$, so there is a linear relation in the first group. $a_1\theta_0^2+a_2\theta_1\theta_3+a_3\theta_2^2=0$ (*). (*) is a quadratic form with respect to

$\{\theta_k\}$. If all $a_k \neq 0$, then the rank of (*) is 4, and if $a_1=0$ or $a_3=0$, then the rank of (*) is 3.

Case of rank 3. If $a_1=0$, then $\theta_1\theta_3=c\theta_2^2$ (c: const.) and $\operatorname{div}(\theta_3/\theta_2)=\operatorname{div}(\theta_2/\theta_1)=P_1P_2^{-1}$, It contradicts. So $a_3=0$. Consequentry (*) is $a_1\theta_0^2+a_2\theta_1\theta_3=0$ (**). Then we have $\operatorname{div}(\theta_1)=6P_2$ and $\operatorname{div}(\theta_3)=6P_1$. This shows that the 1-gap sequence of P_1 and P_2 is $\{1, 2, 4, 7\}$.

Case of rank 4. We investigate the divisor of θ_k and we know that 1-gap sequence of P_1 and P_2 is $\{1, 2, 3, 4\}$.

Furthermore we know that P_1 and P_2 are the fixed points of T. So they are 1-Weierstrass points (resp. non 1-Weierstrass point) in the case of rank 3 (resp. the case of rank 4).

Next, we obtain the following 20 differentials of H^3 as products of θ_k.

$$\theta^3,\quad \theta_0\theta_1\theta_3,\quad \theta_0\theta_2^2,\quad \theta_1^2\theta_2,\quad \theta_2\theta_3^2\in H_0^3$$

$$\theta_0^2\theta_1,\quad \theta_0\theta_2\theta_3,\quad \theta_1^2\theta_3,\quad \theta_1\theta_2^2,\quad \theta_3^3\in H_1^3$$

$$\theta_0^2\theta_2,\quad \theta_0\theta_1^2,\quad \theta_0\theta_3^2,\quad \theta_2^3,\quad \theta_1\theta_2\theta_3\in H_2^3$$

$$\theta_0^2\theta_3,\quad \theta_0\theta_1\theta_2,\quad \theta_1^3,\quad \theta_1\theta_3^2,\quad \theta_2^2\theta_3\in H_3^3.$$

From Table 2, $n_0^3=4$, $n_1^3=4$, $n_2^3=3$, $n_3^3=4$, so there is a linear relation in each group. But we may consider only one linear relation in the third group from (*) or (**). We have

$$b_1\theta_0\theta_1^2+b_2\theta_0\theta_3^2+b_3\theta_2^3+b_4\theta_1\theta_2\theta_3=0 \qquad (\#).$$

By the way we know that the canonical image of M in $\mathbf{P}^3$ is contained in the intersection of the quadric and cubic defined by (*) or (**) and (#).

(i) Case of rank 3. We put $\theta_0=x$, $\theta_2=y$, $\theta_1=z$, $\theta_3=u$ and normalize $a_1=1$, $a_2=-1$ so we have

$$x^2-zu=0 \quad (**) \qquad b_1xz^2+b_2xu^2+b_3y^3+b_4yzu=0 \quad (\#).$$

We put $u=1$ and we represent the surface in affine space. Then $b_1x^5+b_2x+b_3y^3+b_4yx^2=0$. Clearly $b_1\neq 0$ and $b_3\neq 0$ otherwise the genus $g<4$. So $y^3+b_4yx^2+x^5+b_2x=0$. Namely we can normalize the equation of the surface as follows:

$$x^2-zu=0, \qquad xz^2+axu^2+y^3+byzu=0$$

or

$$y^3+byx^2+x^5+ax=0.$$

(ii) Case of rank 4. We put $\theta_0=x$, $\theta_2=y$, $\theta_1=z$, $\theta_3=u$ and normalize

$a_1=a_3=1,\ a_2=-1.$

$$x^2-zu-y^2=0\quad(*)\qquad b_1xz^2+b_2xu^2+b_3y^3+b_4yzu=0\quad(\#)$$

We put $u=1$ and we represent the surface in affine space. Then $b_1x(x^2+y^2)^2+b_2x+b_3y^3+b_4y(x^2+y^2)=0$. Clearly $b_1\neq 0$ otherwise the genus $g<4$. Namely we can normalize the equation of the surface as follows;

$$x^2-zu+y^2=0,\qquad xz^2+axu^2+by^3+cyzu=0$$

or

$$x(x^2+y^2)^2+ax+by^3+cy(x^2+y^2)=0\,.$$

Appendix (The integral solution of some trigonometric "diophantine" equations of §2 and §3.)

1.

$$1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})=\varepsilon(1-\varepsilon^{(N-1)/2})/(1-\varepsilon)\,,\qquad(1)$$

Put $\nu_2=\lambda$, $\nu_3=\mu$ and assume $\lambda\leqq\mu$. Here, from $\varepsilon=\cos(2\pi/N)+i\sin(2\pi/N)$, we have

$$\varepsilon/(1-\varepsilon)=-1/2+i\sin(2\pi/N)/2(1-\cos(2\pi/N))$$

$$\varepsilon^\lambda/(1-\varepsilon^\lambda)=-1/2+i\sin(2\lambda\pi/N)/2(1-\cos(2\lambda\pi/N))$$

$$\varepsilon^\mu/(1-\varepsilon^\mu)=-1/2+i\sin(2\mu\pi/N)/2(1-\cos(2\mu\pi/N))$$

$$\varepsilon^{(N-1)/2}/(1-\varepsilon)=i/2\sin(\pi/N)\,.$$

Hence, the above equation (1) is deformed as the following equation

$$\sin(2\lambda\pi/N)/2(1-\cos(2\lambda\pi/N))+\sin(2\mu\pi/N)/2(1-\cos(2\mu\pi/N))=1/2\sin(\pi/N)$$

or

$$\cot(\lambda\pi/N)+\cot(\mu\pi/N)-\operatorname{cosec}(\pi/N)=0\,.$$

Step 1. Assume $2\leqq\lambda\leqq\mu$.
Since $\cot x$ is monotone decreasing in $0<x<\pi$,

$$\cot(\lambda\pi/N)+\cot(\mu\pi/N)-\operatorname{cosec}(\pi/N)\leqq 2\cot(2\pi/N)-\operatorname{cosec}(\pi/N)$$

$$=(\cos(2\pi/N)-\cos(\pi/N))/\sin(\pi/N)\cos(\pi/N)<0\,.$$

Therefore there is no solution in this case.
Step 2. We may assume $\lambda=1$ from step 1.

$$\cot(\mu\pi/N)=\operatorname{cosec}(\pi/N)-\cot(\pi/N)=(1-\cos(\pi/N))/\sin(\pi/N)$$

$$\cos(\mu\pi/N)\sin(\pi/N)+\sin(\mu\pi/N)\cos(\pi/N)=\sin(\mu\pi/N)$$

$$\sin((\mu+1)\pi/N)=\sin(\mu\pi/N)\,.$$

Therefore we have $\mu=(N-1)/2$.

2.

$$1+\varepsilon/(1-\varepsilon)+\varepsilon^{\nu_2}/(1-\varepsilon^{\nu_2})+\varepsilon^{\nu_3}/(1-\varepsilon^{\nu_3})+\varepsilon^{4}/(1-\varepsilon^{\nu_4})=-1\,, \qquad (2)$$

Put $\nu_2=\lambda$, $\nu_3=\mu$, $\nu_4=\nu$ and assume $\lambda\leqq\mu\leqq\nu$. Here, from $\varepsilon=\cos(2\pi/N)+i\sin(2\pi/N)$, (2) is deformed as below;

$$\sin(2\pi/N)/(1-\cos(2\pi/N))+\sin(2\lambda\pi/N)/(1-\cos)2\lambda\pi/N)$$
$$+\sin(2\mu\pi/N)/(1-\cos(2\mu\pi/N))+\sin(2\nu\pi/N)/(1-\cos(2\nu\pi/N))=0$$

or

$$\cot(\pi/N)+\cot(\lambda\pi/N)+\cot(\mu\pi/N)+\cot(\nu\pi/N)=0\,.$$

Since the function $\cot x$ is monotone decreasing in $0<x<\pi$,

$$\cot(\lambda\pi/N)+\cot(\mu\pi/N)+\cot(\nu\pi/N)\geqq 3\cot\nu\pi/N)\,.$$

Step 1. Assume $\lambda\leqq\mu\leqq\nu\leqq N-3$ $(N\geqq 4)$. Then,

$$\cot(\pi/N)+\cot(\lambda\pi/N)+\cot(\mu\pi/N)+\cot(\nu\pi/N)\geqq\cot(\pi/N)+3\cot(\nu\pi/N)$$
$$\geqq\cot(\pi/N)+3\cot((N-3)\pi/N)=\cot(\pi/N)-3\cot(3\pi/N)$$
$$=(\sin(3\pi/N)\cos(\pi/N)-3\cos(3\pi/N)\sin(\pi/N))/\sin(\pi/N)\sin(3\pi/N)$$
$$=(\sin(3\pi/N)\cos(\pi/N)-\cos(3\pi/N)\sin(\pi/N)$$
$$-2\cos(3\pi/N)\sin(\pi/N)/\sin(\pi/N)\sin(3\pi/N)$$
$$=\{\sin(2\pi/N)-(\sin(4\pi/N)-\sin(2\pi/N))\}/\sin(\pi/N)\sin(3\pi/N)$$
$$=(2\sin(2\pi/N)-\sin(4\pi/N))/\sin(\pi/N)\sin(3\pi/N)$$
$$=2\sin(2\pi/N)(1-\cos(2\pi/N))/\sin(\pi/N)\sin(3\pi/N)>0\,.$$

Therefore, there is no solution under the assumption $\lambda\leqq\mu\leqq\nu\leqq N-3$.
Step 2. Assume $\nu=N-2$. Since $(N,\nu)=1$, N must be odd. Then

$$\cot(\pi/N)+\cot(\lambda\pi/N)+\cot(\mu\pi/N)+\cot((N-2)\pi/N)=0$$

or $\quad \cot(\lambda\pi/N)+\cot(\mu\pi/N)-\cot(2\pi/N)+\cot(\pi/N)=0$

or $\quad \cot(\lambda\pi/N)+\cot(\mu\pi/N)+\operatorname{cosec}(2\pi/N)=0\,.$

If $\mu=N-2$,

$$\cot(\lambda\pi/N)+\cot((N-2)\pi/N)+\operatorname{cosec}(2\pi/N)=0$$

or $\quad \cot(\lambda\pi/N)-\cot(2\pi/N)+\operatorname{cosec}(2\pi/N)=0$ or $\cot(\lambda\pi/N)+\tan(\pi/N)=0$

or $\quad \cos(\lambda\pi/N)\cos(\pi/N)+\sin(\lambda\pi/N)\sin(\pi/N)=0$ or $\cos((\lambda-1)\pi/N)=0\,.$

But since N is odd, there is no solution.

If $\mu \leqq N-4$,

$$\cot(\lambda\pi/N)+\cot(\mu\pi/N)+\operatorname{cosec}(2\pi/N)\geqq -2\cot(4\pi/N)+\operatorname{cosec}(2\pi/N)$$

$$=(\cos(2\pi/N)-\cos(4\pi/N))/(\sin(2\pi/N)\cos(2\pi/N))>0\,.$$

Therefore, there is no solution.

If $\mu=N-3$,

$$\cot(\lambda\pi/N)+\cot((N-3)\pi/N)+\operatorname{cosec}(2\pi/N)=0$$

or $$\cot(\lambda\pi/N)-\cot(3\pi/N)+\operatorname{cosec}(2\pi/N)=0\,.$$

We put $F(\lambda)=\cot(\lambda\pi/N)-\cot(3\pi/N)+\operatorname{cosec}(2\pi/N)$. Then $F(\lambda)$ is monotone decreasing function of λ.

$$F(N-5)=-\cot(5\pi/N)-\cot(3\pi/N)+\operatorname{cosec}(2\pi/N)$$

$$=(\sin(5x)\sin(3x)-\sin(2x)\sin(8x))/\sin(2x)\sin(3x)\sin(5x)\quad (x=\pi/N)\,.$$

The numerator$=(1/2)(2\cos(6x)\cos(4x)-\cos(6x)-\cos(8x))$.

Put $\cos(2x)=X$, then

The numerator$=f(X)=8X^5-4X^4-12X^3+4X^2+(9/2)X-(1/2)$.

$$f'(X)=40X^4-16X^3-36X^2+8X+\frac{9}{2},\quad f(1)=0,\quad f'(1)=\frac{1}{2}>0.$$

Hence $f(X)<0$ for X nealy equal to 1. Therefore $F(N-5)<0$ for large N.

$$F(N-6)=-\cot(6\pi/N)-\cot(3\pi/N)+\operatorname{cosec}(2\pi/N)$$

$$=(\sin(6x)\sin(3x)-\sin(9x)\sin(2x))/\sin(2x)\sin(3x)\sin(6x)\quad (x=\pi/N)$$

$$=4\sin(2x)\sin(3x)(\sin^2(3x)-\sin^2(2x))/\sin(2x)\sin(3x)\sin(6x)$$

$$=4(\sin^2(3x)-\sin^2(2x))/\sin(6x)>0\,.$$

Therefore, for large N, there is not integer solution. If N is small, we can investigate that there is no integer solution by direct computation.

Step 3. Assume $\nu=N-1$. Then

$$\cot(\pi/N)+\cot(\lambda\pi/N)+\cot(\mu\pi/N)+\cot((N-1)\pi/N)=0$$

$$\cot(\lambda\pi/N)+\cot(\mu\pi/N)=0\qquad\text{so}\quad \sin((\lambda+\mu)\pi/N)=0.$$

Hence we obtain that $\lambda+\mu=N$.

3.

$$\varepsilon^{q}-\varepsilon^{q+(N-1)/2}+\varepsilon^{-2q-(N-3)/2}-\varepsilon^{-2q+1}+\varepsilon^{(N-1)q/2}-\varepsilon^{(N-1)q/2+1}=0\,. \tag{3}$$

We may assume $0 \leqq q \leqq N-1$ clearly. From $\varepsilon = \cos(2\pi/N) + i\sin(2\pi/N)$,

$$\varepsilon^{q} - \varepsilon^{q+(N-1)/2}$$

$$= \cos(2q\pi/N) - \cos((2q+N-1)\pi/N) + i\{\sin(2q\pi/N) - \sin((2q+N-1)\pi/N)\}$$

$$= 2\sin((4+N-1)\pi/2N)\sin((N-1)\pi/2N)$$

$$-i\{2\cos((4q+N-1)\pi/2N)\sin((N-1)\pi/2N)\}.$$

$$\varepsilon^{-2q-(N-3)/2} - \varepsilon^{-2q+1}$$

$$= \cos((4q+N-3)\pi/N) - \cos(2(2q-1)\pi/N)$$

$$-i\sin((4q+N-3)\pi/N) - \sin(2(2q-1)\pi/N)\}$$

$$= -2\sin((8q+N-5)\pi/2N)\sin((N-1)\pi/2N)$$

$$-i\{2\cos((8q+N-5)\pi/2N)\sin((N-1)\pi/2N)\}.$$

$$\varepsilon^{(N-1)q/2} - \varepsilon^{(N-1)q/2+1}$$

$$= \cos((N-1)q\pi/N) - \cos(((N-1)q+2)\pi/N)$$

$$+i\{\sin((N-1)q\pi/N) - \sin(((N-1)q+2)\pi/N)\}$$

$$= 2\sin(((N-1)q+1)\pi/N)\sin(\pi/N) - i\{2\cos(((N-1)q+1)\pi/N)\sin(\pi/N)\}.$$

Hence, (Real part of the left side of (3))/2

$$= \sin((4q+N-1)\pi/2N)\sin((N-1)\pi/2N)$$

$$-\sin((8q+N-5)\pi/2N)\sin((N-1)\pi/2N) + \sin((N-1)q+1)\pi/N)\sin(\pi/N)$$

$$= 2\cos((6q+N-3)\pi/2N)\sin((-q+1)\pi/N)\sin((N-1)\pi/N)$$

$$+\sin(((N-1)q+1)\pi/N)\sin(\pi/N)$$

$$= 2\sin((-6q+3)\pi/2N)\sin((-q+1)\pi/N)\cos(\pi/2N)$$

$$+2\sin(q\pi + (-q+1)\pi/N)\sin(\pi/2N)\cos(\pi/2N)$$

$$= 2\cos(\pi/2N)\{\sin((-6q+3)\pi/2N)\sin((-q+1)\pi/N)$$

$$+\sin(q\pi + (-q+1)\pi/N)\sin(\pi/2N)\} = 0.$$

Since $\cos(\pi/2N) \neq 0$, we have

$$\sin((-6q+3)\pi/2N)\sin(-q+1)\pi/N)$$

$$+\sin(q\pi + (-q+1)\pi/N)\sin(\pi/2N) = 0. \tag{3.1}$$

By the same way, from (Imaginary part of the left side of (3))=0,

$$\sin((-6q+3)\pi/2N)\cos(-q+1)\pi/N)$$
$$+\cos(q\pi+(-q+1)\pi/N)\sin(\pi/2N)=0\,. \qquad (3.2)$$

If q is odd, (3.1) and (3.2) have the forms

$$\sin((-q+1)\pi/N)\{\sin((-6q+3)\pi/2N)-\sin(\pi/2N)\}=0 \qquad (3.1)'$$
$$\cos((-q+1)\pi/N)\{\sin((-6q+3)\pi/2N)-\sin(\pi/2N)\}=0\,. \qquad (3.2)'$$

Thus

$$\sin((6q-3)\pi/2N)+\sin(\pi/2N)=0 \quad \text{or}$$
$$2\sin((3q-1)\pi/N)\cos((3q-2)\pi/2N)=0\,.$$

Hence we have

$$q=(2N+1)/3 \quad \text{or} \quad q=(N+2)/3\,.$$

If q is even, (3.1) and (3.2) have the forms

$$\sin((-q+1)\pi/N)\{\sin((-6q+3)\pi/2N)+\sin(\pi/2N)\{=0 \qquad (3.1)''$$
$$\cos((-q+1)\pi/N)\{\sin((-6q+3)\pi/2N)+\sin(\pi/2N)\}=0\,. \qquad (3.2)''$$

Thus

$$\sin((6q-3)\pi/2N)-\sin(\pi/2N)=0 \quad \text{or}$$
$$2\sin((3q-2)\pi/2N)\cos((3q-1)\pi/2N)=0\,.$$

Hence we have

$$q=2(N+1)/3 \quad \text{or} \quad q=(N+1)/3\,.$$

4.

$$\varepsilon^{-3q+2}/(1-\varepsilon)+\varepsilon^{\nu q}/(1-\varepsilon^{\nu})+\varepsilon^{(N-\nu)q}/(1-\varepsilon^{N-\nu})+\varepsilon^{(N-1)}/(1-\varepsilon^{N-1})=0\,. \qquad (4)$$

We may assume $0\leqq q\leqq N-1$. By simple calculations we have (Real part of the left side of (4))/4

$$=\cos((\nu-4q+3)\pi/N)\sin(\nu\pi/N)\sin((-2q+1)\pi/N)$$
$$+\cos((\nu+1)\pi/N)\sin(\pi/N)\sin(\nu(2q-1)\pi/N)=0\,, \qquad (4.1)$$

and (Imaginary part of the left side of (4))/4

$$=\sin((\nu-4q+3)\pi/N)\sin(\nu\pi/N)\sin((-2q+1)\pi/N)$$
$$+\sin((\nu+1)\pi/N)\sin(\pi/N)\sin(\nu(2q-1)\pi/N)=0. \qquad (4\cdot 2)$$

From (4.1) and (4.2), we have

$$\sin((\nu-4q+3)\pi/N)\cos((\nu+1)\pi/N)-\cos((\nu-4q+3)\pi/N)\sin((\nu+1)\pi/N)=0.$$

Thus

$$\sin(2(q-1)\pi/N)=0 \quad \text{or}$$

$$2\sin((2q-1)\pi/N)\cos((2q-1)/N)=0\,.$$

If $\sin((2q-1)\pi/N)=0$, then we have $q=(N+1)/2$.

If $\cos((2q-1)\pi/N)=0$, we have $q=(N+2)/4$ or $q=(3N+2)/4$ but these do not satisfy our condition. Indeed, for example $q=(N+2)/4$, (4.1) and (4.2) have the forms

$$\cos((\nu+1)\pi/N)\{\sin(\nu\pi/N)+\sin(\pi/N)\sin(\nu\pi/2)\}=0\,,$$

$$\sin((\nu+1)\pi/N)\{\sin(\nu\pi/N)+\sin(\pi/N)\sin(\nu\pi/2)\}=0\,.$$

Thus

$$\sin(\nu\pi/N)+\sin(\pi/N)\sin(\nu\pi/2)=0\,. \tag{4.3}$$

Since N is even $(N=4q+2)$, ν is odd. If $\nu\equiv 1(4)$, then $\sin(\nu\pi/2)=1$ and so $\sin(\nu\pi/N)+\sin(\pi/N)\sin(\nu\pi/2)=\sin(\nu\pi/N)+\sin(\pi/N)>0$. Hence (4.3) has no solution. Also the case $q=(3N+2)/4$ is the same as above.

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