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SOME BOUNDS FOR THE SPECTRAL RADIUS OF A COXETER TRANSFORMATION

By

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Let Δ be a finite quiver (=oriented, connected graph) without oriented cycles. Let k be any field. The path algebra $k[\Delta]$ is a hereditary algebra, see [7]. The study of this kind of algebras had played a central role in the development of the Representation Theory of Algebras, see [6, 4, 13, 11].

For a representation X of $k[\Delta]$, we denote by $\underline{\dim}\, X=(\dim_k X(i))_{i\in\Delta_0}$ the dimension vector of X, where Δ_0 is the set of vertices of Δ. The *Coxeter matrix* ϕ_Δ satisfies

$$\underline{\dim}\, \tau X=(\underline{\dim}\, X)\phi_\Delta$$

where τX denotes the Auslander-Reiten translate of the non-projective indecomposable representation X. The *spectral radius* $\rho(\phi_\Delta)$ of the Coxeter matrix ϕ_Δ, contains relevant information about the behaviour of the translation τ, see [5, 11].

In this work, we consider some elementary relations between the spectral radii $\rho(\phi_{\bar{\Delta}})$ and $\rho(\phi_\Delta)$ for a *Galois covering* $\pi:\bar{\Delta}\to\Delta$. In particular, we show that for any covering $\pi:\bar{\Delta}\to\Delta$ defined by the action of a residually finite group and any finite subgraph F of $\bar{\Delta}$, we have $\rho(\phi_F)\leqq\rho(\phi_\Delta)$.

In [12], we have explored the relations between the spectral radii $r(\Delta)$ and $r(\bar{\Delta})$ of the adjacency matrices $A_{\bar{\Delta}}$ and A_Δ, for a Galois covering $\pi:\bar{\Delta}\to\Delta$. In section 2, we show how to use these results to get some interesting bounds for $\rho(\phi_\Delta)$.

Finally, we get some applications. In relation with a problem posed by Kerner, we show that

$$\frac{g(\Delta)}{\rho(\phi_\Delta)}\leqq\frac{|\Delta_0|}{2},$$

where $g(\Delta)=|\Delta_1|-|\Delta_0|+1$ denotes the *genus* of the underlying graph of Δ.

1. Galois covering and Coxeter matrices.

1.1. Let n be the number of vertices of the quiver Δ.

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For each vertex $i \in \Delta_0$, we denote by P_i the indecomposable projective $k[\Delta]$-module associated with i.

The *Cartan matrix* C_Δ of $k[\Delta]$ is the $n \times n$-matrix whose i-th column is the dimension vector $(\underline{\dim}\, P_i)^T$. This matrix is invertible.

The *Coxeter matrix* ϕ_Δ of $k[\Delta]$ is defined as

$$\phi_\Delta = -C_\Delta^{-T} C_\Delta\,,$$

where M^T denotes the transpose of M. We consider ϕ_Δ as a linear map, $\phi_\Delta : \boldsymbol{C}^{\Delta_0} \to \boldsymbol{C}^{\Delta_0}$, $\phi_\Delta(v) = v\phi_\Delta$. We recall that ϕ_Δ is characterized by $\phi_\Delta(\underline{\dim}\, P_i) = -\underline{\dim}\, I_i$, where I_i denotes the indecomposable injective $k[\Delta]$-module associated with i.

1.2. The *spectrum* $\mathrm{Spec}\,(\phi_\Delta)$ of ϕ_Δ is the set of eigenvalues of ϕ_Δ. The *spectral radius* $\rho(\phi_\Delta)$ is

$$\rho(\phi_\Delta) = \max\{|\lambda|\,;\ \lambda \text{ is an eigenvalue of } \phi_\Delta\}.$$

By [5, 11], $\rho(\phi_\Delta)$ is an eigenvalue of ϕ_Δ and there exists a corresponding eigenvector y^+ with non-negative coordinates.

As observed in [14], given a full subquiver Δ' of Δ, we get $\rho(\phi_{\Delta'}) \leqq \rho(\phi_\Delta)$.

1.3. Let $\pi : \bar{\Delta} \to \Delta$ be an onto morphism of quivers. Then π is said to be a *Galois covering* defined by the action of a group G if the following is satisfied:

i) G is a group of automorphisms of $\bar{\Delta}$, acting freely on $\bar{\Delta}$; that is, if $g(i) = i$ (resp. $g(\alpha) = \alpha$) for some vertex i (resp. arrow α), then $g = 1$.

ii) For any $g \in G$, $\pi g = \pi$.

iii) For any vertex i (resp. arrow α) of $\bar{\Delta}$, $\pi^{-1}\pi(i) = Gi$ (resp. $\pi^{-1}\pi(\alpha) = G\alpha$).

A Galois covering $\pi : \bar{\Delta} \to \Delta$, induces a Galois covering of algebras $k(\pi) : k[\bar{\Delta}] \to k[\Delta]$. Conversely, a Galois covering functor $F : k[\bar{\Delta}] \to k[\Delta]$ induces a Galois covering of quivers, see [8, 2].

1.4. Let $\pi : \bar{\Delta} \to \Delta$ be a Galois covering defined by the action of a group G. Let $F = k(\pi) : k[\bar{\Delta}] \to k[\Delta]$ be the induced functor. Following [8, 2], we can define the *push-down* functor, $F_\lambda : \mathrm{mod}\, k[\bar{\Delta}] \to \mathrm{mod}\, k[\Delta]$, and the *pull-up* functor, $F_. : \mathrm{mod}\, k[\Delta] \to \mathrm{Mod}\, k[\bar{\Delta}]$. In case the group G is finite, we get induced linear maps

$$f_\lambda : \boldsymbol{C}^{\bar{\Delta}_0} \longrightarrow \boldsymbol{C}^{\Delta_0} \qquad \text{with } f_\lambda(v)(\pi(i)) = \sum_{g \in G} v(g(i))$$

and

$$f_. : \boldsymbol{C}^{\bar{\Delta}_0} \longrightarrow \boldsymbol{C}^{\Delta_0} \qquad \text{with } f_.(z)(i) = z(\pi(i))\,.$$

We observe that

$$\phi_\Delta f_\lambda = f_\lambda \phi_{\bar\Delta} \text{ [evaluate in the basis } \{\underline{\dim}\, P_i\, ;\, i\in\bar\Delta_0\}]$$

and

$$\phi_\Delta f_{\cdot} = f_{\cdot} \phi_{\bar\Delta} \text{ [evaluate in the basis } \{\underline{\dim}\, P_j\, ;\, j\in\Delta_0\}],$$

see also [2].

1.5. PROPOSITION. *Let* $\pi:\bar\Delta\to\Delta$ *be a Galois covering defined by the action of a finite group* G. *Then* $\operatorname{Spec}(\phi_{\bar\Delta})\subset\operatorname{Spec}(\phi_\Delta)$ *and* $\rho(\phi_\Delta)=\rho(\phi_{\bar\Delta})$.

PROOF. Let $\lambda\in\operatorname{Spec}(\phi_\Delta)$. Let $0\neq x\in \boldsymbol{C}^{\Delta_0}$ be such that $\phi_\Delta(x)=\lambda x$. Consider the vector $0\neq\bar x=f_{\cdot}(x)\in\boldsymbol{C}^{\bar\Delta_0}$. By (1.4), $\phi_{\bar\Delta}(\bar x)=\lambda\bar x$. Hence, $\lambda\in\operatorname{Spec}(\phi_{\bar\Delta})$. In particular, $\rho(\phi_\Delta)\leqq\rho)\phi_{\bar\Delta})$.

Since the eigenvector $y^+\in\boldsymbol{C}^{\bar\Delta_0}$ has non-negative coordinates, then $0\neq f_\lambda(y^+)\in\boldsymbol{C}^{\Delta_0}$. By (1.4), this is an eigenvector of ϕ_Δ with eignvalue $\rho(\phi_{\bar\Delta})$. Therefore, $\rho(\phi_\Delta)=\rho(\phi_{\bar\Delta})$. □

1.6. PROPOSITION. *Let* $\pi:\bar\Delta\to\Delta$ *be a Galois covering defined by the action of a residually finite group* G. *Let* F *be any finite induced subquiver of* $\tilde\Delta$, *then* $\rho(\phi_F)\leqq\rho(\phi_\Delta)$.

PROOF. First, we show the existence of a factorization of π

$$\begin{array}{ccc} & & \bar\Delta \\ & \overset{\bar\pi}{\swarrow} & \downarrow \pi \\ \Delta' & \xrightarrow[\pi']{} & \Delta \end{array}$$

where π' and $\bar\pi$ are Galois coverings, Δ' is finite, $\bar\pi(F)$ is a full subquiver of Δ', and the induced morphism $\bar\pi|:F\to\Delta'$ is injective. Indeed, the set $S=\{g\in G\, ;\, g\neq 1,\ g(F'_0)\cap F'_0\neq\emptyset\}$ is finite, where F' is the full induced subquiver of $\tilde\Delta$ with set of vertices $F_0\cup\{i\in\tilde\Delta_0$; there exists $j\in F_0$ such that i and j joined by an arrow in $\bar\Delta\}$. Since G acts freely on $\bar\Delta$. Hence there exists a normal subgroup $H\lhd G$ with finite index and such that $S\cap H=\emptyset$. The covering $\bar\pi:\tilde\Delta\to\Delta'$ defined by the action of H satisfies the desired properties.

By (1.2) and (1.5), we have

$$\rho(\phi_F)=\rho(\phi_{\bar\pi(F)})\leqq\rho(\phi_{\Delta'})=\rho(\phi_\Delta).$$

□

1.7. COROLLARY. *Let* $\pi:\tilde\Delta\to\Delta$ *be the universal Galois covering of* Δ. *For any finite induced subquiver* F *of* Δ', *we have* $\rho(\phi_F)\leqq\rho(\phi_\Delta)$.

PROOF. The universal covering π is defined by the action of a free group Π (the fundamental group). Thus π is residually finite. □

2. Coxeter matrices and adjacency matrices.

2.1. Let Δ be a finite quiver as above and $\pi: \bar{\Delta} \to \Delta$ be a Galois covering. The set of vertices $\bar{\Delta}_0$ is at most countable, thus we assume that either $\bar{\Delta}_0 = \{1, \cdots, n\}$ for some $n \in \boldsymbol{N}$ or $\bar{\Delta}_0 = \boldsymbol{N}$. The *adjacency matrix* of $\bar{\Delta}$, $A_{\bar{\Delta}} = (a_{ij})$ is the matrix whose (i, j)-th entry a_{ij} is the number of edges between the vertices i and j if $i \neq j$ and a_{ii} is twice the number of loops at i. Similarly we define the adjacency matrix A_Δ. Following [10, 12], we consider $A_{\bar{\Delta}}$ as a linear operator $A_{\bar{\Delta}}: l^2_{\bar{\Delta}} \to l^2_{\bar{\Delta}}$, where $l^2_{\bar{\Delta}}$ is the Hilbert space of all sequences $(x_i)_{i \in \bar{\Delta}_0}$ of complex numbers such that $\sum_{i \in \bar{\Delta}_0} |x_i|^2$ converges.

We recall that the *spectrum* $\rho(\bar{\Delta})$ *of the quiver* $\bar{\Delta}$ is the set of complex numbers λ such that $A_{\bar{\Delta}} - \lambda I$ is not an invertible operator, where I denotes the identity operator in $l^2_{\bar{\Delta}}$. The *spectral radius* $r(\bar{\Delta})$ of $\bar{\Delta}$ is defined as $r(\bar{\Delta}) = \sup\{|\lambda| : \lambda \in \sigma(\bar{\Delta})\}$.

THEOREM [10, 12]. *Let* $\pi: \bar{\Delta} \to \Delta$ *be a Galois covering of* Δ. *Then*

i) $r(\bar{\Delta}) = \sup\{r(F)$; *F is a finite induced subquiver of* $\bar{\Delta}\}$

ii) $r(\bar{\Delta}) \leqq r(\Delta)$. □

2.2. We recall now a basic relation between the spectral radius $\rho(\phi_\Delta)$ of the Coxeter matrix and the spectral radius $r(\Delta)$ of the adjacency matrix A_Δ.

PROPOSITION [11]. *Assume that* Δ *is a finite tree, whose underlying graph is not a Dynkin type. Then there exists a real number* $\lambda \geqq 1$ *such that*

$$r(\Delta) = \lambda + \lambda^{-1} \quad \textit{and} \quad \rho(\phi_\Delta) = \lambda^2 .$$

Sketch of the proof: For any $\mu \neq 0$, we have

$$\det(\mu^2 I - \phi_\Delta) = \mu^n \det((\mu + \mu^{-1})I - A_\Delta) .$$

Hence μ^2 is an eigenvalue of ϕ_Δ if and only if $\mu + \mu^{-1}$ is an eigenvalue of A_Δ. Moreover, by [1], $1 \leqq \rho(\phi_\Delta)$ is an eigenvalue of ϕ_Δ. □

2.3. We show how to use the above results to get lower bounds for $\rho(\phi_\Delta)$ for a general quiver Δ.

THEOREM. *Let* Δ *be a finite quiver without oriented cycles, whose underly-*

ing graph is not of Dynkin type. Let $\pi: \tilde{\Delta} \to \Delta$ *be the universal covering. Then there is a real number* $\lambda \geqq 1$ *such that*

$$r(\tilde{\Delta}) = \lambda + \lambda^{-1} \quad \textit{and} \quad \rho(\phi_\Delta) \geqq \lambda^2 .$$

PROOF. If Δ is a tree, the result is just (2.2). If Δ is a cycle, then the underlying graph of $\tilde{\Delta}$ is of the form

$$\cdots \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \cdots .$$

Therefore, $r(\tilde{\Delta}) = 2$ and $\rho(\phi_\Delta) = 1$.

Assume that Δ is not a tree nor a cycle. Then there is a sequence $(F_m)_m$ of induced finite subquivers of $\tilde{\Delta}$, such that the underlying graph of F_m is not of Dynkin type, F_m is contained in F_{m+1} and $\lim_{m \to \infty} r(F_m) = r(\tilde{\Delta})$.

Since $\tilde{\Delta}$ is an infinite tree, for each $m \in \boldsymbol{N}$ there is a real number $\lambda_m \geqq 1$ such that $r(F_m) = \lambda_m + \lambda_m^{-1}$ and $\rho(\phi_{F_m}) = \lambda_m^2$. By (2.1), $(\lambda_m)_m$ is a bounded squence. Let $\lambda = \sup_m \{\lambda_m\}$. Hence $r(\tilde{\Delta}) = \lambda + \lambda^{-1}$ and by (1.6)

$$\lambda^2 = \sup_m \{\rho(\phi_{F_m})\} \leqq \rho(\phi_\Delta) .$$

□

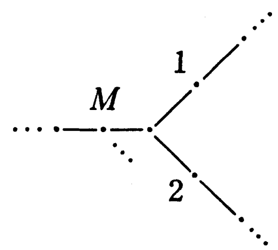
2.4. We get an explicit bound for $\rho(\phi_\Delta)$ as an application of (2.3).

PROPOSITION. *Let* Δ *be a quiver without vertices of degree* 1. *Let* M_Δ *be the maximum of the degrees of vertices of* Δ. *Then*

$$\rho(\phi_\Delta) \geqq M_\Delta - 1 .$$

PROOF. Let $\pi: \tilde{\Delta} \to \Delta$ be the universal covering of Δ. It is not hard to see that $\tilde{\Delta}$ contains an induced subquiver with underlying graph S_M, where $M = M_\Delta$.

S_M :

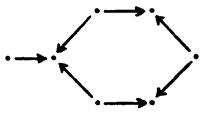


In (2.5) we will show that $r(S_M) = (M-1)^{1/2} + (M-1)^{-1/2}$.

By (2.1), $r(S_M) \leqq r(\tilde{\Delta})$. Therefore, the result follows by (2.3). □

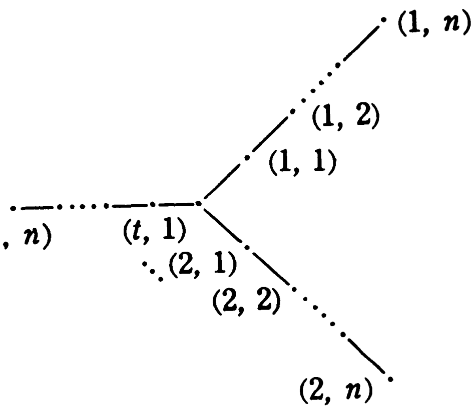
COROLLARY. *Let* Δ *be a quiver and denote by* Δ' *the maximal induced subquiver of* Δ *without vertices of degree* 1. *Then* $\rho(\phi_\Delta) \geqq M_{\Delta'} - 1$. □

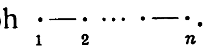
The bound of the proposition does not hold in the general situation. For *example*:

Δ:  $\rho(\phi_\Delta)=1.8832\cdots<2=M_\Delta-1.$

2.5. LEMMA. *Let S_t be the infinite graph defined in* (2.4), *then* $r(S_t)=(t-1)^{1/2}+(t-1)^{-1/2}$.

PROOF. The case $t=2$ is well known. Assume $t\geqq 3$. For any $n\in\boldsymbol{N}$, consider the finite star $S_t^{(n)}$



Let L_n be the graph 

Let $p_n(x)$ (resp. $q_n(x)$) be the characteristic polynomial of the adjacency matrix of $S_t^{(n)}$ (resp. L_n). An easy calculation shows that $p_n=xq_n^t-tq_{n-1}q_n^{t-1}$.

Let $x=\mu+\mu^{-1}$, then $q_n(x)=(\mu-\mu^{-1})^{-1}(\mu^{n+1}-\mu^{-n-1})$. This can be deduced by induction using [9]. Hence,

$$p_n(x)=\frac{1}{(\mu-\mu^{-1})}q_n^{t-1}(x)[\mu^n(\mu^2-(t-1))+\mu^{-n-2}((t-1)\mu^2-1)].$$

Let $\mu_0=(t-1)^{1/2}$ and $2<\lambda_0=\mu_0+\mu_0^{-1}$. Then for any $\lambda\geqq\lambda_0$, we have $p_n(\lambda)>0$. From this we deduce that

$$r(S_t)=\sup_n\{r(S_t^{(n)})\}\leqq\lambda_0\,.$$

If $2<\lambda<\lambda_0$ with $\lambda=\mu+\mu^{-1}$, then we may assume that $1<\mu<\mu_0$, and $p_n(\lambda)<0$ for n big enough. Hence, $r(S_t)=\lambda_0$. □

For results similar to this lemma see [9].

3. A relation between $g(\Delta)$ and $\rho(\phi_\Delta)$.

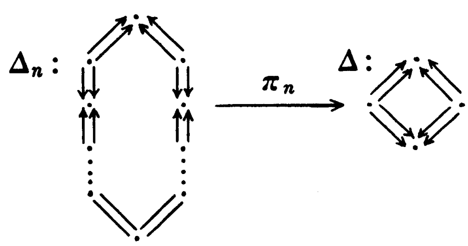
3.1. Let Δ be a finite quiver. The *genus* $g(\Delta)$ of Δ is the rank of the fundamental group of Δ. It is well known that

$$g(\Delta)=|\Delta_1|-|\Delta_0|+1\,,$$

where Δ_1 is the set of arrows of Δ.

Recently, O. Kerner asked if there was some constant upper bound for the ratio $g(\Delta)/\rho(\phi_\Delta)$ (in fact, he asked for a bound of the ratio $\dim H_1(k[\Delta])/\rho(\phi_\Delta)$, where $H_1(k[\Delta])$ denotes the first cohomology group of $k[\Delta]$. It is known that $g(\Delta)\leqq\dim H_1(k[\Delta])$). We answer this question in the negative and we give a linear bound in the number of vertices $|\Delta_0|$.

3.2. Consider Galois coverings $\pi_n: \Delta_n\to\Delta$ as follows



where Δ_n has $4n$ vertices. By (1.5), $\rho(\phi_{\Delta_n})=\rho(\phi_\Delta)=7+4\sqrt{3}$. On the other hand $g(\Delta_n)=4n+1$, which shows that $g(\Delta_n)/\rho(\phi_{\Delta_n})$ grows linearly with $|(\Delta_n)_0|$.

3.3. PROPOSITION. *Let Δ be a finite quiver. Then*

$$\frac{g(\Delta)}{\rho(\phi_\Delta)}\leqq\frac{|\Delta_0|}{2}$$

PROOF. Let Δ' be the maximal induced subquiver of Δ without vertices of degree 1. Clearly, $g(\Delta')=g(\Delta)$. By (1.2), $g(\Delta)/\rho(\phi_\Delta)\leqq g(\Delta')/\rho(\phi_{\Delta'})$ and $|\Delta'_0|/2\leqq|\Delta_0|/2$.

Therefore, we may assume that Δ has not vertices of degree 1.

Let M_Δ be the maximal of the degrees of vertices of Δ. By (2.4), $\rho(\phi_\Delta)\geqq M_\Delta-1$.

On the other hand, $|\Delta_1|=\frac{1}{2}\sum_{i\in\Delta_0}\text{degree }(i)\leqq\frac{M_\Delta|\Delta_0|}{2}$.

Therefore,

$$g(\Delta)=|\Delta_1|-|\Delta_0|+1\leqq\frac{(M_\Delta-2)|\Delta_0|+2}{2}\,.$$

Hence

$$\frac{g(\Delta)}{\rho(\phi_\Delta)} \leqq \frac{(M_\Delta - 2)|\Delta_0| + 2}{2(M_\Delta - 1)} \leqq \frac{|\Delta_0|}{2}. \qquad \square$$

REMARK. The bound in (3.3) is in general not optimum. Easy calculations provide some improvements. For example, if $M_\Delta=3$ and $|\Delta_0|\geqq 6$, then $g(\Delta)/\rho(\phi_\Delta)\leqq|\Delta_0|/3$.

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