

CYCLIC-PARALLEL REAL HYPERSURFACES OF QUATERNIONIC PROJECTIVE SPACE

By

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Abstract. We classify cyclic-parallel real hypersurfaces of quaternionic projective space.

1. Introduction.

Let M be a connected real hypersurface of a quaternionic projective space QP^m , $m \geq 2$, endowed with the metric of constant quaternionic sectional curvature 4. If ζ denotes the unit local normal vector field and $\{J_1, J_2, J_3\}$ a local basis of the quaternionic structure of QP^m (see [2]), then $U_i = -J_i\zeta$, $i=1, 2, 3$, are vector fields tangent to M .

It is known, [4], that there do not exist parallel real hypersurfaces of QP^m , $m \geq 2$. A real hypersurface of QP^m is called cyclic-parallel if it satisfies

$$(1.1) \quad \sigma(g((\nabla_X A)Y, Z)) = 0$$

for any X, Y, Z tangent to M , where A denotes the Weingarten endomorphism of M and σ the cyclic sum.

Our purpose is to classify such real hypersurfaces by mean of the following

THEOREM. *A real hypersurface M of QP^m , $m \geq 2$, is cyclic-parallel if and only if it is congruent to an open subset of a tube of radius r , $0 < r < \Pi/2$, over QP^k , $k \in \{0, \dots, m-1\}$.*

In the Theorem, QP^k is considered canonically and totally geodesically embedded in QP^m .

2. Preliminaries.

Let X be a vector field tangent to M . We write $J_i X = \phi_i X + f_i(X)\zeta$, $i=1, 2, 3$, where $\phi_i X$ denotes the tangent component of $J_i X$ and $f_i(X) = g(X, U_i)$. From this, see [3], we have

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$$(2.1) \quad g(\phi_i X, Y) + g(X, \phi_i Y) = 0, \quad \phi_i U_i = 0, \quad \phi_j U_k = -\phi_k U_j = U_t$$

for any X, Y tangent to M , $i=1, 2, 3$, (j, k, t) being a circular permutation of $(1, 2, 3)$.

From the expression of the curvature tensor of QP^m (see [2]) the equation of Codazzi is given by

$$(2.2) \quad (\nabla_X A)Y - (\nabla_Y A)X = \sum_{i=1}^3 \{f_i(X)\phi_i Y - f_i(Y)\phi_i X + 2g(X, \phi_i Y)U_i\}$$

for any X and Y tangent to M . Moreover

$$(2.3) \quad \nabla_X U_i = q_k(X)U_j - q_j(X)U_k + \phi_i AX$$

for any X tangent to M , (i, j, k) being a circular permutation of $(1, 2, 3)$ and q_i , $i=1, 2, 3$, certain local 1-forms on M (see [2]).

In the following we shall denote by $\mathfrak{D}^\perp = \text{Span}\{U_1, U_2, U_3\}$ and by \mathfrak{D} its orthogonal complement in TM .

3. Proof of Theorem.

From (1.1) and applying twice (2.2) we find

$$(3.1) \quad (\nabla_X A)Y = -\sum_{i=1}^3 \{f_i(Y)\phi_i X + g(Y, \phi_i X)U_i\}.$$

From (3.1), (2.1) and (2.3) we have for any X, Y, Z tangent to M

$$(3.2) \quad (\nabla_Z (\nabla_X A))Y - (\nabla_{\nabla_Z X} A)Y = -\sum_{i=1}^3 \{g(\phi_i X, Y)\phi_i AZ + g(Y, \phi_i AZ)\phi_i X + f_i(X)g(AZ, Y)U_i - 2f_i(Y)g(AX, Z)U_i + f_i(Y)f_i(X)AZ\}.$$

If we exchange Z and X , from (3.2) we obtain

$$(3.3) \quad (R(Z, X)A)Y = -\sum_{i=1}^3 \{f_i(X)g(AZ, Y)U_i - f_i(Z)g(AX, Y)U_i + g(\phi_i X, Y)\phi_i AZ - g(\phi_i Z, Y)\phi_i AX + g(\phi_i AZ, Y)\phi_i X - g(\phi_i AX, Y)\phi_i Z + f_i(Y)f_i(X)AZ - f_i(Y)f_i(Z)AX\}.$$

Let now $\{E_j, j=1, \dots, 4m-1\}$ be an orthonormal frame on M . From (3.3) we have

$$(3.4) \quad \sum_{j=1}^{4m-1} g((R(E_j, X)A)Y, E_j) = -\sum_{i=1}^3 \{f_i(X)f_i(AY) - g(\phi_i X, Y)\text{trace}(A\phi_i) - 2f_i(Y)f_i(AX) - g(A\phi_i Y, \phi_i X) + \text{trace}(A)f_i(Y)f_i(X)\}.$$

But the left hand side of (3.4) is symmetric respect to X and Y . This gives

$$(3.5) \quad -3\sum_{i=1}^3 f_i(X)f_i(AY) = -3\sum_{i=1}^3 f_i(Y)f_i(AX) + 2\sum_{i=1}^3 g(\phi_i Y, X)\text{trace}(A\phi_i).$$

From (2.1) it is easy to see that $\text{trace}(A\phi_i) = 0$. This implies

$$(3.6) \quad \sum_{i=1}^3 f_i(X)f_i(AY) = \sum_{i=1}^3 f_i(Y)f_i(AX)$$

for any X, Y tangent to M . From (3.6) we obtain that $g(A\mathfrak{D}, \mathfrak{D}^\perp) = \{0\}$. Thus, [1], M is congruent to an open subset of either a tube of radius r , $0 < r < \Pi/2$, over QP^k $k \in \{0, \dots, m-1\}$ or a tube of radius r , $0 < r < \Pi/4$ over CP^m .

If we consider the second case, the principal curvatures of M are, [3], $\cot(r)$, $-\tan(r)$, $2 \cot(2r)$, $-2 \tan(2r)$ with respective multiplicities $2(m-1)$, $2(m-1)$, 1 and 2. If X is a unit vector field such that $AX = -\tan(r)X$, from (2.3) the \mathfrak{D} -component of $(\nabla_X A)U_2$ is $-\tan(r)(-2 \tan(2r) - \cot(r))\phi_2 X$. But if M is cyclic-parallel from (3.1) $(\nabla_X A)U_2$ must be equal to $-\phi_2 X$. This implies that $\cot^2(r) = -1$ which is impossible.

On the other hand it is easy to prove that the tubes over QP^k are cyclic-parallel and this finishes the proof.

References

- [1] J. Berndt, Real hypersurfaces in quaternionic space forms, *J. reine angew. Math.*, **419** (1991), 9-26.
- [2] S. Ishihara, Quaternion Kählerian manifolds, *J. Differential Geom.*, **9** (1974), 483-500.
- [3] A. Martínez and J.D. Pérez, Real hypersurfaces in quaternionic projective space, *Ann. di Mat.*, **145** (1986), 355-384.
- [4] J.S. Pak, Real hypersurfaces in quaternionic Kaehlerian manifolds with constant Q -sectional curvature, *Kodai Math. Sem. Rep.*, **29** (1977), 22-61.

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