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RECOLLEMENT AND IDEMPOTENT IDEALS

Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

By

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The notion of quasi-hereditary algebras was introduced by E. Cline, B. Parshall and L. Scott [3, 4, 8 and 9]. A quasi-hereditary algebra is defined by a chain of particular idempotent ideals, and induces a sequence of recollements of their derived categories. In case A is a semiprimary ring, V. Dlab and C. M. Ringel [5] studied the notion of a quasi-hereditary ring. The notion of recollement was introduced by A. A. Beilinson, J. Bernstein and P. Deligne [2]. In [7] we studied localization of triangulated categories and derived categories, and showed that recollement is equivalent to bilocalization.

Recall that an ideal I of a ring A is called idempotent if $I=AeA$ for some idempotent e of A; in particular, I is a minimal idempotent ideal provided that e is primitive. An ideal J of A is said to be a heredity ideal of A if $J^2=J$, $J(Rad\ A)J=0$, and J_A is projective. Then, in case of A being a semiprimary ring, J is a heredity ideal if and only if there exists an idempotent e of A such that: (1) $J=AeA$; (2) $Ae\otimes_{eAe}eA\cong AeA$; (3) eAe is a semisimple ring [5, 9]. In this case, E. Cline, B. Parshall and L. Scott showed that $\{\boldsymbol{D}^b(\mathrm{Mod}\ A/AeA), \boldsymbol{D}^b(\mathrm{Mod}\ A), \boldsymbol{D}^b(\mathrm{Mod}\ eAe)\}$ is recollement [9].

In this note, we give necessary and sufficient conditions for $\{\boldsymbol{D}^b(\mathrm{Mod}\ A/AeA), \boldsymbol{D}^b(\mathrm{Mod}\ A), \boldsymbol{D}^b(\mathrm{Mod}\ eAe)\}$ to be recollement in case of A is left noetherian or semiprimary. In particular, we study when a minimal idempotent ideal satisfies recollement conditions. Throughout this note, we assume that all rings have unity and that all modules are unital. For a ring A, Mod A (resp. A-Mod) is the category of right (resp., left) A-modules, and mod A (resp., A-mod) is the category of finitely presented right (resp., left) A-modules.

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THEOREM 1. *Suppose A is a left noetherian or semiprimary ring. Let e be an idempotent of A. The following assertions are equivalent*:

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(1) $\{\boldsymbol{D}^b(\mathrm{Mod}\, A/AeA),\ \boldsymbol{D}^b(\mathrm{Mod}\, A),\ \boldsymbol{D}^b(\mathrm{Mod}\, eAe)\}$ *is recollement,*

(2) (i) $\mathrm{Tor}_i^A(A/AeA, A/AeA)=0$ *for all* $i>0$; (ii) {(*a*) *or* (*c*)} *and* {(*b*) *or* (*d*)}.

(3) (i) $\mathrm{Ext}_A^i(A/AeA_A, A/AeA_A)=0$ *for all* $i>0$; (ii) (*a*) *and* {(*b*) *or* (*d*)},

(4) (i) $\mathrm{Ext}_A^i({}_AA/AeA, {}_AA/AeA)=0$ *for all* $i>0$; (ii) (*b*) *and* {(*a*) *or* (*c*)},

(5) (i) $Ae\otimes_{eAe}eA\cong AeA$ *and* $\mathrm{Tor}_i^{eAe}(Ae, eA)=0$ *for all* $i>0$; (ii) {(*a*) *or* (*c*)} *and* {(*b*) *or* (*d*)},

where (*a*) $\mathrm{pdim}\, A/AeA_A<\infty$, (*b*) $\mathrm{pdim}\, {}_AA/AeA<\infty$, (*c*) $\mathrm{pdim}\, Ae_{eAe}<\infty$, *and* (*d*) $\mathrm{pdim}_{eAe}eA<\infty$.

PROOF. First, we show that if A is left noetherian or semiprimary, then we have $\mathrm{wdim}_AA/AeA=\mathrm{pdim}_AA/AeA$ and $\mathrm{wdim}_{eAe}eA=\mathrm{pdim}_{eAe}eA$. If A is left noetherian, then ${}_AAeA$ is a finitely generated left A-module. Therefore we have an epimorphism ${}_AAe^{(n)}\to{}_AAeA$ for some $n\in\boldsymbol{N}$. This implies that eA is a finitely generated left eAe-module. By [1, Theorem 4], we have $\mathrm{wdim}_AA/AeA=\mathrm{pdim}_AA/AeA$ and $\mathrm{wdim}_{eAe}eA=\mathrm{pdim}_{eAe}eA$. If A is semiprimary, then we have also same results by [1, Proposition 7]. According to [7, Section 2 and 3], it suffices to show that the condition (i) in (2)-(5) hold, in order to show that (1) implies the other assertions. Conversely, if the functor $\boldsymbol{D}^b(\mathrm{Mod}\, A/AeA)\to\boldsymbol{D}^b(\mathrm{Mod}\, A)$ is fully faithful, then $0\to\boldsymbol{D}^b(\mathrm{Mod}\, A/AeA)\to\boldsymbol{D}^b(\mathrm{Mod}\, A)\to\boldsymbol{D}^b(\mathrm{Mod}\, eAe)\to 0$ is exact in the sense of [2]. According to [7, Section 2], (a) and (b) are equivalent to (c) and (d), respectively. And (ii) of the other assertions imply that $\{\boldsymbol{D}^b(\mathrm{Mod}\, A/AeA), \boldsymbol{D}^b(\mathrm{Mod}\, A), \boldsymbol{D}^b(\mathrm{Mod}\, eAe)\}$ is recollement (see [7, Sections 2, 3 and Proposition 5.9] for details).

(1)⇒(2): $\boldsymbol{D}^b(\mathrm{Mod}\, A/AeA)\to\boldsymbol{D}^b(\mathrm{Mod}\, A)$ has a left adjoint, say G. Then $G\cong\boldsymbol{L}^{-b}(-\otimes_AA/AeA)$ (see [7, Section 3] or [8, Proof of (2.1) Theorem]). Therefore we have the following isomorphism in $\boldsymbol{D}^b(\mathrm{Mod}\, A/AeA)$:

$$A/AeA\cong\boldsymbol{L}^{-b}(-\otimes_AA/AeA)(A/AeA).$$

In particular, we have

$$\mathrm{Tor}_i^A(A/AeA, A/AeA)=0 \qquad \text{for all } i>0.$$

(2)⇒(1): According to [7, Proposition 5.3] or [8, Proof of (2.1) Theorem], we have a fully faithful functor

$$\boldsymbol{D}^b(\mathrm{Mod}\, A/AeA)\longrightarrow\boldsymbol{D}^b(\mathrm{Mod}\, A).$$

(1)⇔(5): See [8, (2.1) Theorem] and [9, Theorem 2.1].

(1)⇒(3): This is trivial by the following isomorphisms:

$$\begin{aligned}\mathrm{Ext}_A^i(A/AeA_A,\ A/AeA_A)&\cong\mathrm{Hom}_{D^b(\mathrm{Mod}\,A)}(A/AeA_A,\ A/AeA_A[i])\\&\cong\mathrm{Hom}_{D^b(\mathrm{Mod}\,A/AeA)}(A/AeA_A,\ A/AeA_A[i])\\&=0\qquad\text{for all } i>0\,.\end{aligned}$$

(3)⇒(1): By Rickard's results there exists a fully faithful functor $\boldsymbol{D}^-(\mathrm{Mod}\,A/AeA)\to\boldsymbol{D}^-(\mathrm{Mod}\,A)$, in particular, a fully faithful functor $\boldsymbol{D}^b(\mathrm{Mod}\,A/AeA)\to\boldsymbol{D}^b(\mathrm{Mod}\,A)$ (see [6], [10] and [11]).

(4)⇒(2): Considering (3)⇒(1) in case of the left module categories (we need not assume that A is right noetherian), $\{\boldsymbol{D}^b(A/AeA\text{-Mod}),\ \boldsymbol{D}^b(A\text{-Mod}),\ \boldsymbol{D}^b(eAe\text{-Mod})\}$ is recollement. As well as (1)⇒(2), we get $\mathrm{Tor}_i^A(A/AeA,\ A/AeA)=0$ for all $i>0$.

(2)⇒(4): Since the condition (2) is symmetric, $\{\boldsymbol{D}^b(A/AeA\text{-Mod}),\ \boldsymbol{D}^b(A\text{-Mod}),\ \boldsymbol{D}^b(eAe\text{-Mod})\}$ is recollement (we need not assume that A is right noetherian). well as (1)⇒(3), we get $\mathrm{Ext}_A^i({}_AA/AeA,\ {}_AA/AeA)=0$ for all $i>0$.

REMARK. (2)-(5) in the above theorem are also equivalent for right noetherian rings.

Recall that a ring A is called a noetherian algebra if its center $\boldsymbol{Z}(A)$ is a noetherian ring, and A is a finitely generated $\boldsymbol{Z}(A)$-module.

PROPOSITION 2. *Let A be a noetherian algebra, and e an idempotent. The following assertions are equivalent*:

(1) $\{\boldsymbol{D}^b(\mathrm{mod}\,A/AeA),\ \boldsymbol{D}^b(\mathrm{mod}\,A),\ \boldsymbol{D}(\mathrm{mod}\,eAe)$ *is recollement,*

(2) $\{\boldsymbol{D}^b(\mathrm{Mod}\,A/AeA),\ \boldsymbol{D}^b(\mathrm{Mod}\,A),\ \boldsymbol{D}^b(\mathrm{Mod}\,eAe)\}$ *is recollement.*

PROOF. In general, if R is a right coherent ring, then we have $\boldsymbol{D}^b{}_{\mathrm{mod}\,R}(\mathrm{Mod}\,R)\cong\boldsymbol{D}^b(\mathrm{mod}\,R)$. Also, for a given $X\in\mathrm{mod}\,R$, if $\mathrm{Ext}_R^i(X,\ Y)=0$ for all $i>n$ and $Y\in\mathrm{mod}\,R$, then $\mathrm{pdim}\,X_R\leq n$.

(1)⇒(2): Let F and G be right and left adjoint functors of $\boldsymbol{D}^b(\mathrm{mod}\,A/AeA)\to\boldsymbol{D}^b(\mathrm{mod}\,A)$, respectively. Since A is noetherian and A/AeA is a finitely generated A-module, we have $G\cong\boldsymbol{L}^{-b}(-\otimes_A A/AeA)$, and $\mathrm{Tor}_i^A(A/AeA,\ A/AeA)=0$ for all $i>0$ as well as (1)⇒(2) in the proof of theorem 1. Moreover $\mathrm{Tor}_i^A(\mathrm{mod}\,A,\ A/AeA)=0$ implies $\mathrm{Tor}_i^A(\mathrm{Mod}\,A,\ A/AeA)=0$ for all i, in particular, $\mathrm{pdim}\,{}_AA/AeA<\infty$. For given $X\in\mathrm{mod}\,A$, we have the following isomorphisms:

$$\begin{aligned}\mathrm{Ext}_A^i(A/AeA,\ X)&\cong\mathrm{Hom}_{D^b(\mathrm{mod}\,A)}(G(A/AeA),\ X[i])\\&\cong\mathrm{Hom}_{D^b(\mathrm{mod}\,A/AeA)}(A/AeA,\ FX[i])\\&\cong H^i(FX[i])\qquad\text{for all } i.\end{aligned}$$

Since $FX[i]$ is contained in $\boldsymbol{D}^b(\text{mod } A)$, we get pdim $A/AeA_A<\infty$. Hence $\{\boldsymbol{D}^b(\text{Mod } A/AeA),\ \boldsymbol{D}^b(\text{Mod } A),\ \boldsymbol{D}^b(\text{Mod } eAe)\}$ is recollement by Theorem 1.

(2)$\Rightarrow$(1): Let E and H be right and left adjoint functors of $\boldsymbol{D}^b(\text{Mod } A/AeA) \to \boldsymbol{D}^b(\text{Mod } A)$, respectively. It is clear that $\boldsymbol{D}^b(\text{mod } A/AeA) \to \boldsymbol{D}^b(\text{mod } A)$ has a left adjoint. Since A is a noetherian algebra, and A/AeA is finitely generated, $\text{Ext}_A^i(A/AeA, X)$ is a finitely generated A/AeA-module for all $X \in \text{mod } A$. Then it is easy to see that $\text{Im } H|_{D^b(\text{mod } A)}$ is contained in $\boldsymbol{D}^b{}_{\text{mod } A}(\text{Mod } A)$. By the above equivalence, $\boldsymbol{D}^b(\text{mod } A/AeA) \to \boldsymbol{D}^b(\text{mod } A)$ has a right adjoint. We are done by Theorem 1.

Let A be a left (or right) noetherian or semiprimary ring. An ideal I of A is called a recollement ideal of A if $I=AeA$ with some idempotent e of A which satisfies the equivalent conditions (2)-(5) of Theorem 1. The next proposition is useful to exhibiting examples of recollement ideals.

PROPOSITION 3. *Let R be a commutative ring, and A and B R-algebras. Suppose A is a left or right noetherian ring and B is a finitely generated projective R-module. If I is a recollement ideal of A, then $I \otimes_R B$ is a recollement ideal of $A \otimes_R B$.*

PROOF. First, $A \otimes_R B$ is a left or right noetherian ring, because B is a finitely generated R-module. Since B is R-projective, we have pdim $I_A \geqq$ pdim $I \otimes_R B_{A \otimes_R B}$ and pdim ${}_R I \geqq \text{pdim}_{A \otimes_R B} I \otimes_R B$. And let $P\cdot$ be a projective resolution of A/I. Then we have

$$\begin{aligned}\text{Tor}_i^{A \otimes_R B}(A/I \otimes_R B,\ A/I \otimes_R B) &\cong \text{H}_i(P\cdot \otimes_R B \otimes_{A \otimes_R B} A/I \otimes_R B) \\ &\cong \text{H}_i(P\cdot \otimes_A A/I) \otimes_R B \\ &\cong \text{Tor}_i^A(A/I,\ A/I) \otimes_R B \\ &= 0 \qquad \text{for all } i>0\,.\end{aligned}$$

LEMMA 4. *If A is a local semiprimary ring, then* pdim M *is* 0 *or* ∞, *for all modules M.*

PROPOSITION 5. *Suppose A is a semiprimary ring. Let I be a minimal idempotent ideal of A. Then I is a recollement ideal of A if and only if I is projective as both a left and right A-module.*

PROOF. If $I=AeA$ is projective as both a left and right A-module, then it is easy to see that A/AeA satisfies the condition (2) of Theorem 1. Conversely, if $I=AeA$ is a recollement ideal, then AeA has finite projective dimension. Let $P\cdot$ be a projective resolution of Ae as right eAe-modules. Then given any left A-module X, we get

$$\mathrm{Tor}_i^A(AeA, X)\cong \mathrm{Tor}_i^A(Ae\otimes_{eAe}eA, X)\cong \mathrm{H}_i(P\cdot\otimes_{eAe}eA\otimes_A X)$$
$$\cong \mathrm{Tor}_i^{eAe}(Ae, eX).$$

For every left eAe-module Y, there exists a left A-module X such that Y is isomorphic to eX. Then Ae has finite projective dimension in Mod eAe. Since I is a minimal idempotent ideal of A, eAe is a local semiprimary ring. Therefore Ae is a projective right eAe-module by Lemma 4. Hence AeA is a projective right A-module by the above isomorphisms. Similarly, AeA is also a projective left A-module.

According to the above proposition, it suffices to find idempotent ideals which are two-sided projective, when we want to find minimal recollement ideals. But the following proposition implies that heredity ideals are best possible in case of rings of finite global dimension.

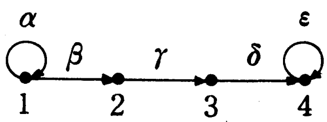
PROPOSITION .6 *Suppose A is a semiprimary ring of finite global dimension. Let I be a minimal idempotent ideal. Then I is a recollement ideal if and only if I is a heredity ideal.*

PROOF. Let I be AeA with some idempotent e of A, $P\cdot$ a projective resolution of eAe/eJe as right eAe-modules. The $P\cdot\otimes_{eAe}eA$ is a projective resolution of $eA/eJeA$ as right A-modules, where J is the radical of A. Therefore, we get

$$\mathrm{Tor}_i^{eAe}(eAe/eJe, eX)\cong \mathrm{H}_i(P\cdot\otimes_{eAe}eA\otimes_A X)$$
$$\cong \mathrm{Tor}_i^A(eA/eJeA, X)$$

According to assumption, pdim $eA/eJeA<\infty$, and pdim $eAe/eJe<\infty$. Since eAe is a local semiprimary ring, eAe/eJe is a projective eAe-module by Lemma 4. Hence $eJe=0$.

EXAMPLES. (*a*) Let A be a finite dimensional algebra over a field k which has a quiver with relations:



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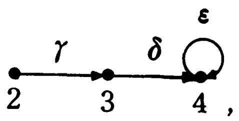
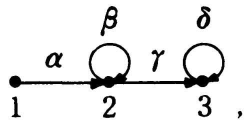
with $\alpha^2=\varepsilon^2=\gamma\beta=0$. Then Ae_1A is projective as both sides. Moreover, e_1Ae_1 is isomorphic to $k[x]/(x^2)$ as a ring, and A/Ae_1A has the following quiver with relations:



with $\varepsilon^2=0$. Hence we have pdim $A=$gldim $e_1Ae_1=$gldim $A/Ae_1A=\infty$.

(*b*) Let A be a finite dimensional algebra over a field k which has a quiver with relations:



with $\beta\alpha=\delta\gamma=\beta^2=\delta^2=0$. Then $A(e_1+e_2)A$ is a recollement ideal. But Ae_2A is not a recollement ideal because of pdim $Ae_2A_A=\infty$.

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