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RING EXTENSIONS AND ENDOMORPHISM RINGS OF A MODULE

By

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In this paper we consider two conditions of a module related with a ring extension. The one is

$$(T)\quad M\otimes_S R\langle\oplus M\oplus\cdots\oplus M$$

that is, $R\supset S$ is a ring extension and M a right R-module such that $M\otimes_S R$ is a direct summand of a finite direct sum of M as a right R-module. The second is

$$(H)\quad \mathrm{Hom}({}_QP,\ {}_QM)\langle\oplus M\oplus\cdots\oplus M$$

that is, $P\supset Q$ is a ring extension and M a left P-module such that $\mathrm{Hom}({}_QP, {}_QM)$ is a direct summand of a finite direct sum of M as a left P-module. In § 1 we show that above two conditions are closely related with each other when $P=\mathrm{End}(M_S)$, $Q=\mathrm{End}(M_R)$ and when $R=\mathrm{End}({}_QM)$, $S=\mathrm{End}({}_PM)$, Propositions 1.1 and 1.2. In § 2 we apply the results in § 1 to H-separable extensions. We can give alternative proof of Sugano's theorem on H-separable extensions in [4]. It is easily seen that under the former condition (T) if M is a generator as an S-module then M is a generator as an R-module. Similarly we see that under the latter condition (H) if M is a Q-cogenerator then M is a P-cogenerator. But it seems too strong. In § 3 we treat about relative (co-)generators. Throughout this paper all rings have an identity, subrings contain this element, modules are unitary.

1. On conditions (T) and (H).

Let $R\supset S$ be a ring extension and M a right R-module. Let P and Q be the endomorphism rings of M as an S-module and as an R-module respectively, which operate on left side of M. Assume now the condition

$$(T)\quad M\otimes_S R\langle\oplus M\oplus\cdots\oplus M.$$

Then there exist R-homomorphisms f_i: $M\otimes_S R\to M$ and g_i: $M\to M\otimes_S R$ such

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that $\sum g_i \circ f_i = 1_{M\otimes_S R}$, the identity map of $M\otimes_S R$. Now applying Theorem 1.2 in [2] to (T), we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(M_R, M\otimes_S R_R)\otimes_Q M & \longrightarrow & \mathrm{Hom}({}_Q\mathrm{Hom}(M\otimes_S R_R, M_R), {}_Q Q)\otimes_Q M \\ \downarrow & & \downarrow \\ M\otimes_S R & \longrightarrow & \mathrm{Hom}({}_Q\mathrm{Hom}(M\otimes_S R_R, M_R), {}_Q M). \end{array}$$

All arrows are (Ω, R)-isomorphisms where Ω is $\mathrm{End}(M\otimes {}_S R_R)$. Note that $\mathrm{Hom}(M\otimes_S R_R, M_R)\cong \mathrm{Hom}(M_S, M_S)=P$ and P may consider as a subring of Ω by $p(x\otimes r)=p(x)\otimes r$ for $p\in P$, $x\otimes r\in M\otimes_S R$, in fact, P is a right P-direct summand of Ω. Therefore the maps in the above diagram are all (P, Q)-isomorphisms and in particular we have

$$M\otimes_S R\cong \mathrm{Hom}({}_Q P, {}_Q M) \tag{1.1}$$

as left P- and right R-modules. On the other hand from (T) we have

$$\begin{aligned} &\mathrm{Hom}(M\otimes_S R_R, M_R) \oplus \mathrm{Hom}((\oplus M)_R, M_R)\cong \oplus \mathrm{Hom}(M_R, M_R) \\ &= Q\oplus \cdots \oplus Q \end{aligned}$$

as left Q-modules. Therefore P is left Q-finitely generated projective. Now assume furthermore that R is left S-finitely generated projective, then from (1.1) we have

$$\mathrm{Hom}({}_Q P, {}_Q M) \oplus M\otimes_S(\oplus S)\cong M\oplus \cdots \oplus M \tag{1.2}$$

as left P-modules. Also if S is a left S-direct summand of R then M is a left P-direct summand of $\mathrm{Hom}({}_Q P, {}_Q M)$, that is, M is relative (P, Q)-injective. We have proved the following proposition.

PROPOSITION 1.1. *Under the above notations,* (T) *is equivalent to* (1.1) *and* P *is left* Q*-finitely generated projective. If furthermore* R *is left* S*-finitely generated projective we have* (1.2), *and if* S *is a left* S*-direct summand of* R *then* M *is relative* (P, Q)*-injective.*

Next, we start with a ring extension $P\supset Q$. Let $P\supset Q$ be a ring extension and M a left P-module. Let R^* and S^* be the endomorphism rings of left Q-module M and left P-module M respectively, which operate on right side of M. Consider the condition

$$(H)\quad \mathrm{Hom}({}_Q P, {}_Q M) \oplus M\oplus \cdots \oplus M.$$

Then by the same way as above we have the following commutative diagram with (P, R^*)-isomorphic arrows

$$\begin{array}{ccc} M\otimes_{S*}\mathrm{Hom}({}_PM,\ {}_P\mathrm{Hom}({}_QP,\ {}_QM)) & \longrightarrow & M\otimes_{S*}\mathrm{Hom}(\mathrm{Hom}({}_P\mathrm{Hom}({}_QP,\ {}_QM)_PM)_{S*},\ S^*_{S*}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}({}_QP,\ {}_QM) & \longrightarrow & \mathrm{Hom}(\mathrm{Hom}({}_P\mathrm{Hom}({}_QP,\ {}_QM),\ {}_PM)_{S*},\ M_{S*}). \end{array}$$

Note that $\mathrm{Hom}({}_PM,\ {}_P\mathrm{Hom}({}_QP,\ {}_QM))\cong\mathrm{Hom}({}_QM,\ {}_QM)=R^*$ and we have from the left vertical map of the above diagram

(1.3) $$M\otimes_{S*}R^*\cong\mathrm{Hom}({}_QP,\ {}_QM)$$

as left P- and right R^*-modules. Also we have from (H)

$$R^*\cong\mathrm{Hom}({}_PM,\ {}_P\mathrm{Hom}({}_QP,\ {}_QM))\langle\!\oplus\mathrm{Hom}({}_PM,\ {}_P(\oplus M))$$

$$\cong S^*\oplus\cdots\oplus S^*$$

as left S^*-modules, and R^* is left S^*-finitely generated projective. If furthermore P is left Q-finitely generated projective then we have from (1.3)

(1.4) $$M\otimes_{S*}R^*\langle\!\oplus M\oplus\cdots\oplus M$$

as right R^*-modules. Also if Q is a left Q-direct summand of P then M is a right R^*-direct summand of $M\otimes_{S*}R^*$, that is, M is relative $(R^*,\ S^*)$-projective. Therefore we have following proposition.

PROPOSITION 1.2. *Under the above notations,* (H) *is equivalent to* (1.3) *and* R^* *is left* S^**-finitely generated projective. If furthermore* P *is left* Q*-finitely generated projective we have* (1.4) *and if* Q *is a left* Q*-direct summand of* P *then* M *is relative* $(R^*,\ S^*)$*-projective.*

Next two propositions are characterizations of the conditions (T) and (H) respectively.

PROPOSITION 1.3. *Notations are the same as above. Then following conditions are equivalent for a right* R*-module* M.

(1) *M satisfies* (T).

(2) $\mathrm{Hom}(M_R,\ M\otimes_S R_R)\otimes_Q\mathrm{Hom}(M\otimes_S R_R,\ M_R)\cong\Omega$.

(3) *For every right* R*-module* X *we have*

$$\mathrm{Hom}(M_S,\ X_S)\cong\mathrm{Hom}(M_R,\ X_R)\otimes_Q P.$$

PROOF. (1) ⇒ (2). There is a natural map from $\mathrm{Hom}(M_R,\ M\otimes_S R_R)\otimes_Q\mathrm{Hom}(M\otimes_S R_R,\ M_R)$ to Ω defined by $g\otimes f\rightarrow g\circ f$ for $g\in\mathrm{Hom}(M_R,\ M\otimes_S R_R)$, $f\in\mathrm{Hom}(M\otimes_S R_R,\ M_R)$. The inverse map is given by $\omega\rightarrow\sum\omega g_i\otimes f_i$, $\omega\in\Omega$.

(2) ⇒ (1). Choose $\sum k_j\otimes h_j$, $k_j\in\mathrm{Hom}(M_R,\ M\otimes_S R_R)$ and $h_j\in\mathrm{Hom}(M\otimes_S R_R,$

M_R), corresponding to 1 of Ω, then $\{h_j, k_j\}$ gives (T).

(1) ⇒ (3). We have seen that (T) is equivalent to $M\otimes_S R\cong \mathrm{Hom}({}_QP, {}_QM)$ and P is Q-finitely generated projective. Now let X be a right R-module, then we have

$$\mathrm{Hom}(M_R, X_R)\otimes_Q P\cong \mathrm{Hom}(\mathrm{Hom}({}_QP, {}_QM)_R, X_R)\cong \mathrm{Hom}(M\otimes_S R_R, X_R)$$
$$\cong \mathrm{Hom}(M_S, X_S).$$

(3) ⇒ (2). Take $M\otimes_S R$ as X. Then we have

$$\mathrm{Hom}(M_R, M\otimes_S R_R)\otimes_Q \mathrm{Hom}(M\otimes_S R_R, M_R)\cong \mathrm{Hom}(M_R, M\otimes_S R_R)\otimes_Q P$$
$$\cong \mathrm{Hom}(M_S, M\otimes_S R_S)\cong \mathrm{Hom}(M\otimes_S R_R, M\otimes_S R_R)=\Omega.$$

PROPOSITION 1.4. *Let $P\supset Q$ be a ring extension and M a left P-module. Then the following are equivalent for a left P-module M.*

(1) *M satisfies (H).*

(2) $\mathrm{Hom}({}_P\mathrm{Hom}({}_QP, {}_QM), {}_PM)\otimes_{S^*} \mathrm{Hom}({}_PM, {}_P\mathrm{Hom}({}_QP, {}_QM))$
$\cong \mathrm{End}({}_P\mathrm{Hom}({}_QP, {}_QM))$.

(3) *For every left P-module Y we have*

$$\mathrm{Hom}({}_PY, {}_PM)\otimes_{S^*} R^*\cong \mathrm{Hom}({}_QY, {}_QM).$$

The proof is similar to that of Proposition 1.3. Note that homomorphisms operate on right sides of modules.

There are remarkable isomorphisms in our situation.

PROPOSITION 1.5. *Assume that the condition (T) holds for $R\supset S$ and M. Let $P=\mathrm{End}(M_S)$ and $Q=\mathrm{End}(M_R)$, then we have*

(1) $\mathrm{Hom}(M_R, M\otimes_S R_R)\cong \mathrm{Hom}({}_QP, {}_QQ)$.

(2) $\Omega\cong \mathrm{Hom}({}_QP, {}_QP)$.

PROOF. (1). There is a natural map from $\mathrm{Hom}(M_R, M\otimes_S R_R)$ to $\mathrm{Hom}({}_Q\mathrm{Hom}(M\otimes_S R_R, M_R), {}_QQ)$ defined by $g\to(f\to f\circ g)$ for $g\in \mathrm{Hom}(M_P, M\otimes_S R_R)$, $f\in \mathrm{Hom}(M\otimes_S R_R, M_R)$. The inverse map is given by $\varphi\to\sum g_i\varphi(f_i)$ for $\varphi\in \mathrm{Hom}({}_Q\mathrm{Hom}(M\otimes_S R_R, M_R), {}_QQ)$.

(2). We have following sequence of isomorphisms.

$$\Omega\cong \mathrm{Hom}(M_R, M\otimes_S R_R)\otimes_Q \mathrm{Hom}(M\otimes_S R_R, M_R)\cong \mathrm{Hom}({}_QP, {}_QQ)\otimes_Q P$$
$$\cong \mathrm{Hom}({}_QP, {}_QP).$$

Note that the composition map is a ring isomorphism.

PROPOSITION 1.6. *Assume that the condition* (H) *holds for* $P\supset Q$ *and* M. *Let* $R^*=\mathrm{End}({}_QM)$ *and* $S^*=\mathrm{End}({}_PM)$, *then we have*

(1) $\mathrm{Hom}({}_P\mathrm{Hom}({}_QP,\ {}_QM),\ {}_PM)\cong\mathrm{Hom}({}_{S^*}R^*,\ {}_{S^*}S^*)$

(2) $\mathrm{Hom}({}_P\mathrm{Hom}({}_QP,\ {}_QM),\ {}_P\mathrm{Hom}({}_QP,\ {}_QM))\cong\mathrm{Hom}({}_{S^*}R^*,\ {}_{S^*}R^*)$.

The proof is similar to that of Proposition 1.5.

Now assume that M is faithfully balanced as an R- and as an S-module respectively, that is, if $Q=\mathrm{End}(M_R)$ then $\mathrm{End}({}_QM)=R$, and if $P=\mathrm{End}(M_S)$ then $\mathrm{End}({}_PM)=S$. Then combining Propositions 1.1 and 1.2 we have

THEOREM 1.7. *If* M *is faithfully balanced as an* R- *and as an* S-*module respectively, then the following are equivalent.*

(1) $\{R,\ S,\ M\}$ *satisfies* (T) *and* R *is left* S-*finitely generated projective.*

(2) $\{P,\ Q,\ M\}$ *satisfies* (H) *and* P *is left* Q-*finitely generated projective.*

2. Application to H-separable extensions

Now we consider an H-separable extension $R\supset S$ and a right R-module M.

PROPOSITION 2.1. *Let* $R\supset S$ *be an* H-*separable extension and* M *a right* R-*module. Put* $P=\mathrm{End}(M_S)$ *and* $Q=\mathrm{End}(M_R)$. *Then we have*

$$P <_{\oplus} Q\oplus\cdots\oplus Q$$

as two-sided Q-*modules, that is,* P *is* Q-*centrally projective.*

PROOF. As $R\supset S$ is H-separable we have

$$R\otimes_S R <_{\oplus} R\oplus\cdots\oplus R$$

as two-sided R-modules. Tensoring with M over R we have

$$M\otimes_S R <_{\oplus} M\oplus\cdots\oplus M$$

as left Q- and right R-modules. Then we have following isomorphisms and direct summand relation.

$$P=\mathrm{Hom}(M_S,\ M_S)\cong\mathrm{Hom}(M\otimes_S R_R,\ M_R) <_{\oplus} \mathrm{Hom}((\oplus M)_R,\ M_R)$$

$$\cong\oplus\mathrm{Hom}(M_R,\ M_R)=Q\oplus\cdots\oplus Q$$

as two-sided Q-modules.

Put $R^*=\mathrm{End}({}_QM)$ and $S^*=\mathrm{End}({}_PM)$ then from the above proposition we have

$$\mathrm{Hom}({}_QP,\ {}_QM) <_{\oplus} \mathrm{Hom}({}_Q(\oplus Q),\ {}_QM)\cong M\oplus\cdots\oplus M$$

as left Q- and right R^*-modules. Now assume that R is left S-finitely generated projective. Then by Propositions 1.1 and 1.2 R^* is left S^*-finitely generated projective. Therefore we have

$$R^*\oplus\cdots\oplus R^*\cong \mathrm{Hom}({}_QM,\ {}_Q(\oplus M))\oplus\!\!> \mathrm{Hom}({}_QM,\ {}_Q\mathrm{Hom}({}_QP,\ {}_QM))$$

$$\cong \mathrm{Hom}({}_QM,\ {}_QM\otimes_{S^*}R^*)\cong \mathrm{Hom}({}_QM,\ {}_QM)\otimes_{S^*}R^*=R^*\otimes_{S^*}R^*$$

as two-sided R^*-modules. We have proved the following theorem.

THEOREM 2.2. ([4] Theorem 1) *Let $R\supset S$ be an H-separable extension and M a right R-module. Put* $Q=\mathrm{End}(M_R)$, $R^*=\mathrm{End}({}_QM)$, $P=\mathrm{End}(M_S)$ *and* $S^*=\mathrm{End}({}_PM)$. *Then if R is left S-finitely generated projective R^* is an H-separable extension of S^* and R^* is left S^*-finitely generated projective. And if S is a left S-direct summand of R then S^* is a left S^*-direct summand of R^*.*

The last assertion is as follows. If S is a left S-direct summand of R then by Proposition 1.1 $\mathrm{Hom}({}_QP,\ {}_QM)\oplus\!\!> M$ as left P-modules. Then we have

$$S^*=\mathrm{Hom}({}_PM,\ {}_PM)<\!\!\oplus \mathrm{Hom}({}_PM,\ {}_P\mathrm{Hom}({}_QP,\ {}_QM))\cong \mathrm{Hom}({}_QM,\ {}_QM)=R^*$$

as left S^*-modules.

3. Relative (co-) generator.

It is easily seen that if an R-module M is an S-generator and satisfies (T) then M is an R-generator. So the condition (T) may consider as M has a property such as relative generator. In this connection we have following proposition.

PROPOSITION 3.1. *Let $R\supset S$ be a ring extension and M a right R-module. Then the following conditions are equivalent for M.*

(1) *Let X, Y be any right R-modules and let f, g be R-homomorphisms of X to Y such that there exists an S-homomorphism h_0 of M to X with $fh_0\neq gh_0$. Then there exists an R-homomorphism h of M to X with $fh\neq gh$.*

(2) *$M\otimes_SR$ is an epimorphic image of a (finite or infinite) direct sum of M.*

(3) $\mathrm{Tr}_X(M_S)=\mathrm{Tr}_X(M_R)$ *for every right R-module X where* $\mathrm{Tr}_X(M_R)$ *is the trace of M in X i.e.* $\mathrm{Tr}_X(M_R)=\sum(h(M),\ h\in\mathrm{Hom}(M_R,\ X_R))$ *and* $\mathrm{Tr}_X(M_S)=\sum(h(M),\ h\in\mathrm{Hom}(M_S,\ X_S))$ (cf. [1]). *When this is the case if M is a generator as an S-module then M is also a generator as an R-module.*

PROOF. (1)$\Rightarrow$(2). Condition (1) is equivalent to that if an R-homomorphism $f: X\to Y$ satisfies $fh_0\neq 0$ for some S-homomorphism $h_0: M\to X$ there exists an R-homomorphism $h: M\to X$ with $fh\neq 0$. Now assume that $\mathrm{Tr}_{M\otimes_S R}(M_R)\subsetneqq M\otimes_S R$ and consider the natural map $f: M\otimes_S R\to M\otimes_S R/\mathrm{Tr}_{M\otimes_S R}(M_R)$. Let $h_0: M\to M\otimes_S R$ be the S-homomorphism defined by $h_0(x)=x\otimes 1$, $x\in M$. If $fh_0=0$ then $f(M\otimes_S R)=f(h_0(M))R=0$. So $fh_0\neq 0$ but $fh=0$ for all $h\in\mathrm{Hom}(M_R, X_R)$ contradicts.

(2) $\Rightarrow$ (3). As is easily seen that there hold following relations for any right R-module X

$$\mathrm{Tr}_X(M\otimes_S R_R)=\mathrm{Tr}_X(M_S)R\supset\mathrm{Tr}_X(M_S)\supset\mathrm{Tr}_X(M_R).$$

Now assume that there exists an R-epimorphism φ of $\oplus M$ to $M\otimes_S R$. Then for any $h\in\mathrm{Hom}(M\otimes_S R_R, X_R)$ and $\xi\in M\otimes_S R$ there exist $x_i\in M$ with $h(\xi)=h(\varphi(\sum x_i))$. Therefore $\mathrm{Tr}_X(M\otimes_S R_R)\subset\mathrm{Tr}_X(M_R)$.

(3) $\Rightarrow$ (1). Let f be an R-homomorphism from X to Y such that there exists an S-homomorphism h_0 from M to X with $fh_0\neq 0$. Then since $\mathrm{Ker}\, f\not\supset\mathrm{Tr}_X(M_S)=\mathrm{Tr}_X(M_R)$ there exists an R-homomorphism h of M to X with $fh\neq 0$.

Now assume that M is a generator as an S-module. Let X be any right R-module. Then there exists an S-epimorphism from $\oplus M$ to X, $\oplus M\to X\to 0$. Tensoring with R over S and combine with the epimorphism $X\otimes_S R\to X\to 0$ we have an epimorphism $\oplus(M\otimes_S R)\to X\to 0$. Now if M generates $M\otimes_S R$ then M generates X as an R-module. This completes the proof.

Dually we can prove the following proposition.

PROPOSITION 3.2. *Let $P\supset Q$ be a ring extension and M a left P-module. Then the following conditions are equivalent for M.*

(1). *Let X, Y be left P-modules and let f, g be P-homomorphisms of Y to X such that there exists a Q-homomorphism h_0 of X to M with $fh_0\neq gh_0$ then there exists a P-homomorphism h of X to M with $fh\neq gh$. In this time homomorphisms operate on right sides of modules.*

(2). *There exists a P-monomorphism from* $\mathrm{Hom}({}_QP, {}_QM)$ *to a (finite or infinite) direct product of M.*

(3). $\mathrm{Rej}_X({}_QM)=\mathrm{Rej}_X({}_PM)$ *for every left P-module X where* $\mathrm{Rej}_X({}_PM)$ *is the reject of M in X i.e.* $\mathrm{Rej}_X({}_PM)=\cap\mathrm{Ker}\, h$, $h\in\mathrm{Hom}({}_PX, {}_PM))$ *and* $\mathrm{Rej}_X({}_QM)$ *is that of Q-modules M and X.* (cf. [1]) *When this is the case if M is a cogenerator as a Q-module then M is also a cogenerator as a P-module.*

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