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EQUIVARIANT MINIMAL IMMERSIONS OF COMPACT RIEMANNIAN HOMOGENEOUS SPACES INTO COMPACT RIEMANNIAN HOMOGENEOUS SPACES

By

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§ 1. Introduction.

Let M and N be two compact connected Riemannian manifolds. A smooth mapping $F: M\to N$ is called harmonic if it is an extremal of the energy. Moreover, if harmonic mapping $F: M\to N$ is an isometric immersion, then F is called minimal. The existence and construction of minimal immersions and harmonic mappings are interesting and important problems in various situations. T. Takahashi proved the following theorem.

THEOREM (T. TAKAHASHI [5]). *A compact homogeneous Riemannian manifold with irreducible linear isotropy group admits a minimal immersion in a Euclidean sphere.*

In § 3, we construct minimal immersions and harmonic mappings of compact Riemannian homogeneous spaces into Grassmann manifolds (Theorem 3.1). Applying this, we prove

THEOREM A.

(1) *A compact Riemannian homogeneous space of dimension*$\geqq 2$ *with irreducible linear isotropy group admits an equivariant minimal immersion into a Grassmann manifold.*

(2) *There exists a nonconstant equivariant harmonic mapping from a compact Riemannian homogeneous space with non trivial isotropy group into a Grassmann manifold.*

In Theorem A, we can restrict ambient manifolds to projective spaces or $G_{2,n}(\boldsymbol{R})$ if the domain manifolds are compact irreducible symmetric spaces.

More precisely,

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THEOREM B.

(1) *Let M be a compact irreducible Riemannian symmetric space of dimension* $\geqq 2$.

(i) *M admits an equivariant minimal immersion into a real projective space or* $G_{2,n}(\boldsymbol{R})$.

(ii) *M admits an equivariant minimal immersion into a complex projective space.*

(iii) *M admits an equivariant minimal immersion into a quaternion projective space.*

(2) *Let M be a compact Riemannian symmetric space with non trivial isotropy group.*

(i) *There exists a nonconstant equivariant harmonic mapping from M into a real projective space or* $G_{2,n}(\boldsymbol{R})$.

(ii) *There exists a nonconstant equivariant harmonic mapping from M into a complex projective space.*

(iii) *There exists a nonconstant equivariant harmonic mapping from M into a quaternion projective space.*

On the other hand, it is an important problem to know whether a given minimal submanifold is stable or not. Simons [4] proved that there are no stable minimal submanifolds in S^n. Nagura [3] studied on the spectra of the Jacobi differential operator for minimally immersed spheres into spheres. In § 4, the problem of computing the spectra of the Jacobi differential operator of equivariant minimal immersions of compact Riemannian homogeneous spaces into compact Riemannian homogeneous spaces is reduced to the eigenvalue problems for certain linear mappings of finite dimensional vector spaces applying the representation theory of compact Lie groups (Theorem 4.2).

In § 5, applying the results in § 3 and § 4, we study the equivariant minimal immersions of S^2 into Grassmann manifolds.

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§ 2. Preliminaries.

2.1. Let G (resp. U) be a compact connected Lie group with Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{u}$) and K (resp. L) be a closed subgroup of G (resp. U) with Lie algebra $\mathfrak{k}$ (resp. $\mathfrak{l}$). Then $M=G/K$ (resp. $N=U/L$) is a compact Riemannian homogeneous space with G (resp. U)-invariant Riemannian metric. Since K (resp. L) is compact, M (resp. N) is reductive, that is, there exist an $\mathrm{Ad}(K)$ (resp. $\mathrm{Ad}(L)$)-invariant subspace $\mathfrak{m}$ (resp. $\mathfrak{p}$) such that

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{m} \text{ (direct sum)} \quad (\text{resp. } \mathfrak{u}=\mathfrak{l}+\mathfrak{p}).$$

We call $\mathfrak{m}$ (resp. $\mathfrak{p}$) a Lie subspace of M (resp. N). We identify the tangent space $T_o(M)$ (resp. $T_o(N)$) at $o=\pi(e)$ with $\mathfrak{m}$ (resp. $\mathfrak{p}$) in a natural manner, where π is the natural projection of G (resp. U) onto M (resp. N). The differential mapping $k_*(k\in K)$ (resp. $l_*(l\in L)$) acting on $T_o(M)$ (resp. $T_o(N)$) corresponds to $\mathrm{Ad}(k)$ (resp. $\mathrm{Ad}(l)$) on $\mathfrak{m}$ (resp. $\mathfrak{p}$), that is,

$$k_*\pi_*X=\pi_*\,\mathrm{Ad}(k)X \qquad \text{for each } X\in\mathfrak{m}.$$

Hence we have

$$\frac{d}{dt}(\exp tY)_*\pi_*X_{|t=0}=\pi_*[Y, X] \qquad \text{for } Y\in\mathfrak{k},\ X\in\mathfrak{m}. \tag{2.1}$$

Let $F\colon M\to N$ be an equivariant mapping, that is, there exists an analytic homomorphism $\rho: G\to U$ with $\rho(K)\subset L$ such that $F(gK)=\rho(g)L$ for each $g\in G$. We get

$$F_*X=(\rho_*X)_{\mathfrak{p}} \qquad \text{for each } X\in\mathfrak{m}. \tag{2.2}$$

2.2. In this subsection, we review some elementary results on representation theory of compact connected Lie groups without proof. Let G be a compact connected Lie group. The following lemmas are well-known.

LEMMA 2.1. *Let (ρ, V) be a real irreducible representation of G. (ρ^C, V^C) is not a complex irreducible representation of G if and only if there exists a complex irreducible representation (τ, W) of G such that $(\rho, V)=(\tau_R, W_R)$ where we denote (ρ^C, V^C) (resp. (τ_R, W_R)) the complex (resp. real) representation of G obtained by extension (resp. restriction) of the coefficient field of (ρ, V) (resp. (τ, W)) to $\boldsymbol{C}$ (resp. $\boldsymbol{R}$).*

LEMMA 2.2. *Let (ρ, V) be a complex irreducible representation of G. (ρ_R, V_R) is not a real irreducible representation of G if and only if there exisis a real irreducible representation (τ, W) of G such that $(\rho, V)=(\tau^C, W^C)$.*

LEMMA 2.3. *Let (ρ, V) be a complex irreducible representation of G. (ρ^H, V^H) is not a quaternion irreducible representation of G if and only if there exists a quaternion irreducible representation (τ, W) of G such that $(\rho, V)=(\tau_C, W_C)$, where we denote (ρ^H, V^H) (resp. (τ_C, W_C)) the quaternion (resp. complex) representation* *of G obtained by extension (resp. restriction) of the coefficient field of (ρ, V) (resp. (τ, W)) to $\boldsymbol{H}$ (resp. $\boldsymbol{C}$).*

LEMMA 2.4. *Let (ρ, V) be a quaternion irreducible representation of G. (ρ_C, V_C) is not a complex irreducible representation of G if and only if there exists a complex irreducible representation (τ, W) of G such that $(\rho, V)=(\tau^H, W^H)$.*

§3. A construction of equivariant minimal immersions and harmonic mappings of compact Riemannian homogeneous spaces into Grassmann manifolds.

3.1. Let G be a compact connected Lie group and K be a closed subgroup of G. Then $M=G/K$ is a compact Riemannian homogeneous space wi3h G-invariant Riemannian metric $\langle , \rangle$. For a field $E=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$, put

$$U(n, E)=\begin{cases} O(n) & (E=\boldsymbol{R}), \\ U(n) & (E=\boldsymbol{C}), \\ Sp(n) & (E=\boldsymbol{H}). \end{cases}$$

Let $F: M=G/K \to G_{n,m}(E)=U(n+m, E)/U(n, E)\times U(m, E)$ $(n\geqq 1,\ m\geqq 1)$ be an equivariant mapping. Then there exists an analytic homomorphism $\rho: G\to U(n+m, E)$ with $\rho(K)\subset U(n, E)\times U(m, E)$ such that $F(gK)=\rho(g)U(n, E)\times U(m, E)$ for each $g\in G$. We shall call $G_{n,m}(E)$ E-Grassmann manifold. Put $V=E^{n+m}$, $V_1=E^n$, $V_2=E^m$. Then $V=V_1+V_2$ (direct sum) and the Lie algebra $\mathfrak{u}$ of $U(n+m, E)$ acts on V, naturally. Put $\mathfrak{k}=\mathrm{Lie}(U(n, E)\times U(m, E))$ and

$$\mathfrak{p}=\{A\in\mathfrak{u}\,;\ AV_1\subset V_2,\ AV_2\subset V_1\}.$$

Then $\mathfrak{u}=\mathfrak{k}+\mathfrak{p}$ is the canonical decomposition of $\mathfrak{u}$. Put

$$\mathrm{Hom}_K(V_1, V_2)=\{A\in\mathrm{Hom}(V_1, V_2)\,;\ \rho(k)A=A\rho(k) \text{ for each } k\in K\}.$$

We explain that F is E-full. Let V_1' and V_2' be subspaces of V_1 and V_2 respectively. Put $n'=\dim_E V_1'$ and $m'=\dim_E V_2'$. Then $U(n'+m', E)$ is considered as a closed subgroup of $U(n+m, E)$ in a natural manner. So $G_{n',m'}(E)$ is a totally geodesic submanifold of $G_{n,m}(E)$. The mapping F is said to be E-full when the image $F(M)$ is not contained in these totally geodesic submanifolds $G_{n',m'}(E)$ with $n'+m'<n+m$.

THEOREM 3.1. *If $\mathrm{Hom}_K(V_1, V_2)=\{0\}$ and V_i $(i=1, 2)$ is not G-invariant, then F is a nonconstant harmonic mapping. Furthermore, if G acts on V irreducibly, then F is E-full. Moreover, if K acts on $T_o(M)$ irreducibly, then F is a minimal immersion of a multiple of the G-invariant Riemannian metric $\langle , \rangle$*

on M.

PROOF. Let $H\in\mathfrak{p}$ denote the tension field of F at o. Then $H\rho(k)=\rho(k)H$ for each $k\in K$. From assumption, we have $H=0$. So F is a nonconstant harmonic mapping.

We assume that K acts on $T_o(M)$ irreducibly. We define a symmetric linear transformation A of $T_o(M)$ by

$$\langle X, AY\rangle=(F_*X, F_*Y) \qquad \text{for } X, Y\in T_o(M),$$

where $(\,,)$ denote a $U(n+m, E)$-invariant Riemannian metric on $G_{n,m}(E)$. Since A is a K-homomorphism, A is a scalar operator by the irreducibility of K. The scalar is clearly nonnegative. So if F were not an isometric (more precisely, homothetic) immersion, then $F_*=0$. Thus V_i is G-invariant from (2.2) and the connectedness of G. So F is an isometric minimal immersion. If G acts on V irreducibly, then F is clearly E-full. Q. E. D.

Example (Equivariant minimal immersion of S^2 into Grassmann manifold) Let (ρ, V) be any $SU(2)$-E-irreducible representation. Put $K=S(U(1)\times U(1))$. Let $V=\sum_i W_i$ be a K-E-irreducible decomposition of V. We have

$$W_i\cong W_j \ (K\text{-isomorphic}) \Longleftrightarrow i=j \ (\text{see §5, Lemma 5.1}). \tag{3.1}$$

Let V_i $(i=1, 2)$ be a K-E-invariant subspace of V such that $V_i\neq\{0\}$, V and $V=V_1+V_2$ (direct sum). Put $n=\dim_E V_1$, $m=\dim_E V_2$. If we put $F: S^2=SU(2)/K\to G_{n,m}(E)$; $gK\mapsto\rho(g)U(n, E)\times U(m, E)$ for $g\in G$, then F is a full minimal immersion from (3.1) and Theorem 3.1. ■

3.2. In this subsection, we apply Theorem 3.1. Let M $(\neq\{\text{single point}\})$ be a compact Riemannian homogeneous space. The identity component G of the group of all isometries of M is compact. The action of G on M is effective and transitive. The subgroup $K=\{g\in G\,;\, g\cdot o=o\}$ of G is closed and called isotropy group of M at o.

A G-E-irreducible representation (ρ, V) is called an E-spherical representation of the pair (G, K), if $V_K=\{v\in V\,;\,\rho(k)v=v$ for each $k\in K\}\neq\{0\}$ $(E=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H})$. The dimension of V and V_K is called the degree and the multiplicity of (ρ, V), respectively.

LEMMA 3.1. *If $K\neq\{e\}$, then there exists an E-spherical representation (ρ, V) such that $V_K\neq V$.*

PROOF. We may assume that $E=\boldsymbol{C}$ by Lemma 2.2 and Lemma 2.3. Let

$L^2(G/K)$ denote the space of complex valued functions f on G/K with

$$\int_{G/K} |f(x)|^2 dx < \infty .$$

Put

$$L^2(G, K) = \{f \in L^2(G/K);\ f(kx) = f(x) \text{ for each } k \in K,\ x \in M\}.$$

Since $K \neq \{e\}$, we get $L^2(G/K) \neq L^2(G, K)$. If $V = V_K$ for each $\boldsymbol{C}$-spherical representation (ρ, V), then we have $L^2(G/K) = L^2(G, K)$ from Peter-Weyl theorem (see [6], p. 20). This is a contradiction. Q. E. D.

The manifold M is said to have irreducible linear isotropy group, if K acts on $T_o(M)$ irreducibly.

LEMMA 3.2. *We assume that M has irreducible linear isotropy group.*

(1) *The degree of any nontrivial $\boldsymbol{R}$-spherical representation of (G, K) is greater than or equal to* $\dim M+1$.

(2) *If* $\dim M \geqq 2$ *then the degree of any nontrivial $\boldsymbol{C}$-spherical representation is greater than or equal to* 2.

PROOF. (1) is obtained from Theorem of T. Takahashi [5]. But, for the sake of completeness, we give a proof. For each nontrivial $\boldsymbol{R}$-spherical representation (ρ, V) of (G, K) put

$$F: M = G/K \longrightarrow V; \qquad gK \longmapsto \rho(g)v ,$$

where $v \in V_K$ and $\|v\| = 1$ with respect to a G-invariant inner product on V. Then we can prove that F is an immersion in the same way in the proof of Theorem 3.1. Clearly the image $F(M)$ is contained in the unit hypersphere in V. So we get the conclusion. (2) is obtained from (1) and Lemma 2.2. Q.E.D.

POPOSITION 3.1. (1) *A compact Riemannian homogeneous space of dimension $\geqq 2$ with irreducible linear isotropy group admits an equivariant minimal immersion into an E-Grassmann manifold.*

(2) *There exists a nonconstant equivariant harmonic mapping from a compact Riemannian homogeneous space with non trivial isotropy group into an E-Grassmann manifold.*

PROOF. (1) Take (ρ, V) as in Lemma 3.1. Put $V_1 = V_K$ and $V_2 = V_K^{\perp}$ with respect to a G-invariant inner product on V. Then the assertion follows from Theorem 3.1.

(2) It is obtained from Theorem 3.1 and Lemma 3.1 in the same way of (1). Q. E. D.

If M is a compact Riemannian symmetric space, then (G, K) is a compact symmetric pair (see §4 for definition).

LEMMA 3.3. *Let* (G, K) *be a compact symmetric pair.*

(1) *The multiplicity of any* $\boldsymbol{C}$*-spherical representation of* (G, K) *equals to* 1.

(2) *The multiplicity of any* $\boldsymbol{R}$*-spherical representation of* (G, K) *equals to* 1 *or* 2.

(3) *Any* $\boldsymbol{H}$*-spherical representation of* (G, K) *is obtained from the extension of coefficient field of a* G-$\boldsymbol{C}$*-irreducible representation to* $\boldsymbol{H}$ *and the multiplicity is equal to* 1.

PROOF. (1) We refer to [6], p.104, Theorem 5.5. (2) is obtained from (1) and Lemma 2.1. (3) is obtained from (1) and Lemma 2.4. Q. E. D.

LEMMA 3.4. *Let* M *be a compact irreducible Riemannian symmetric space of dimension*$\geqq 2$. *Then the degree of any nontrivial* $\boldsymbol{H}$*-spherical representation of* (G, K) *is greater than or equal to* 2.

PROOF. It is obtained from Lemma 3.2(2) and Lemma 3.3(3). Q. E. D.

PROPOSITION 3.2. (1) *Let* M *be a compact irreducible Riemannian symmetric space of dimension*$\geqq 2$.

(i) M *admits an equivariant minimal immersion into a real projective space or* $G_{2,n}(\boldsymbol{R})$.

(ii) M *admits an equivariant minimal immersion into a complex projective space.*

(iii) M *admits an equivariant minimal immersion into a quaternion projective space.*

(2) *Let* M *be a compact Riemannian symmetric space with non trivial isotropy group.*

(i) *There exists a nonconstant equivariant harmonic mapping from* M *into a real projective space or* $G_{2,n}(\boldsymbol{R})$.

(ii) *There exists a nonconstant equivariant harmonic mapping from* M *into a complex projective space.*

(iii) *There exists a nonconstant equivariant harmonic mapping from* M *into a quaternion projective space.*

PROOF. (1) It is obtained from Theorem 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4 in the same way of Proposition 3.1.

(2) It is obtained from Theorem 3.1 and Lemma 3.3 in the same way of Proposition 3.1. Q. E. D.

§ 4. The Jacobi differential operators of equivariant minimal immersions.

Let G (resp. U) be a compact connected Lie group with Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{u}$ and K (resp. L) be a closed subgroup of G (resp. U) with Lie algebra $\mathfrak{k}$ (resp. $\mathfrak{l}$). Take a bi-invariant Riemannian metric metric $\langle\, ,\rangle$ on G (resp. U) and denote also by $\langle\, ,\rangle$ the induced $\mathrm{Ad}(K)$ (resp. $\mathrm{Ad}(L)$)-invariant inner product on $\mathfrak{m}=\mathfrak{k}^{\perp}$ (resp. $\mathfrak{p}=\mathfrak{l}^{\perp}$). Thus $M=(M^m, \langle\, ,\rangle)=G/K$ (resp. $N=(N^n, \langle\, ,\rangle)=U/L$) is a compact Riemannian homogeneous space with Lie subspace $(\mathfrak{m}, \langle , \rangle)$ (resp. $(\mathfrak{p}, \langle , \rangle)$). We denote by ∇ and R the covariant derivative and the Riemannian curvature tensor of M, respectively. We denote by $\bar{\nabla}$ and $\bar{R}$ for N in the same way. For each $X\in\mathfrak{g}$, we define a Killing vector field $X^*\in\mathfrak{X}(M)$ by

$$X^*_x=\frac{d}{dt}\exp tX\cdot x\,|_{t=0}\in T_x(M).$$

We have by the Koszul formula ([1], p. 48, (2))

$$(4.1)\qquad (\nabla_{X^*}Y^*)_o=-[X_{\mathfrak{m}}, Y_{\mathfrak{k}}]-\frac{1}{2}[X_{\mathfrak{m}}, Y_{\mathfrak{m}}]_{\mathfrak{m}} \qquad \text{for } X, Y\in\mathfrak{g}.$$

From the above equation, we have

$$(4.2)\qquad \nabla_{g_*v}X=\frac{d}{dt}(\exp(-t\,\mathrm{Ad}(g)v))_*X_{g\exp tv\cdot K}\,|_{t=0}+\frac{1}{2}g_*[v, g_*^{-1}X_{gK}]_{\mathfrak{m}}$$

$$\text{for } v\in\mathfrak{m},\ g\in G,\ X\in\mathfrak{X}(M),$$

$$(4.3)\qquad R(X, Y)Z=-\frac{1}{2}[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}}-\frac{1}{4}[[Y, Z]_{\mathfrak{m}}, X]_{\mathfrak{m}}$$

$$+\frac{1}{4}[[X, Z]_{\mathfrak{m}}, Y]_{\mathfrak{m}}-[[X, Y]_{\mathfrak{k}}, Z] \qquad \text{for } X, Y, Z\in\mathfrak{m}.$$

Let $F: M\to N$ be an equivariant isometric immersion. Then there exists an analytic homomorphism $\rho: G\to U$ with $\rho(K)\subset L$ such that $F(gK)=\rho(g)L$ for each $g\in G$. Let A and B denote the shape operator and the second fundamental form of F, respectively. Take an orthonormal basis $\{X_i\}_{1\leq i\leq p}$ of $\mathfrak{g}$ with $\{X_i\}_{1\leq i\leq m}\subset\mathfrak{m}$ and $\{X_j\}_{m+1\leq j\leq p}\subset\mathfrak{k}$.

PROPOSITION 4.1. (1)

$$B(X, Y)=-[(\rho_*X)_{\mathfrak{l}}, (\rho_*Y)_{\mathfrak{l}}]-\frac{1}{2}[(\rho_*X)_{\mathfrak{t}}, (\rho_*Y)_{\mathfrak{t}}]_{\mathfrak{t}}$$

$$+\frac{1}{2}(\rho_*([X, Y]_{\mathfrak{m}}))_{\mathfrak{t}} \qquad \textit{for } X, Y\in\mathfrak{m}\,.$$

(2) *F is minimal if and only if* $\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{l}}, (\rho_*X_i)_{\mathfrak{t}}]=0$.

PROOF. (1) is obtained from (2.2) and (4.1). (2) is clear from (1). Q.E.D.

From now on, we assume that F is minimal. We define symmetric linear transformations $\bar{R}_x$ and $\tilde{A}_x$ on the normal space $N_x(M)$ at x as follows:

$$\bar{R}_x(v)=\sum_{i=1}^{m}(\bar{R}(e_i, v)e_i)^{\perp}, \qquad \tilde{A}_x(v)=\sum_{1=1}^{m}B(e_i, A^v e_i)$$

for $v\in N_x(M)$, where $\{e_i\}_{1\leq i\leq m}$ is an orthonormal basis of $T_x(M)$. Clearly, we get $\tilde{A}=0$, if F is totally geodesic. Let $N(M)$ denote the normal bundle of M and $\Gamma(N(M))$ denote the vector space of all C^∞-sections of $N(M)$. Let Δ denote the negative of the rough Laplacian of the normal connection $\nabla^{\perp}$ of $N(M)$, that is,

$$-\Delta V=\sum_{1\leq i,j\leq m}g^{ij}\nabla^{\perp}_{E_i}\nabla^{\perp}_{E_j}V-\sum_{1\leq i,j\leq m}g^{ij}\nabla^{\perp}_{\nabla_{E_i}E_j}V \qquad \text{for } V\in\Gamma(N(M)),$$

where $\{E_i\}_{1\leq i\leq m}$ is a local frame field of M, $g_{ij}=\langle E_i, E_j\rangle$ and $(g^{ij})_{1\leq i,j\leq m}=(g_{ij})^{-1}_{1\leq i,j\leq m}$. Since the Jacobi differential operator

$$J=\Delta+\bar{R}-\tilde{A}$$

is a strongly elliptic linear differential operator, it has discrete eigenvalues:

$$Spec\,(J)=\{\lambda_1\leqq\lambda_2\leqq\cdots\to\infty\}\,.$$

The minimal immersion F is said to be stable if the second variation of the volume of F is nonnegative for every variation. The minimal immersion F is stable if and only if $\lambda_1\geqq 0$ (see [4], pp. 73–74). Since the orthogonal complement $\mathfrak{m}^{\perp}$ in $\mathfrak{p}$ is identified with $N_o(M)$ in a natural manner, we may consider $\bar{R}$ and $\tilde{A}$ as symmetric linear transformations on $\mathfrak{m}^{\perp}$.

The pair (U, L) is called a compact symmetric pair, if there exists an involutive antomorphism θ of U such that L lies between the identity component $(L_\theta)_0$ of L_θ and $L_\theta=\{l\in U\,;\,\theta(l)=l\}$. In this case, an Ad$(U)$ and θ-invariant inner product $\langle\,,\rangle$ on $\mathfrak{u}$ naturally induces a U-invariant Riemannian metric on N. Since $\mathfrak{p}=\{X\in\mathfrak{u}\,;\,\theta(X)=-X\}$, we have $[\mathfrak{p}, \mathfrak{p}]\subset\mathfrak{l}$.

LEMMA 4.1.

(1) $\bar{R}(v)=\sum_{i=1}^{m}[F_*X_i, [F_*X_i, v]_{\mathfrak{l}}]_{\mathfrak{m}^\perp}+(1/4)\sum_{i=1}^{m}[F_*X_i, [F_*X_i, v]_{\mathfrak{p}}]_{\mathfrak{m}^\perp}$ *for* $v\in\mathfrak{m}^\perp$.
If (U, L) *is a compact symmetric pair, then*

$$\bar{R}(v)=\sum_{i=1}^{m}[F_*X_i, [F_*X_i, v]]_{\mathfrak{m}^\perp} \qquad \text{for } v\in\mathfrak{m}^\perp .$$

(2) $\tilde{A}(v)=-\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{l}}+(1/2)(\rho_*X_i)_{\mathfrak{p}}, [(\rho_*X_i)_{\mathfrak{l}}+(1/2)(\rho_*X_i)_{\mathfrak{l}}, v]_{F_*\mathfrak{m}}]_{\mathfrak{m}^\perp}$
for $v\in\mathfrak{m}^\perp$.
If (U, L) *is a compact symmetric pair, then*

$$\tilde{A}(v)=-\sum_{i=1}^{m}[\rho_*X_i, [\rho_*X_i, v]_{F_*\mathfrak{m}}]_{\mathfrak{m}^\perp} \qquad \text{for } v\in\mathfrak{m}^\perp .$$

PROOF. (1) follows from (4.3).
(2) By using (4.1), we have

$$\begin{aligned}\langle A^v X_i, X_j\rangle &=\langle v, (\bar{\nabla}_{(\rho_*X_i)^*}(\rho_*X_j)^*)_o\rangle \\ &=-\left\langle v, [(\rho_*X_i)_{\mathfrak{p}}, (\rho_*X_j)_{\mathfrak{l}}]+\frac{1}{2}[(\rho_*X_i)_{\mathfrak{p}}, (\rho_*X_j)_{\mathfrak{p}}]_{\mathfrak{p}}\right\rangle .\end{aligned}$$

Hence we have by Proposition 4.1 (1)

$$\langle\tilde{A}(v), w\rangle=-\sum_{i=1}^{m}\left\langle\left[(\rho_*X_i)_{\mathfrak{l}}+\frac{1}{2}(\rho_*X_i)_{\mathfrak{p}}, \left[(\rho_*X_i)_{\mathfrak{l}}+\frac{1}{2}(\rho_*X_i)_{\mathfrak{p}}, v\right]_{F_*\mathfrak{m}}\right], w\right\rangle$$

for $v, w\in\mathfrak{m}^\perp$. Q. E. D.

The group K acts on $\mathfrak{m}^\perp$ by $\mathrm{Ad}(\rho(k))(k\in K)$. We denote by $(\mathrm{Ad}\circ\rho)^\perp$ this action of K on $\mathfrak{m}^\perp$. We identify $\Gamma(N(M))$ with

$$C^\infty(G\,;\,\mathfrak{m}^\perp)_K=\{\varphi\in C^\infty(G\,;\,\mathfrak{m}^\perp)\,;\,\varphi(gk)=(\mathrm{Ad}\circ\rho)^\perp(k^{-1})\varphi(g) \text{ for } g\in G,\ k\in K\}$$

by the following correspondence:

$$C^\infty(G\,;\,\mathfrak{m}^\perp)_K\ni\varphi\longmapsto\tilde{\varphi}\in\Gamma(N(M))\,;\,\tilde{\varphi}(gK)=\rho(g)_*\varphi(g) \qquad \text{for } g\in G\,.$$

We define an action L_x (resp. $\rho(x)_*$) of G on $C^\infty(G\,;\,\mathfrak{m}^\perp)_K$ (resp. $\Gamma(N(M))$) as follows:

$$(L_x\varphi)(g)=\varphi(x^{-1}g) \qquad \text{for } \varphi\in C^\infty(G\,;\,\mathfrak{m}^\perp)_K,\ x,\ g\in G\,,$$

$$(\rho(x)_*V)_{gK}=\rho(x)_*V_{x^{-1}gK} \qquad \text{for } V\in\Gamma(N(M)),\ x,\ g\in G\,.$$

The action L_x $(x\in G)$ on $C^\infty(G\,;\,\mathfrak{m}^\perp)_K$ corresponds to $\rho(x)_*$ on $\Gamma(N(M))$. We also denote by J, Δ and $\tilde{A}$ the operators on $C^\infty(G\,;\,\mathfrak{m}^\perp)_K$ corresponding to the operators J, Δ and $\tilde{A}$ on $\Gamma(N(M))$, respectively. Let $C=-\sum_{i=1}^{p}X_i^2$ denote the negative of the Casimir differential operator of G.

LEMMA 4.2.

$$\Delta\varphi=C\varphi+\sum_{i=m+1}^{p}[\rho_*X_i, [\rho_*X_i, \varphi]]-2\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{l}}, X_i\varphi]_{\mathfrak{m}^\perp}$$

$$-\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{l}}, [(\rho_*X_i)_{\mathfrak{l}}, \varphi]]_{\mathfrak{m}^\perp}-\tilde{A}(\varphi)-\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{p}}, X_i\varphi]_{\mathfrak{m}^\perp}$$

$$-\frac{1}{2}\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{l}}, [(\rho_*X_i)_{\mathfrak{p}}, \varphi]_{\mathfrak{p}}]_{\mathfrak{m}^\perp}-\frac{1}{2}\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{p}}, [(\rho_*X_i)_{\mathfrak{l}}, \varphi]]_{\mathfrak{m}^\perp}$$

$$-\frac{1}{4}\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{p}}, [(\rho_*X_i)_{\mathfrak{p}}, \varphi]_{\mathfrak{p}}]_{\mathfrak{m}^\perp} \qquad \textit{for } \varphi\in C^\infty(G\,;\,\mathfrak{m}^\perp)_K.$$

If (U, L) *is a compact symmetric pair, then*

$$\Delta\varphi=C\varphi+\sum_{i=m+1}^{p}[\rho_*X_i, [\rho_*X_i, \varphi]]-2\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{l}}, X_i\varphi]_{\mathfrak{m}^\perp}$$

$$-\sum_{i=1}^{m}[(\rho_*X_i)_{\mathfrak{l}}, [(\rho_*X_i)_{\mathfrak{l}}, \varphi]]_{\mathfrak{m}^\perp}-\tilde{A}(\varphi) \qquad \textit{for } \varphi\in C^\infty(G\,;\,\mathfrak{m}^\perp)_K\,.$$

PROOF. For $V=\tilde{\varphi}\in\Gamma(N(M))$, we have

$$-(\Delta V)(o)=\Big(\sum_{i=1}^{m}\bar{\nabla}_{(\rho_*X_i)^*}\bar{\nabla}_{(\rho_*X_i)^*}V\Big)^{\perp}(o)+\tilde{A}(V_o)\,.$$

Put $W_i=\bar{\nabla}_{(\rho_*X_i)^*}V$ $(1\leqq i\leqq m)$. Then we have by (2.1) and (4.2)

$$W_{i,\exp t\rho_*X_iL}=\frac{d}{ds}(\exp(-s\rho_*X_i))_*V_{\exp(t+s)\rho_*X_iL|s=0}$$
$$+(\exp t\rho_*X_i)_*[(\rho_*X_i)_{\mathfrak{l}}, (\exp(-t\rho_*X_i)_*V_{\exp t\rho_*X_iL}]$$
$$+\frac{1}{2}(\exp t\rho_*X_i)_*[(\rho_*X_i)_{\mathfrak{p}}, (\exp(-t\rho_*X_i))_*V_{\exp t\rho_*X_iL}]_{\mathfrak{p}}.$$

In particular

$$W_{i,o}=\frac{d}{ds}(\exp(-s\rho_*X_i))_*V_{\exp s\rho_*X_iL|s=0}+[(\rho_*X_i)_{\mathfrak{l}}, V_o]+\frac{1}{2}[(\rho_*X_i)_{\mathfrak{p}}, V_o]_{\mathfrak{p}}.$$

Thus we have by (2.1) and (4.2)

$$(\bar{\nabla}_{(\rho_*X_i)^*}\bar{\nabla}_{(\rho_*X_i)^*}V)(o)$$
$$=(\bar{\nabla}_{\rho_*X_i)^*}W_i)_o$$
$$=\frac{d}{dt}(\exp(-t\rho_*X_i))_*W_{i,\exp t\rho_*X_iL|t=0}+[(\rho_*X_i)_{\mathfrak{l}}, W_{i,o}]+\frac{1}{2}[(\rho_*X_i)_{\mathfrak{p}}, W_{i,o}]_{\mathfrak{p}}$$
$$=\frac{\partial^2}{\partial t\partial s}(\exp(-(t+s)\rho_*X_i))_*V_{\exp(t+s)\rho_*X_iL|t=s=0}$$
$$+2\Big[(\rho_*X_i)_{\mathfrak{l}}, \frac{d}{dt}(\exp(-t\rho_*X_i))_*V_{\exp t\rho_*X_iL|t=0}\Big]+[(\rho_*X_i)_{\mathfrak{l}}, [(\rho_*X_i)_{\mathfrak{l}}, V_o]]$$

$$+\left[(\rho_*X_{ii})_{\mathfrak{p}},\ \frac{d}{dt}(\exp(-t\rho_*X_i))_*V_{\exp t\rho_*X_iL|t=0}\right]_{\mathfrak{p}}+\frac{1}{2}[(\rho_*X_i)_{\mathfrak{l}},\ [(\rho_*X_i)_{\mathfrak{p}},\ V_o]_{\mathfrak{p}}]$$

$$+\frac{1}{2}[(\rho_*X_i)_{\mathfrak{p}},\ [(\rho_*X_i)_{\mathfrak{l}},\ V_o]]_{\mathfrak{p}}+\frac{1}{4}[(\rho_*X_i)_{\mathfrak{p}},\ [(\rho_*X_i)_{\mathfrak{p}},\ V_o]_{\mathfrak{p}}]_{\mathfrak{p}}\,.$$

So we get

$$-(\Delta\varphi)(e)=\Big(\sum_{i=1}^m X_i^2\varphi\Big)(e)+2\sum_{i=1}^m[(\rho_*X_i)_{\mathfrak{l}},\ (X_i\varphi)(e)]_{\mathfrak{m}^\perp}$$

$$+\sum_{i=1}^m[(\rho_*X_i)_{\mathfrak{l}},\ [(\rho_*X_i)_{\mathfrak{l}},\ \varphi(e)]]_{\mathfrak{m}^\perp}+\tilde{A}(\varphi(e))+\sum_{i=1}^m[(\rho_*X_i)_{\mathfrak{p}},\ (X_i\varphi)(e)]_{\mathfrak{m}^\perp}$$

$$+\frac{1}{2}\sum_{i=1}^m[(\rho_*X_i)_{\mathfrak{l}},\ [(\rho_*X_i)_{\mathfrak{p}},\ \varphi(e)]_{\mathfrak{p}}]_{\mathfrak{m}^\perp}+\frac{1}{2}\sum_{i=1}^m[(\rho_*X_i)_{\mathfrak{p}},\ [\rho_*X_i)_{\mathfrak{l}},\ \varphi(e)]]_{\mathfrak{m}^\perp}$$

$$+\frac{1}{4}\sum_{i=1}^m[(\rho_*X_i)_{\mathfrak{p}},\ [(\rho_*X_i)_{\mathfrak{p}},\ \varphi(e)]_{\mathfrak{p}}]_{\mathfrak{m}^\perp}\,.$$

Since $\Delta L_x=L_x\Delta$ for $x\in G$ by the equivariance of F, we get the conclusion.

Q. E. D.

We define operators $J_i\colon C^\infty(G\,;\,\mathfrak{m}^\perp)_K\to C^\infty(G\,;\,\mathfrak{m}^\perp)_K$ $(i=1,\ 2)$ by

$$J_1\varphi=\sum_{i=1}^p[\rho_*X_i,\ X_i\varphi]_{\mathfrak{m}^\perp}+\sum_{i=1}^p[\rho_*X_i,\ [\rho_*X_i,\ \varphi]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp}\,,$$

$$J_2\varphi=\sum_{i=1}^p[(\rho_*X_i)_{\mathfrak{p}},\ X_i\varphi]_{\mathfrak{m}^\perp}+\sum_{i=1}^p[(\rho_*X_i)_{\mathfrak{p}},\ [\rho_*X_i,\ \varphi]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp}$$

for $\varphi\in C^\infty(G\,;\,\mathfrak{m}^\perp)_K$.

REMARK. It follows that $\sum_{i=1}^p[\rho_*X_i,\ X_i\varphi]_{\mathfrak{m}^\perp}$, $\sum_{i=1}^p[\rho_*X_i,\ [\rho_*X_i,\varphi]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp}$, $\sum_{i=1}^p[(\rho_*X_i)_{\mathfrak{p}},\ X_i\varphi]_{\mathfrak{m}^\perp}$ and $\sum_{i=1}^p[(\rho_*X_i)_{\mathfrak{p}},\ [\rho_*X_i,\ \varphi]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp}\in C^\infty(G\,;\,\mathfrak{m}^\perp)_K$ for $\varphi\in C^\infty(G\,;\,\mathfrak{m}^\perp)_K$. Moreover each of the above four operators is commutative with L_x for $x\in G$. If $(U,\ L)$ is a compact symmetric pair, then $J_2=0$. ■

THEOREM 4.1.

(1) $J\varphi=C\varphi-2J_1\varphi+[\sum_{i=1}^p(\mathrm{ad}\,\rho_*X_i)^2\varphi]_{\mathfrak{m}^\perp}+J_2\varphi-\sum_{i=1}^p[\rho_*X_i,\ [(\rho_*X_i)_{\mathfrak{p}},\ \varphi]_{F_*\mathfrak{m}}]_{\mathfrak{m}^\perp}+(1/2)\sum_{i=1}^p[(\rho_*X_i)_{\mathfrak{p}},\ [(\rho_*X_{ii})_{\mathfrak{p}},\ \varphi]_{F_*\mathfrak{m}}]_{\mathfrak{m}^\perp}$ *for* $\varphi\in C^\infty(G\,;\,\mathfrak{m}^\perp)_K$.

(2) $J_1=0$ *if and only if* $[\rho_*X,\ v]_{\mathfrak{m}^\perp}=0$ *for* $X\in\mathfrak{m}$, $v\in\mathfrak{m}^\perp$.

(3) $J_2=0$ *if and only if* $[(\rho_*X)_{\mathfrak{p}},\ v]_{\mathfrak{m}^\perp}=0$ *for* $X\in\mathfrak{m}$, $v\in\mathfrak{m}^\perp$.

(4) *If* $(U,\ L)$ *is a compact symmetric pair, then*

$$J\varphi=C\varphi-2J_1\varphi+\Big[\sum_{i=1}^p(\mathrm{ad}\ \rho_*X_i)^2\varphi\Big]_{\mathfrak{m}^\perp}\quad for\ \ \varphi\in C^\infty(G\,;\,\mathfrak{m}^\perp)_K\,.$$

PROOF. (1) follows from Lemma 4.1, Lemma 4.2 and minimal condition

(Proposition 4.1(2)). (2) and (3) are obtained in the same way of [3], I, p. 138, Proposition 4.2.2. (4) follows from (1). Q. E. D.

REMARK. It follows that $C\varphi$, $[\sum_{i=1}^{p}(\mathrm{ad}\,\rho_*X_i)^2\varphi]_{\mathfrak{m}^\perp}$, $\sum_{i=1}^{p}[\rho_*X_i, [(\rho_*X_i)_{\mathfrak{p}}, \varphi]_{F_*\mathfrak{m}}]_{\mathfrak{m}^\perp}$ and $\sum_{i=1}^{p}[(\rho_*X_i)_{\mathfrak{p}}, [(\rho_*X_i)_{\mathfrak{p}}, \varphi]_{F_*\mathfrak{m}}]_{\mathfrak{m}^\perp}\in C^\infty(G;\mathfrak{m}^\perp)_K$ for $\varphi\in C^\infty(G;\mathfrak{m}^\perp)_K$. Moreover each of the above four operators is commutative with L_x for $x\in G$. ■

Let $D(G)$ be the set of equivalence classes of finite dimensional $\boldsymbol{C}$-irreducible representations of G. Let $c_\sigma(\geqq 0)$ be the eigenvalue of the negative of the Casimir operator of $(\sigma, W)\in D(G)$. (By the formula of Freudenthal, we can determine c_σ (see [6], p. 205).) For $(\sigma, W)\in D(G)$, put

$$(W^*\otimes(\mathfrak{m}^\perp)^C)_0=\{\alpha\in W^*\otimes(\mathfrak{m}^\perp)^C\,;\,(\sigma^*\otimes(\mathrm{Ad}\circ\rho)^\perp)(k)\alpha=\alpha \text{ for } k\in K\}.$$

Put

$$D(G\,;K,(\mathrm{Ad}\circ\rho)^\perp)=\{(\sigma, W)\in D(G)\,;\,(W^*\otimes(\mathfrak{m}^\perp)^C)_0\neq\{0\}\}.$$

For $(\sigma, W)\in D(G\,;K,(\mathrm{Ad}\circ\rho)^\perp)$, we define a symmetric linear mapping $J_\sigma\in \mathrm{End}((W^*\otimes\mathfrak{m}^\perp)^C)_0)$ as follows:

$$\begin{aligned}J_\sigma=c_\sigma 1-2&\Big\{\sum_{i=1}^{p}\sigma^*(X_i)\otimes[\rho_*X_i, *]_{\mathfrak{m}^\perp}+1_{W^*}\otimes\sum_{i=1}^{p}[\rho_*X_i, [\rho_*X_i, *]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp}\Big\}\\ &+1_{W^*}\otimes\Big[\sum_{i=1}^{p}(\mathrm{ad}\,\rho_*X_i)^2*\Big]_{\mathfrak{m}^\perp}\\ &+\Big\{\sum_{i=1}^{p}\sigma^*(X_i)\otimes[(\rho_*X_i)_{\mathfrak{p}}, *]_{\mathfrak{m}^\perp}+1_{W^*}\otimes\sum_{i=1}^{p}[(\rho_*X_i)_{\mathfrak{p}}, [\rho_*X_i, *]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp}\Big\}\\ &+1_{W^*}\otimes\Big\{\frac{1}{2}\sum_{i=1}^{p}[(\rho_*X_i)_{\mathfrak{p}}, [(\rho_*X_i)_{\mathfrak{p}}, *]_{F_*\mathfrak{m}}]_{\mathfrak{m}^\perp}-\sum_{i=1}^{p}[\rho_*X_i, [(\rho_*X_i)_{\mathfrak{p}}, *]_{F_*\mathfrak{m}}]_{\mathfrak{m}^\perp}\Big\}.\end{aligned}$$

Clearly, if (U, L) is a compact symmetric pair, then

$$\begin{aligned}J_\sigma=c_\sigma 1-2&\Big\{\sum_{i=1}^{p}\sigma^*(X_i)\otimes[\rho_*X_i, *]_{\mathfrak{m}^\perp}+1_{W^*}\otimes\sum_{i=1}^{p}[\rho_*X_i, [\rho_*X_i, *]_{\mathfrak{m}^\perp}]_{\mathfrak{m}^\perp}\Big\}\\ &+1_{W^*}\otimes\Big[\sum_{i=1}^{p}(\mathrm{ad}\,\rho_*X_i)^2*\Big]_{\mathfrak{m}^\perp}.\end{aligned}$$

By virtue of the Peter-Weyl theorem for homogeneous vector bundles, the problem of computing the spectra of J is reduced to the eigenvalue problem of the linear mapping J_σ with $(\sigma, W)\in D(G\,;K,(\mathrm{Ad}\circ\rho)^\perp)$ (see [3], I, §5).

THEOREM 4.2. *For $(\sigma, W)\in D(G\,;K,(\mathrm{Ad}\circ\rho)^\perp)$, let $\{\lambda_{\sigma;1},\cdots,\lambda_{\sigma;m_\sigma}\}$ be the eigenvalues of J_σ, where $m_\sigma=\dim(W^*\otimes(\mathfrak{m}^\perp)^C)_0$. Then*

$$Spec(J)=\bigcup_{(\sigma,W)\in D(G;K,(\mathrm{Ad}\circ\rho)^\perp)}\{\underbrace{\lambda_{\sigma;1},\cdots,\lambda_{\sigma;1}}_{d_\sigma},\cdots,\underbrace{\lambda_{\sigma;m_\sigma},\cdots,\lambda_{\sigma;m_\sigma}}_{d_\sigma}\},$$

where $d_\sigma = \dim W$.

§ 5. Equivariant minimal immersions of S^2 into Grassmann manifolds.

In this section, we denote by G (resp. K) the special unitary group $SU(2)$ of degree 2 (resp. the closed subgroup $S(U(1)\times U(1))$ of $SU(2)$). Basic notations are same in § 4. We define an Ad(G)-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ by

$$\langle X, Y\rangle = -2\,\mathrm{Tr}\,(XY) \qquad \text{for } X, Y \in \mathfrak{g}\,.$$

We also denote by $\langle\ ,\ \rangle$ the induced G-invariant Riemannian metric on $S^2 = G/K$. Then the Riemannian manifold $(S^2, \langle\ ,\ \rangle)$ is of constant curvature 1. We choose an orthonomal basis $\{e_i\}_{1\le i\le 3}$ of $\mathfrak{g}$ as follows:

$$e_1 = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}, \qquad e_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad e_3 = \frac{1}{2}\begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}.$$

First we write down all G-E-irreducible representations:

LEMMA 5.1. *G-E-irreducible representation (ρ, V) is one of the following:*
(1) *The case of $E=\boldsymbol{C}$:*
There exists an orthonormal basis $\{f_k\}_{0\le k\le n}$ of V such that

$$\rho(e_3)f_k = \frac{\sqrt{-1}}{2}(-n+2k)f_k\,,$$

$$\rho(e_1)f_k = \frac{\sqrt{-1}}{2}\{\sqrt{(n-k)(k+1)}\,f_{k+1} + \sqrt{k(n-k+1)}\,f_{k-1}\}\,,$$

$$\rho(e_2)f_k = \frac{1}{2}\{-\sqrt{(n-k)(k+1)}\,f_{k+1} + \sqrt{k(n-k+1)}\,f_{k-1}\}$$

for $0\le k\le n$.

(2) *The case of $E=\boldsymbol{R}$:*
(2-a) *There exists an orthonormal basis $\{h_0\}\cup\{f_k, g_k\}_{1\le k\le n}$ of V such that*

$$\rho(e_3)h_0 = 0\,,$$

$$\rho(e_3)f_k = k g_k\,,$$

$$\rho(e_3)g_k = -k f_k\,,$$

$$\rho(e_1)h_0 = \sqrt{\frac{n(n+1)}{2}}\,g_1\,,$$

$$\rho(e_1)f_k = \frac{1}{2}\sqrt{(n-k)(n+k+1)}\,g_{k+1} + \frac{1}{2}\sqrt{(n+k)(n-k+1)}\,g_{k-1}\,,$$

$$\rho(e_1)g_k=-\frac{1}{2}\sqrt{(n-k)(n+k+1)}f_{k+1}-\frac{1}{2}\sqrt{(n+k)(n-k+1)}f_{k-1},$$

$$\rho(e_2)h_0=-\sqrt{\frac{n(n+1)}{2}}f_1,$$

$$\rho(e_2)f_k=-\frac{1}{2}\sqrt{(n-k)(n+k+1)}f_{k+1}+\frac{1}{2}\sqrt{(n+k)(n-k+1)}f_{k-1},$$

$$\rho(e_2)g_k=-\frac{1}{2}\sqrt{(n-k)(n+k+1)}g_{k+1}+\frac{1}{2}\sqrt{(n+k)(n-k+1)}g_{k-1}$$

for $1\leqq k\leqq n$, *where we put* $f_0=\sqrt{2}h_0$, $g_0=0$.

(2-b) *There exists an orthonormal basis* $\{f_k, g_k\}_{0\leq k\leq 2n-1}$ *of* V *such that*

$$\rho(e_3)f_k=\frac{1}{2}(-2n+1+2k)g_k,$$

$$\rho(e_3)g_k=-\frac{1}{2}(-2n+1+2k)f_k,$$

$$\rho(e_1)f_k=\frac{1}{2}\sqrt{(2n-1-k)(k+1)}\,g_{k+1}+\frac{1}{2}\sqrt{k(2n-k)}\,g_{k-1},$$

$$\rho(e_1)g_k=-\frac{1}{2}\sqrt{(2n-1-k)(k+1)}f_{k+1}-\frac{1}{2}\sqrt{k(2n-k)}f_{k-1},$$

$$\rho(e_2)f_k=-\frac{1}{2}\sqrt{(2n-1-k)(k+1)}f_{k+1}+\frac{1}{2}\sqrt{k(2n-k)}f_{k-1},$$

$$\rho(e_2)g_k=-\frac{1}{2}\sqrt{(2n-1-k)(k+1)}g_{k+1}+\frac{1}{2}\sqrt{k(2n-k)}g_{k-1}$$

for $0\leqq k\leqq 2n-1$.

(3) *The case of* $E=\boldsymbol{H}$:

(3-a) *There exists an orthonormal basis* $\{f_k\}_{0\leq k\leq n-1}$ *of* V *such that*

$$\rho(e_3)f_k=\frac{\boldsymbol{i}}{2}(-2n+1+2k)f_k,$$

$$\rho(e_1)f_k=\frac{\boldsymbol{i}}{2}(1-\delta_{k,n-1})\{\sqrt{(2n-1-k)(k+1)}f_{k+1}+\sqrt{k(2n-k)}f_{k-1}\}$$
$$+\frac{\delta_{k,n-1}}{2}\{n(-1)^{n+1}\boldsymbol{k}f_{n-1}+\boldsymbol{i}\sqrt{(n-1)(n+1)}f_{n-2}\},$$

$$\rho(e_2)f_k=\frac{1}{2}(1-\delta_{k,n-1})\{-\sqrt{(2n-1-k)(k+1)}f_{k+1}+\sqrt{k(2n-k)}f_{k-1}\}$$
$$+\frac{\delta_{k,n-1}}{2}\{-n(-1)^{n+1}\boldsymbol{j}f_{n-1}+\sqrt{(n-1)(n+1)}f_{n-2}\}$$

for $0 \leqq k \leqq n-1$.

(3-b) *There exists an orthonormal basis* $\{f_k\}_{0\leq k\leq 2n}$ *of* V *such that*

$$\rho(e_3)f_k = \boldsymbol{i}(-n+k)f_k\,,$$

$$\rho(e_1)f_k = \frac{\boldsymbol{i}}{2}\{\sqrt{(2n-k)(k+1)}\,f_{k+1} + \sqrt{k(2n-k+1)}\,f_{k-1}\},$$

$$\rho(e_2)f_k = \frac{1}{2}\{-\sqrt{(2n-k)(k+1)}\,f_{k+1} + \sqrt{k(2n-k+1)}\,f_{k-1}\}$$

for $0 \leqq k \leqq 2n$, *where we denote the differential representation of the representation* ρ *of* G *by the same symbol* ρ.

PROOF. (1) is obtained from Theorem 1.3, p. 599, [2]. (2) is obtained from (1), Lemma 2.1 and Lemma 2.2:

(2-a) When $n=2m$ in (1), put

$$p_k = \begin{cases} f_{k+m} - f_{-k+m} & (k=\text{odd}), \\ f_{k+m} + f_{-k+m} & (k=\text{even}), \end{cases}$$

$$q_k = \begin{cases} \sqrt{-1}(f_{k+m} + f_{-k+m}) & (k=\text{odd}), \\ \sqrt{-1}(f_{k+m} - f_{-k+m}) & (k=\text{even}), \end{cases} \qquad h_0 = \sqrt{2}f_m = \frac{1}{\sqrt{2}}p_0.$$

and

$$W = \boldsymbol{R}h_0 + \sum_{k=1}^{m}(\boldsymbol{R}p_k + \boldsymbol{R}q_k).$$

Then W is a G-$\boldsymbol{R}$-irreducible representation. We rewrite $m=n$, $p_k=f_k$, $q_k=g_k$ and $W=V$. We get (2-a).

(2-b) When $n=2m-1$ in (1), $((\rho_{2m-1})_{\boldsymbol{R}}, (V_{2m-1})_{\boldsymbol{R}})$ is a G-$\boldsymbol{R}$-irreducible representation. Put $g_k=\sqrt{-1}f_k$. We rewrite $m=n$. We get (2-b).

(3) is obtained from (1), Lemma 2.3 and Lemma 2.4:

(3-a) When $n=2m-1$ in (1), we define a conjugate G-linear mapping $\mathfrak{J}$ by

$$\mathfrak{J}f_k = (-1)^k f_{2m-1-k}\,.$$

Since $\mathfrak{J}^2=-1$, (ρ_{2m-1}, V_{2m-1}) is considered as a G-$\boldsymbol{H}$-representation (σ_m, W_m). $\{f_k\}_{0\leq k\leq m-1}$ is an orthonormal basis of (σ_m, W_m) and $f_{2m-1-k}=(-1)^k\boldsymbol{j}f_k$. We rewrite $m=n$ and $(\sigma_n, W_n)=(\rho_n, V_n)$. We get (3-a).

(3-b) When $n=2m$ in (1), $((\rho_{2m})^{\boldsymbol{H}}, (V_{2m})^{\boldsymbol{H}})$ is a G-$\boldsymbol{H}$-irreducible representation. We rewrite $m=n$. We get (3-b). Q. E. D.

For G-E-irreducible representation (ρ, V), put

$$X=\begin{cases}\{0, 1, \cdots, n\} & (\text{if } (\rho, V) \text{ is type } (1)),\\ \{1, \cdots, n\} & (\text{if } (\rho, V) \text{ is type } (2\text{-a})),\\ \{0, 1, \cdots, 2n-1\} & (\text{if } (\rho, V) \text{ is type } (2\text{-b})),\\ \{0, 1, \cdots, n-1\} & (\text{if } (\rho, V) \text{ is type } (3\text{-a})),\\ \{0, 1, \cdots, 2n\} & (\text{if } (\rho, V) \text{ is type } (3\text{-b})).\end{cases}$$

For subsets $P(\neq\emptyset)$, $Q(\neq\emptyset)$ of X with $X=P\cup Q$ (disjoint union), put

$$V_P=\begin{cases}\sum_{p\in P}\boldsymbol{C}f_p & (\text{if } (\rho, V) \text{ is type } (1)),\\ \boldsymbol{R}h_0+\sum_{p\in P}(\boldsymbol{R}f_p+\boldsymbol{R}g_p) & (\text{if } (\rho, V) \text{ is type } (2\text{-a})),\\ \sum_{p\in P}(\boldsymbol{R}f_p+\boldsymbol{R}g_p) & (\text{if } (\rho, V) \text{ is type } (2\text{-b})),\\ \sum_{p\in P}\boldsymbol{H}f_p & (\text{if } (\rho, V) \text{ is type } (3\text{-a}) \text{ or } (3\text{-b})),\end{cases}$$

$$V_Q=\begin{cases}\sum_{q\in R}\boldsymbol{C}f_q & (\text{if } (\rho, V) \text{ is type } (1)),\\ \sum_{q\in Q}(\boldsymbol{R}f_q+\boldsymbol{R}g_q) & (\text{if } (\rho, V) \text{ is type } (2\text{-a})),\\ \sum_{q\in Q}(\boldsymbol{R}f_q+\boldsymbol{R}g_q) & (\text{if } (\rho, V) \text{ is type } (2\text{-b})),\\ \sum_{q\in Q}\boldsymbol{H}f_q & (\text{if } (\rho, V) \text{ is type } (3\text{-a}) \text{ or } (3\text{-b})),\end{cases}$$

$$a=\dim_E V_P\,, \qquad b=\dim_Q V_E\,.$$

Then $V_P(\neq\{0\})$ and $V_Q(\neq\{0\})$ are K-E-invariant subspaces of V and

$$V=V_P+V_Q \text{ (direct sum)}, \qquad \mathrm{Hom}_K(V_P, V_Q)=\{0\}.$$

Thus

$$F: S^2=G/K \longrightarrow G_{a,b}(E)=U(a+b)/U(a)\times U(b);$$
$$gK \longmapsto \rho(g)U(a)\times U(b) \quad \text{for } g\in G$$

is a full minimal immersion by Proposition 3.1. Note that S^2, $G_{m,2}(\boldsymbol{R})$, $G_{mn}(\boldsymbol{C})$ are Hermitian symmetric spaces.

PROPOSITION 5.1.

(1) *The case of which* (ρ, V) *is type* (1):

F is totally geodesic if and only if $P=\{even\}$, $Q=\{odd\}$ (*or* $P=\{odd\}$, $Q=\{even\}$),

F is a Kähler immersion if and only if $P=\{0, 1, \cdots, k\}$, $Q=\{k+1, \cdots, n\}$ (*or* $Q=\{0, 1, \cdots, k\}$, $P=\{k+1, \cdots, n\}$).

(2) (2-a) *The case of which* (ρ, V) *is type* (2-a):

F is totally geometric if and only if $P=\{even\}$, $Q=\{odd\}$, *F is a Kähler immersion if and only if* $Q=\{n\}$,

(2-b) *The case of which* (ρ, V) *is type* (2-b):
F is totally geodesic if and only if $P=\{even\}$, $Q=\{odd\}$ (*or* $P=\{odd\}$, $Q=\{even\}$),
F is a Kähler immersiou if and only if $P=\{0\}$ *or* $\{2n-1\}$ (*or* $Q=\{0\}$ *or* $\{2n-1\}$).

(3) (3-a) *The case of which* (ρ, V) *is type* (3-a):
F is not totally geodesic,
(3-b) *The case of which* (ρ, V) *is type* (3-b):
F is totally geodesic if and only if $P=\{even\}$, $Q=\{odd\}$ (*or* $P=\{odd\}$, $Q=\{even\}$).

PROOF. It follows from Proposition 4.1(1). Q.E.D.

For example, when (ρ, V) is type (2-a) and $P=\{1\}$, $Q=\{2\}$, we calculate $Spec(J)$ by using Theorem 4.2. In this case, since F is a Kähler immersion, F is stable (see [4], p. 76, Theorem 3.5.1).

THEOREM 5.1. *The spectra of J is given as follows:*

$$Spec(J)=$$
$$\{(n+3)(n-2)\pm\sqrt{6(n+3)(n-2)}\ (\textit{with multiplicity}\ 2(2n+1));\ n=2, 3, \cdots\}.$$

PROOF. In this case,

$$V_P=\boldsymbol{R}h_0+\boldsymbol{R}f_1+\boldsymbol{R}g_1, \qquad V_Q=\boldsymbol{R}f_2+\boldsymbol{R}g_2.$$

The expression matrix of $\rho(e_i)$ $(1\leqq i\leqq 3)$ with respect to an orthonormal basis $\{h_0, f_1, g_1, f_2, g_2\}$ of V is as follows:

$$\rho(e_1)=\begin{pmatrix} 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \sqrt{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(e_2)=\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\rho(e_3)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

Hence we have

$$\mathfrak{m}^{\perp}=\left\{v(x, y, z, w)=\begin{pmatrix} 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & w \\ 0 & 0 & 0 & w & -z \\ -x & -z & -w & 0 & 0 \\ -y & -w & z & 0 & 0 \end{pmatrix}; x, y, z, w\in \boldsymbol{R}\right\}$$

and

$$[\rho(e_3), v(x, y, z, w)]=v(-2y, 2x, -3w, 3z).$$

Put

$$\mathfrak{m}_1^{\perp}=\{v(x, y, 0, 0); x, y\in \boldsymbol{R}\}, \qquad \mathfrak{m}_2^{\perp}=\{v(0, 0, z, w); z, w\in \boldsymbol{R}\}.$$

Then we have

$$\mathfrak{m}=\mathfrak{m}_1^{\perp}+\mathfrak{m}_2^{\perp} \quad (K\text{-irreducible decomposition})$$

and

$$\sum_{i=1}^{3}[\rho(e_i), [\rho(e_i), *]_{\mathfrak{m}^{\perp}}]_{\mathfrak{m}^{\perp}}=\begin{cases} -7id & \text{on } \mathfrak{m}_1^{\perp}, \\ -12id & \text{on } \mathfrak{m}_2^{\perp}. \end{cases}$$

Put

$$v_{\pm}=\sqrt{2}v(1, \pm\sqrt{-1}, 0, 0)\in(\mathfrak{m}_1^{\perp})^C, \quad w_{\pm}=\sqrt{2}v(0, 0, 1, \pm\sqrt{-1})\in(\mathfrak{m}_2^{\perp})^C.$$

Then

$$[\rho(e_3), v_{\pm}]=\mp 2\sqrt{-1}v_{\pm}, \qquad [\rho(e_3), w_{\pm}]=\mp 3\sqrt{-1}w_{\pm}.$$

Hence we have

$$D(G, K, (\mathrm{Ad}\circ\rho)^{\perp})=\{(\rho_{2n}, V_{2n}); n=2, 3, \cdots\},$$

where we denote (ρ_{2n}, V_{2n}) the complex irreducible representation of G of degree $2n+1$. The expression matrix of $\sum_{i=1}^{3}\rho_{2n}^{*}(e_i)\otimes[\rho(e_i), *]_{\mathfrak{m}^{\perp}}$ with respect to an orthogonal basis $\{f_{n+2}^{*}\otimes v_{-}, f_{n+3}^{*}\otimes w_{-}, f_{n-2}^{*}\otimes v_{+}, f_{n-3}^{*}\otimes w_{-}\}$ of $(V_{2n}^{*}\otimes(\mathfrak{m}^{\perp})^C)_0$ is as follows:

$$\sum_{i=1}^{3}\rho_{2n}^{*}(e_i)\otimes[\rho(e_i), *]_{\mathfrak{m}^{\perp}}=\begin{pmatrix} 4 & \alpha & 0 & 0 \\ \alpha & 9 & 0 & 0 \\ 0 & 0 & 4 & -\alpha \\ 0 & 0 & -\alpha & 9 \end{pmatrix},$$

where we put $\alpha=(1/2)\sqrt{6(n+3)(n-2)}$. Since the eigenvalue of the Casimir operator of (ρ_{2n}, V_{2n}) is $n(n+1)$, we have

$$J_{\rho_{2n}}=\{n(n+1)-12\}id-2\begin{pmatrix}-3 & \alpha & 0 & 0\\ \alpha & -3 & 0 & 0\\ 0 & 0 & -3 & -\alpha\\ 0 & 0 & -\alpha & -3\end{pmatrix}.$$

Hence we get the conclusion by Theorem 4.2. Q. E. D.

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