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PRODUCT THEOREMS IN DIMENSION THEORY

Dedicated to Professor Y. Kodama on the occasion of his 60th birthday

By

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Introduction.

Throughout this paper we assume that all spaces are just topological spaces, otherwise specified. We start from the following theorem:

THEOREM 0 [10]. Let $X\times Y$ be piecewise rectangular. Then,

$$(*)\quad \mathrm{Id}(X\times Y)\leqq \mathrm{Id}\,X+\mathrm{Id}\,Y.$$

Where In Z for a space Z is a dimension function introduced by B. A. Pasynkov [9], and we will give its definition in the following section of this paper as well as the definition of piecewise rectangularity.

COROLLARY 0 [10]. *Let $X\times Y$ be normal, piecewise rectangular, and let each of X and Y satisfy a finite sum theorem for* Ind *(FST*(Ind) *for short). Then we have*

$$(**)\quad \mathrm{Ind}(X\times Y)\leqq \mathrm{Ind}\,X+\mathrm{Ind}\,Y.$$

The proofs for these results have not yet been published. The central ideas for those were presented by the first author at General Topology and Geometric Topology Symposium held at Tsukuba in 1990; the simplest case when $X\times Y$ is compact was talked there. Detailed proofs for Theorem 0 and Corollary 0 were given also by the first author when he visited Tsukuba in 1991 (see [12]). On this occasion we discussed the following conjecture:

CONJECTURE. Let $\Pi=X_1'\times X_2, *\in X_1'$, $X_1=X_1'\setminus\{*\}$, and the product $\Pi_0=X_1\times X_2$ be piecewise rectangular and satisfy the following condition (#).

(#) Every set H is functionally separated from $\{*\}\times X_2$ whenever H is closed in Π and $H\cap(\{*\}\times X_2)=\emptyset$. Then, we have $\mathrm{Id}\,\Pi\leqq \mathrm{Id}\,X_1'+\mathrm{Id}\,X_2$.

In this paper we shall prove this conjecture for the following cases:

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THEOREM 1. *The conjecture is trure, when Π_0 is rectangular.*

THEOREM 2. *It is true when it satisfies the following condition* (##) *as well as the condition* (#):

$$(\#\#)\quad \operatorname{Id} X_1 = \operatorname{Id} X_1'.$$

Moreover, we will show the following theorem.

THEOREM 3. *Let $\Pi = X_1' \times X_2'$, $x_i \in X_i'$, $X_i = X_i' \setminus \{x_i\}$, and the product $\Pi_0 = X_1 \times X_2$ be piecewise rectangular satisfying the following condition* (###):

(###) *Every set H is functionally separated from E_i whenever H is closed in Π, and $H \cap E_i = \emptyset$, where $E_1 = X_1' \times \{x_2\}$ and $E_2 = \{x_1\} \times X_2'$. Then, we have* $\operatorname{Id} \Pi \leqq \operatorname{Id} X_1 + \operatorname{Id} X_2$.

CONVENTIONS. We shall use the following conventions. The set $\partial_F U$ denotes the boundary of the set U in F. For a subset A of a space B the set $[A]_B$ denotes the closure of A in B. Some Greek letters are used to denote some families consisting of subsets of a space (in particular, ω does not mean the first infinite ordinal).

1. Definitions and Preliminaries.

We start from definitions (for the simplicity we only deal with a product with *two factors*, and see [10, 11] for general cases). A subset of a product space $\Pi = X_1 \times X_2$ is said to be a *functionally open rectangle* (FORect, for short) if it is of the form $U_1 \times U_2$, where each U_i is a functionally open in X_i. A *clopen* (that is, both closed and open) subset of a FORect is called a *functionally open rectangular piece* (FORectP, for short). A cover of Π by FORect (resp. FORectP) sets is called *functionally open rectangular* (FORect, for short) (resp. *functionally open piecewise rectangular* (FOPRect, for short).

DEFINITION 0 [10]. A product Π is called *piecewise rectangular* (resp. rectangular) if each finite functionally open cover has a σ-locally finite FOPRect (resp. FORect) refinement.

Let λ and ω be families of subsets of X. Then λ is called *finite relative to* ω if for any $O \in \omega$ the family $\{F \in \lambda : F \cap O \neq \emptyset\}$ is finite. λ is called *uniformly locally finite* (ULF, for short) if λ is finite relative to a *functionally open locally finite* (FOLF, for short) cover of X (see [6], and Remark 2).

Let λ and μ be *closed* families (that is, families of closed subsets) of a space X. Then we shall call λ *breaks* μ if for every $F\in\mu$ and for any two closed subsets A and B of F, which are *functionally separated* (FS, for short) in X, there exists an element $C\in\lambda$ contained in F, which is a partition between A and B in F (see [9]).

DEFINITION 1 [12]. A family λ' consiting of subsets of X *uniformly generates* a family λ if for every $L\in\lambda$ there is a ULF family μ_L consisting of closed subsets of some members of λ' such that $L=\cup\mu_L$.

DEFINITION 2 [9]. We define $\operatorname{Id}X=-1$ if and only if $X=\emptyset$. We put $\operatorname{Id}X\leqq n$ for $n=0, 1, 2, \cdots$, if there are $k+2$ closed families σ_i, $-1\leqq i\leqq k\leqq n$, in X satisfying the following conditions:

a) $\sigma_{-1}=\{\emptyset\}$, $X\in\sigma_k$, $\sigma_{i+1}\supset\sigma_i$, $-1\leqq i\leqq k-1$;

b) σ_i breaks σ_{i+1};

c) For any members A and B of σ_i their union $A\cup B$ is also a member of σ_i (in this case we say that the family σ_i is *additive*).

d) Any closed subset of a member of σ_i is also a member of σ_i (in this case we say that the family σ_i is *monotone*).

The following lemmas are used by the first author to prove Theorem 0, and those proofs can be seen in [12].

LEMMA 1. *Let C and D be disjoint closed subsets of X. Let λ be a locally finite closed cover of X, and assume that for each $F\in\lambda$ there exists a partition P_F in F between $C\cap F$ and $D\cap F$. Then there exists a partition P in X between C and D such that*

$$\cup\{P_F:\ F\in\lambda\}\cup T\supset P,\ \textit{where}$$

$$T=\{x\in X:\ x\in F\cap F' \textit{ for some distinct } F \textit{ and } F' \textit{ of } \lambda\}.$$

LEMMA 2. *Let μ' and λ' be closed families, and μ and λ be the families uniformly generated by them, respectively. Then, λ breaks μ if μ' is additive and λ' breaks μ'.*

LEMMA 3. *If families λ_α, $\alpha\in A$, are ULF in X and the family $\mu=\{\cup\lambda_\alpha:\ \alpha\in A\}$ is also ULF in X, then the whole family $\lambda=\cup\{\lambda_\alpha,\ \alpha\in A\}$ is ULF in X again.*

LEMMA 4. *Let C and D be disjoint closed subsets of a closed subset F of X, and let ω be an open cover of F having the following properties:*

a) *Every member of* ω *is disjoint from either* C *or* D;

b) ω *is a union of countably many ULF subfamilies* ω_i *satisfying that there exists a FOLF cover* Ω *of* X *for which the cover* $\Omega\wedge F=\{U\cap F : U\in\Omega\}$ *refines the countable cover* $\{\cup\omega_i\}$. *Then there exists a closed family* λ, *which is ULF in* X, *satisfying that for each* $L\in\lambda$ *there exists* $O\in\omega$ *with* $\partial_F O\supset L$, *and that the set* $\cup\lambda$ *is partition between* C *and* D *in* F.

Using these lemmas we can show the following corollaries.

COROLLARY 1 (Uniformly locally finite sum theorem). *Let* X *be normal and satisfy* FST(Ind). *Then* Ind $X\leqq n$ *if it can be represented as a union of a ULF covering of at most n-dimensional (in the sense of* Ind*) closed subsets.*

The following corollary has been proved by the first author for the case X is *paracompact*.

COROLLARY 2 (Locally finite sum theorem). *Let* X *be strongly normal (see the final section for its definition) satisfying* FST(Ind). *Then, Corollary* 1 *holds for every locally finite closed cover.*

COROLLARY 3 [B. A. Pasynkov, unpublished]. *Let* G *be a normal topological group satisfying* FST(Ind). *Then* Ind $G=loc$ Ind G.

The following lemma can be quoted from a paper of K. Morita [7].

LEMMA 5. *Let* $X\supset Q_\alpha\supset F_\alpha\supset U_\alpha$, *and* O_α *and* F_α *be functionally open (FO, for short) and functionally closed (FC, for short) subsets, respectively. Then the family* $\{U_\alpha\}$ *is ULF in* X *if the family* $\{O_\alpha\}$ *is locally finite in* X.

LEMMA 6. *Let* S *be a closed subsets of a space* X *with following properties*:

(a) *Every set* H *is FS from* S *in* X, *whenever* H *is closed in* X *and is disjoint from* S;

(b) *There exist two closed, monotone, additive families* λ *and* μ *in* X *satisfying that for any* $F\in\mu$ *there exists a functionally open neighborhood (FONbd, for short)* O *of* S *and a* $P\in\lambda$ *which is a partion between* ${}^OC=C\cap[O]$ *and* ${}^OD=D\cap[O]$ *in* ${}^OF=F\cap[O]$, *whenever* C *and* D *are closed subsets of* F *and are FS in* X;

(c) λ' *breaks* μ', *where* λ' *and* μ' *are the subfamilies of* λ *and* μ *consisting of all the elements disjoint from* S, *respectively. Then* λ *breaks* μ.

PROOF. Let C and D be closed subsets in $F\in\mu$, which are FS in X. Then, we have

d) a FONbd O of S and a $P_1 \in \lambda$ such that P_1 is a partition between ${}^O C$ and ${}^O D$ in ${}^O F$;

e) a FONbd U of S such that two sets $[U]$ and $X \setminus O$ are FS in X. Take $P_2 \in \lambda$ which is a partition between $C_U = C \setminus U$ and $D_U = D \setminus U$ in $F_U = F \setminus U$. Since $F_O = F \setminus O$ and $U_F = F \cap [U]$ are FS in X, so is also F_O and $F_{\partial U} = F \cap \partial U$. Since $F_U \in \mu'$, there exists a partition $P_3 \in \lambda$ between F_O and $F_{\partial U}$ in F_U.

It is not difficult to show that there exist two disjoint open sets G_1 and G_2 in F such that $F \setminus P_3 = G_1 \cup G_2$, $F_U \supset H = G_1 \cap P_3$, ${}^O F \supset K = G_2 \cup P_3$. Hence, it holds that H and K are closed, $P_3 = H \cap K \in \lambda$, $P_1 \cap K$ is a partition between $C \cap K$ and $D \cap K$ in K, and that $P_2 \cap H$ is a partition between $C \cap H$ and $D \cap H$ in H.

By Lemma 1 we have a partition P between C and D in F with $P_1 \cup P_2 \cup P_3 \supset P$. Hence, λ breaks μ, and this completes the proof.

2. Proofs of our theorems.

We start from a construction of the following special closed families.

By the definition of Id it is possible to choose closed families σ_j^i, $j=-1, 0, \cdots, n(i)$, $i=1, 2$, in X_1 and X_2 such that

(a) $\sigma_{-1}^i = \{\emptyset\}$, $X_1' \in \sigma_{n(1)}^1$, $X_2 \in \sigma_{n(2)}^2$, $\sigma_{j+1}^i \supset \sigma_j^i$, $-1 \leqq j \leqq n(i)-1$;

(b) σ_j^i breaks σ_{j+1}^i, $-1 \leqq j \leqq n(i)-1$, $i=1, 2$;

(c) σ_j^i is monotone and additive, $-1 \leqq j \leqq n(i)$, $i=1, 2$.

Put $\sigma_{j(1)}^1 \times \sigma_{j(2)}^2 = \{F^1 \times F^2 : F^i \in \sigma_{j(i)}^i,\ i=1, 2\}$, and

$$\sigma_{-1} = \{\emptyset\}, \qquad \sigma_j = \cup \{\sigma_{j(1)}^1 \times \sigma_{j(2)}^2 : j = j(1) + j(2)\}$$

for $0 \leqq j \leqq n(1)+n(2)$.

Let σ_j^* be the family consisting of all finite unions of closed subsets of elements of the family σ_j, and $\sigma_j^{**} = \{F \in \sigma_j^* : F \cap S = \emptyset\}$. Let Σ_j' be the family uniformly generated by σ_j^{**} in Π, and

$$\Sigma_j = \Sigma_j' \cup \sigma_j^* .$$

The following lemma together with Lemma 6 completes our proof of Theorem 1, since each Σ_j is obviously additive and monotone, and

$$\Sigma_{-1} = \{\emptyset\}, \qquad \Pi \in \Sigma_{n(1)+n(2)}, \qquad \Sigma_{j+1} \supset \Sigma_j .$$

Note that it satisfies the condition (b) in Lemma 6 since (#) holds and we can apply Lemma 1, putting $X=F$, for each element F of σ_j^{**}.

LEMMA 7. *Σ_{j-1}' breaks Σ_j' for* $0 \leqq j \leqq n(1)+n(2)$.

PROOF. By Lemma 2 it is sufficient to prove that Σ_{j-1}' breaks σ_j^{**}. Let $F\in\sigma_j^{**}$. For the sake of simplicity we can assume that $F=G\cup H$, where G and H are closed subsets of rectangles $G^1\times G^2$, $H^1\times H^2$, $G^i\in\sigma_{j(i)}{}^i$, $H^i\in\sigma_{t(i)}{}^i$, $i=1, 2$, respectively, satisfying $j(1)+j(2)=t(1)+t(2)=j$.

When $j(i)=t(i)$, $i=1, 2$, it reduces to the case when F is a closed subset of a single rectangle, since $(G^1\cup H^1)\times(G^2\cup H^2)\supset F$ and $G^i\cup H^i\in\sigma_{j(i)}{}^i$, $i=1, 2$.

When $j(i)\neq t(i)$, $i=1, 2$, then it also reduces to the above case, since $\sigma_{j-1}\ni(G^1\cap H^1)\times(G^2\cap H^2)\supset G\cap H$ and Lemma 1 holds. Thus, let $F\in\sigma_j^{**}$, $F^1\times F^2\supset F$, $F^i\in\sigma_{j(i)}{}^i$, $i=1, 2$, $j(1)+j(2)=j$, and let C, D be closed subsets of F which is FS in Π. Using (#) we have FO sets W, W_1 and W_2 and FC sets H and H_1 in Π such that

(1) $$\Pi\setminus F\supset W_2\supset H\supset W\supset H_1\supset W_1\supset S.$$

There exist FO sets O_C and O_D in Π such that

(2) $$C\cap O_C=\emptyset,\quad D\cap O_D=\emptyset \quad\text{and}\quad O_C\cup O_D=\Pi\setminus H.$$

Since Π_0 is rectangular, there exists a σ-locally finite FORect family ω such that ω refines the binary cover $\{O_C, O_D\}$ so that each of its elements is disjoint from either C or D by (2), and

(3) $\cup\omega=O_C\cup O_D=\Pi\setminus H$. Let $\omega_j=\{O_\alpha : \alpha\in A_j\}$ be LF in Π_0, and $\omega=\bigcup_{j=0}^{\infty}\omega_j$.

Put $O_\alpha=O_\alpha{}^1\times O_\alpha{}^2$, where $Q_\alpha{}^i$ is an FO in X_i, $i=1, 2$. Let $x\in O_\alpha{}^2$. Then $(O_\alpha{}^1\times\{x\})\cap(W\cap(X_1'\times\{x\}))=\emptyset$, since $H\cap O_\alpha{}^1\times\{x\}=\emptyset$. So that if we take a continuous function $f: X_1\times\{x\}\to[0, 1]$, with $f^{-1}(0, 1]=O_\alpha{}^1\times\{x\}$, then the function f', which is equal to 0 at the point $(*, x)$ and coincides with f on $X_1\times\{x\}$, is continuous. Hence, $Q_\alpha{}^1$ is FO set in X_1'. Let $f_\alpha{}^i$ be a continuous function from either X_1' or X_2 to $[0, 1]$ such that $O_\alpha{}^i=(f_\alpha{}^i)^{-1}(0, 1]$, respectively. Let

$F_{\alpha t}{}^i=(f_\alpha{}^i)^{-1}[1/t, 1]$, $V_{\alpha t}{}^i=(f_\alpha{}^i)^{-1}(1/t, 1]$, $t=2, 3, \cdots$. Then, for each set $F_{\alpha t}{}^i\cap F^i$ there exists an open set $G_{\alpha t}{}^i$ in F^i such that

(4) $$V_{\alpha t+1}{}^i\supset[G_{\alpha t}{}^i]\supset G_{\alpha t}{}^i\supset F_{\alpha t}{}^i\cap F^i,\quad \partial_{F^i}(G_{\alpha t}{}^i)\in\sigma_{j(i)-1}{}^i.$$

Then, by Lemma 5 the families $\nu_{jt}=\{V_{\alpha t} : \alpha\in A_j\}$, $j=0, 1, 2, \cdots$, $t=2, 3, \cdots$, are ULF in Π_0, since $O_\alpha\supset F_{\alpha t}\supset[V_{\alpha t}]$, where

(5) $V_{\alpha t}=V_{\alpha t}{}^1\times V_{\alpha t}{}^2$ are FO, and the sets $F_{\alpha t}{}^1\times F_{\alpha t}{}^2$ are FC in both spaces Π and Π_0.

Hence, let ν_{jt} be finite relative to a FOLF cover μ of Π_0. Then, ν_{jt} is

ULF in Π also, since it is finite relative to the FOLF cover $\{Q\setminus H_1: Q\in\mu\}\cup\{W\}$ of Π. Obviously, $[V]$ is disjoint from either C or D for every $V\in\nu_{jt}$. Hence, so is the set

(6) $G_{\alpha t}=G_{\alpha t}{}^1\times G_{\alpha t}{}^2\subset V_{\alpha t+1}$, and the family $\gamma_{jt}=\{G_{\alpha t}: \alpha\in A_j\}$ is ULF in Π by (4) and ν_{jt+1} is ULF in Π.

If $x\in F$ then there exist j and $\alpha\in A_j$ such that $x\in F_{\alpha t}$, and hence $x\in G_{\alpha t}$. It follows that the family $\gamma=\cup\{\gamma_{jt}: j\geqq 0, t\geqq 2\}$ covers F consisting of open sets in $F^*=F^1\times F^2$. Let $G_{jt}=\cup\gamma_{jt}$. Then,

$$G_{jt}\supset(\cup\nu_{jt})\cap F^*, \tag{7}$$

since $G_{\alpha t}{}^i\supset F_{\alpha t}{}^i\cap F^i\supset V_{\alpha t}{}^i\cap F^i$, $i=1, 2$, and $G_{\alpha t}\supset V_{\alpha t}\cap F^*$. Because $V_{\alpha t}$ are FO in Π by (5) and the family ν_{jt} is ULF in Π, it holds that the sets $\cup\nu_{jt}$ are FO in Π, and that the family $\Omega=\{\cup\nu_{jt}: j\geqq 0, t\geqq 2\}\cup\{W\}$ is a FO countable cover of Π. Hence, we can assume that Ω is LF, since any FO countable cover has an LF and FO countable refinement. So that by (6), (7), and Lemma 4 there exists a closed family λ, which is ULF in Π, such that for each $L\in\lambda$ there exists $G_{\alpha t}\in\gamma_{jt}$ satisfying that

$$L\subset(\partial_{F^*}G_{\alpha t})\cap F=F\cap((\partial G^1_{\alpha t}\times G^2_{\alpha t})\cup(G^1_{\alpha t}\times\partial G^2_{\alpha t})),$$

which is an element of $\sigma_{j-1}{}^{**}$ and that $P=\cup\lambda$ is a partition between C and D in F. Obviously, since the set P is a member of $\Sigma_{j-1}{}'$, our proof is completed.

COROLLARY 1. *The inequality* (**) *is valid when* $X\times Y$ *is normal, both factor spaces* X *and* Y *satisfy* FST(Ind), *the one point set* $\{*\}$ *is closed in* Y *and* $X\times(Y\setminus\{*\})$ *is rectangular.*

We will give a proof of Theorem 3. We start from reconstruction of the following special closed families.

By the definition of Id it is possible to choose closed families $\tau_j{}^i$, $j=-1, 0, \cdots, n(i)$, $i=1, 2$ in X_i such that

(a) $\tau_{-1}{}^i=\{\emptyset\}$, $X_i\in\tau_{n(i)}{}^i$, $\tau_{j+1}{}^i\supset\tau_j{}^i$, $-1\leqq j\leqq n(i)-1$;

(b) $\tau_j{}^i$ breaks $\tau_{j+1}{}^i$, $-1\leqq j\leqq n(i)-1$, $i=1, 2$;

(c-d) $\tau_j{}^i$ is monotone and additive, $-1\leqq j\leqq n(i)$, $i=1, 2$.

LEMMA 8. *Let* $\sigma_j{}^i=\{F\in\tau_j{}^i: F$ *is closed in* $X_i{}'$ *and* $x_i\notin F\}$ *for* $-1\leqq j\leqq n(i)-1$, *and let* $\sigma_{n(i)}{}^i$ *be the set of all closed subsets in* $X_i{}'$, $i=1, 2$. *Then,* $\sigma_j{}^i$ *breaks* $\sigma_{j+1}{}^i$ *in* $X_i{}'$. *Hence,* $\mathrm{Id}\,X_i{}'\leqq\mathrm{Id}\,X_i$.

PROOF. It suffices to show it for every element $F\in\sigma_j{}^i$ with $x_i\in F$. For given closed subsets C and D of F, which are FS in X_i', let $f: X_i'\to I$ be a continuous map with $f(D)=0$ and $f(C)=1$. Then, we can assume that C contains a neighborhood of x_i, using the value of $f(x_i)$. Hence, by (b) we have an element $L\in\tau_{j-1}{}^i$, which is a partition between C and D in X_i. It is not difficult to see that $L\in\sigma_{j-1}{}^i$, and is a partition between C and D in X_i'. This completes our proof of this lemma.

Put

$$\sigma_{j(1)}{}^1\times\sigma_{j(2)}{}^2=\{F^1\times F^2:\ F^i\in\sigma_{j(i)}{}^i,\ i=1,\ 2\}\ ,$$

and

$$\sigma_{-1}=\{\emptyset\},\quad \sigma_j=\cup\{\sigma_{j(1)}{}^1\times\sigma_{j(2)}{}^2:\ j=j(1)+j(2)\}$$

for $0\leqq j\leqq n(1)+n(2)$.

Let $\sigma_j{}^*$ be the family consisting of all finite unions of closed subsets of elements of the family σ_j, and let Σ_j be the family uniformly generated by $\sigma_j{}^*$ in Π.

Then the following lemma completes our proof of Theorem 3, since each Σ_j is obviously additive and monotone, and

$$\Sigma_{-1}=\{\emptyset\},\qquad \Pi\in\Sigma_{n(1)+n(2)},\qquad \Sigma_{j+1}\supset\Sigma_j\,.$$

LEMMA 9. *Σ_{j-1} breaks Σ_j for $0\leqq j\leqq n(1)+n(2)$.*

PROOF. By the same argument of the proof of Theorem 1 it suffices to show it for the following case. Let $(x_1,\ x_2)\in F=F_1\times F_2$, $F_i\in\sigma_{j(i)}{}^i$, $i=1,\ 2$, $j(1)+j(2)=j$, and let C, D be closed subsets of F which are FS in Π. Then, take partitions P_i between C_i and D_i in E_i, where $C_i=C\cap E_i$ and $D_i\cap E_i$, $Q_i\in\sigma_{j(i)-1}{}^i$, $P_1=Q_1\times\{x_2\}$, and $P_2=\{x_1\}\times Q_2$. Let K_i and H_i be closed sets in F_i such that

$$C_i\subset K_i,\ D_i\subset H_i,\quad\text{and}\quad K_i\cap H_i=Q_i,\ K_i\cap H_i=F_i\,.$$

Then, apply Lemma 1 for the cover λ consisting four members $F_k=A_1\times A_2$, where each A_i is either K_i or H_i. Note that $T=R_1\cup R_2\in\Sigma_{j-1}$, where $R_1=Q_1\times F_2$, $R_2=F_1\times Q_2$, in this case. Hence, by putting $F=F_k$ we can reduce the general case for the following three special cases.

(i) F is disjoint from the set $E=E_1\cup E_2$. In this case we have a partition $L\in\Sigma_{j-1}$ between C and D in F without any difficulties, since Π_0 is piecewise rectangular and any ULF family in F is also ULF in Π by the condition (###).

(ii) F is disjoint from either E_i (say, E_1), and one of C and D (say, C) is disjoint from E_2. In this case, using the condition (###), we can assume that D contains a neighborhood of the set $F\cap E_2$ in F. Then, this case is reduced to the case (i).

(iii) One of C and D (say, C) is disjoint from E_1 and the other (say, D) is disjoint from E_2. In this case, using the condition (###), we can assume that D contains a neighborhood of the set $F\cap E_1$ in F. Taking a partition between C_2 and D_2 in E_2 (if necessary), this case is reduced to the case (ii). Therefore, in all of these cases we have shown that there exists a partition $L\in\Sigma_{j-1}$ between C and D in F, which completes our proof of Theorem 3.

COROLLARY 2. *The inequality* (**) *is valid when* $\Pi=X\times Y$ *is normal,* $\Pi_0=X_0\times Y_0$ *is open in* Π, *piecewise rectangular normal, both of* X_0 *and* Y_0 *satisfy* $FST(\mathrm{Ind})$, *and* $\mathrm{Ind}\, X_0=\mathrm{Ind}\, X$ *and* $\mathrm{Ind}\, Y_0=\mathrm{Ind}\, Y$, *where* $X_0=X\setminus\{x\}$ *and* $Y_0=Y\setminus\{y\}$. *Without the assumption* $\mathrm{Ind}\, X_0=\mathrm{Ind}\, X$ *and* $\mathrm{Ind}\, Y_0=\mathrm{Ind}\, Y$ *we have the inequality*

$$\mathrm{Ind}(X\times Y)\leqq \mathrm{Ind}\, X_0+\mathrm{Ind}\, Y_0 .$$

REMARK 0. (a) Note that Theorems 0 and 2 also follow from Theorem 3 by adding isolated points to factor spaces X and Y of a piecewise rectangular product $X\times Y$.

(b) Under the following condition (##)′ we can show Theorem 2 in more direct way like that in Theorem 1. This case is, however, contained in our case, since (##)′ together with Lemma 8 implies the condition (##).

(##)′ There exist closed families $\sigma^1{}_i$, $-1\leqq i\leqq n(1)$, consisting of subsets of X_1' such that they satisfy the conditions (a)-(d) in Definition 2 with $k=n(1)$ together with the following condition (e).

(e) For every two sets C and D there is a partition P between them in $F'=F\setminus\{*\}$ such that $[P]_F\in\sigma^1{}_{i-1}$, whenever both C and D are closed subsets of F', FS in X_1, and F is any element of $\sigma^1{}_i$.

3. Remarks and Examples.

REMARK 1. The notion of (piecewise) rectangularity is due to B. A. Pasynkov [9, 10]. When we deal with *normal* spaces, a rectangular product is nothing but an F-product due to J. Nagata [8]. Hence, in this case the priority is due to him (all the cases treated in [4] concerning the inequality (**) are included in this case). It is known, however, that there exist normal piecewise rectangular products which are not rectangular [5, 10, 15, 17] (from the definition we see that every rectangular product is piecewise rectangular).

EXAMPLE 0. The Example 1 in [15] is an example which is non rectangular but satisfies the condition of Theorem 1.

EXAMPLE 1. Without *rectangularity* even Corollary 0 does not hold in general, since it is shown by M. Wage [16] (see also [14]) that there exists a locally compact *perfectly normal* product space $X\times Y$ (hence, it satisfies FST(Ind)) such that $\operatorname{Ind} X=\operatorname{Ind} Y=0<\operatorname{Ind}(X\times Y)$.

EXAMPLE 2. Without FST(Ind) even Corollary 0 does not hold in general, since it is shown by V. V. Filippov [3] that there exist two compact spaces X and Y such that $\operatorname{Ind} X=1$, $\operatorname{Ind} Y=2$, but $\operatorname{Ind}(X\times Y)>3$.

EXAMPLE 3. The class which satisfy FST(Ind) is sufficiently large, since all at most 1-dimensional normal spaces are included in it from the following reason: every (locally) finite union of 0-dimensional subsets is 0-dimensional, since for every normal space the condition $\operatorname{Ind} X=0$ and $\dim X=0$ are equivalent.

In Corollary 0 we assume only that *factor spaces* must satisfy FST(Ind). The following example shows that the assumption that the *product* must satisfy it is much stronger than ours (see also [13, p. 365]).

EXAMPLE 4. There exist two compact spaces X and Y such that both of them satisfy FST(Ind), but their product space $X\times Y$ does not.

Let Z be the famous Lokucievskii's example (e.g. [2, Example 2.2.13]). Put $Z=X\cup Y$ and $X\cap Y=I$, where I is the unit interval and both of X and Y are homeomorphic to the following quotient space K,

$$K=(L\times C)/E,$$

where L is the *one-point* (say *) compactification of the long line L_0, and C is the Cantor set, and E is the equivalence relation on their product $L_0\times C$ corresponding to the following decomposition of $L\times C$: Every one-point subset of $L_0\times C$ and the set $\{*\}\times f^{-1}(t)$, where $t\in I$ and $f: C\to I$ is the continuous map from C onto I defined by matching the end points of each interval removed from I to obtain the Cantor set (e.g. [2, Example 2.2.1]).

Let r be the retraction from K onto I defined by

$$r(s, c)=(*, f(c)) \text{ for } (s, c)\in L_0\times C, \quad \text{and} \quad r(t)=t \text{ for } t\in I.$$

For each $t\in I$ let

$$K_t=r^{-1}(t).$$

(1) Note that $\{K_t : t \in I\}$ is a decomposition of K.

Let $g: X \to K$ and $h: Y \to K$ be homeomorphisms. For each $t \in I$ put

$$X_t = g^{-1}(K_t),, \quad \text{and} \quad Y_t = h^{-1}(K_t).$$

Then, let

$$Z^* = \cup_{t \in I}(X_t \times \{t\} \cup \{t\} \times Y_t), \quad \text{and} \quad I^* = \{(t, t) : t \in I\} .$$

Define $p: Z \to Z^*$ as follows:

$p(x) = (x, t)$ for $x \in X$ and $x \in X_t$, and $p(y) = (t, y)$ for $y \in Y$ and $y \in Y_t$.

By (1) the above definition is well-defined. We shall show that p is a homeomorphism. Since it is one to one, it suffices to show that it is continuous. For any point in $Z \setminus I$ it is easy to see that it is continuous. Hence, we shall consider a point $t \in I$. Take any open neighborhood U of $p(t) = (t, t)$. Then, there exist two sets V and W, open in X and Y respectively, such that

$$(t, t) \in L \cap (V \times W) \subset U .$$

We can also assume that

$$r(g(V) \cup h(W)) \subset r(g(V \cap W)) = r(h(V \cap W))$$

by the definition of the retraction r.

Then, the set $G = V \cup W$ is a neighborhood of t in Z and $p(G) \subset U$. Hence, p is continuous.

Since $\operatorname{Ind} X = \operatorname{Ind} Y = 1$, both spaces satisfy FST(Ind). On the other hand, their product does not satisfy FST(Ind), since their product contains 2-dimensional subset Z^*, which is a union of two 1-dimensional subsets $p(X)$ and $p(Y)$.

REMARK 2. The notion of ULFness is due to M. Katětov [6]. We can shown Corollaries 1 and 2, using the following theorem due to him: A normal spact has the following property (K) if and only if it is *strongly normal* (that is, collectionwise normal and countably paracompact).

(K) The notion of LF ness coincides with the notion of ULFness.

It is indicated by K. Morita [7] that the notion of ULFness is effective to study *the covering dimension of nonnormal spaces* (see also [5]).

EXAMPLE 5. There exist an LF family which is not ULF. Indeed, let X be the famous Bing's example [1, Example 5.1.23] which has a discrete family λ consisting of single points which satisfies that there is no *discrete* family μ consisting open sets U such that $f \in U$ for each $\{f\} \in \lambda$. Since we can show moreover that there is no such *locally finite* family, λ is never ULF from

Lemma 5.

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