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A CLASS OF SPACES WHOSE COUNTABLE PRODUCT WITH A PERFECT PARACOMPACT SPACE IS PARACOMPACT

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1. Introduction.

All spaces are assumed to be T_1-spaces. In particular, paracompact spaces are assumed to be T_2. The letter ω denotes the set of natural numbers.

Let us denote by $\mathcal{P}(\mathcal{L})$ the class of all spaces (regular spaces) whose product with every paracompact (regular Lindelöf) space is paracompact (Lindelöf). On the other hand, let $\mathcal{L}'$ be the class of regular spaces whose product with every regular hereditarily Lindelöf space is Lindelöf. Then it is clear that $\mathcal{L}\subset\mathcal{L}'$. A general problem is to characterize $\mathcal{P}(\mathcal{L})$ (Tamano). T. Przymusiński [13] posed the following problem: If $X\in\mathcal{P}(\mathcal{L})$, then is X^{ω} paracompact (Lindelöf)? Furthermore, E. Michael asked whether $\mathcal{L}'$ is closed with respect to countable products. K. Alster [2], [3] gave a negative answer to E. Michael's problem. He showed that there are a separable metric space M and a regular Lindelöf space X such that for every regular Lindelöf syace Y and $n\in\omega$, the products $Y\times X^{n}$ and X^{ω} are Lindelöf but $M\times X^{\omega}$ is not. However, if X is a separable metric space or X is a regular Čech-complete Lindelöf space or X is a regular C-scattered Lindelöf space, then $X^{\omega}\in\mathcal{L}'$. The first result is due to E. Michael (cf. [10]), the second one is due to Z. Frolik [7] and the third one is due to K. Alster [1].

Let $\mathcal{DC}$ be the class of all T_2-spaces which have a discrete cover by compact sets. The topological game $G(\mathcal{DC}, X)$ was introduced and studied by R. Telgársky [16]. The games are played by two persons called Players I and II. Players I and II choose closed subsets of II's previous play (or of X, if $n=0$): Player I's choice must be in the class $\mathcal{DC}$ and II's choice must be disjoint from I's. We say that Player I *wins* if the intersection of II's choices is empty. Recall from [16] that a space X is said to be a *$\mathcal{DC}$-like space* if Player I has a winning strategy in $G(\mathcal{DC}, X)$. The class of $\mathcal{DC}$-like spaces includes all spaces which admit a σ-closure-preserving closed cover by compact sets, and

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paracompact, σ-C-scattered spaces. R. Telgársky proved that if X is a paracompact (regular Lindelöf) $\mathcal{DC}$-like space, then $X \in \mathcal{P}(\mathcal{L})$. M.E. Rudin and S. Watson [14] proved that the product of countably many scattered paracompact spaces is paracompact. Furthermore, A. Hohti and J. Pelant [9] showed that the product of countably many paracompact, σ-C-scattered spaces is paracompact (cf. [6]). K. Alster [4] also proved that if Y is a perfect paracompact space and X_n is a scattered paracompact space for each $n \in \omega$, then $Y \times \prod_{n \in \omega} X_n$ is paracompact.

In this paper, we discuss paracompact (regular Lindelöf) $\mathcal{DC}$-like spaces and generalize K. Alster's results. More precisely, we show that if Z is a perfect paracompact (regular hereditarily Lindelöf) space and Y_i is a paracompact (regular Lindelöf) $\mathcal{DC}$-like space for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is paracompact (Lindelöf). Therefore, if X is a regular Lindelöf $\mathcal{DC}$-like space, then $X^\omega \in \mathcal{L}'$.

2. Topological games.

The topological game $G(\mathcal{DC}, X)$ is described in the introduction. F. Galvin and R. Telgársky showed that if Player I has a winning strategy in $G(\mathcal{DC}, X)$, then he has a stationary winning strategy in $G(\mathcal{DC}, X)$, i.e., a winning strategy which depends only on II's previous move. More precisely,

LEMMA 2.1. ([8]). *Player I has a winning strategy in $G(\mathcal{DC}, X)$ if and only if there is a function s from 2^X into $2^X \cap \mathcal{DC}$, where 2^X denotes the set of all closed subsets of X, satisfying*

(i) *$s(F) \subset F$ for each $F \in 2^X$,*

(ii) *if $\{F_n : n \in \omega\}$ is a decreasing sequence of closed subsets of X such that $s(F_n) \cap F_{n+1} = \emptyset$ for each $n \in \omega$, then $\bigcap_{n \in \omega} F_n = \emptyset$.*

The following resulls are well known.

LEMMA 2.2 (R. Telgársky [16]). *Le X and Y be spaces, and let $f: X \to Y$ be a perfect mapping from X onto Y. If Y is a $\mathcal{DC}$-like space, then X is also a $\mathcal{DC}$-like space.*

LEMMA 2.3 (R. Telgársky [16]). *If a space X has a countable closed cover by $\mathcal{DC}$-like sets, then X is a $\mathcal{DC}$-like space.*

Recall that a space X is *scattered* if every non-empty subset A of X has an isolated point of A, and *C-scattered* if for every non-empty closed subset A of X, there is a point of A which has a compact neighborhood in A. Clearly scattered spaces and locally compact T_2-spaces are C-scattered. Let X be a space. For each $F\in 2^X$, let

$F^{(1)}=\{x\in F$: x has no compact neighborhood in $F\}$.

Let $X^{(0)}=X$. For each successor ordinal α, let $X^{(\alpha)}=(X^{(\alpha-1)})^{(1)}$. If α is a limit ordinal, let $X^{(\alpha)}=\bigcap_{\beta<\alpha} X^{(\beta)}$. Notice that a space X is C-scattered if and only if $X^{(\alpha)}=\emptyset$ for some ordinal α. If X is C-scattered, let $\lambda=\inf\{\alpha: X^{(\alpha)}=\emptyset\}$. We say that λ is the *C-scattered height* of X. A space X is said to be *σ-scattered* (*σ-C-scattered*) if X is the union of countably many closed scattered (C-scattered) subspaces.

LEMMA 2.4. (R. Telgársky [16]). (a) *If a space X has a σ-closure-preserving closed cover by compact sets, then X is a $\mathcal{D}C$-like space.*

(b) *If X is a paracompact, σ-C-scattered space, then X is a $\mathcal{D}C$-like space.*

LEMMA 2.5. (R. Telgársky [16]). *If X is a paracompact (regular Lindelöf) $\mathcal{D}C$-like space, then $X\in\mathcal{P}(\mathcal{L})$.*

For topological games, the reader is refered to R. Telgársky [16], [17] and Y. Yajima [18].

3. Paracompactness and Lindelöf property.

LEMMA 3.1 (K. Nagami [11]). *For a paracompact (regular Lindelöf) space X, there are a paracompact (regular Lindelöf) space X_0 with dim $X_0\leqq 0$ and a perfect mapping f_X: $X_0\to X$ from X_0 onto X.*

Let A be a set. We denote by $A^{<\omega}$ the set of all finite sequences of elements of A. If $\tau=(a_0, \cdots, a_n)\in A^{<\omega}$ and $a\in A$, then $\tau\oplus a$ denotes the sequence $(a_0, \cdots, a_n, a)$.

The following is the main result in this paper.

THEOREM 3.2. *If Z is a perfect paracompact space and Y_i is a paracompact $\mathcal{D}C$-like space for each $i\in\omega$, then $Z\times\prod_{i\in\omega} Y_i$ is paracompact.*

PROOF. By Lemma 3.1, for each $i\in\omega$, there are a paracompact space $Y_{i,0}$ with $\dim Y_{i,0}\leqq 0$ and a perfect mapping f_i: $Y_{i,0}\to Y_i$ from $Y_{i,0}$ onto Y_i. Let $X=\bigoplus_{i\in\omega} Y_{i,0}\cup\{a\}$, where $a\notin\bigcup_{i\in\omega} Y_{i,0}$. The topology of X is as follows: Every $Y_{i,0}$ is an open-and-closed subspace of X and a is isolated in X. Then X is a paracompact space with $\dim X\leqq 0$. It follows from Lemmas 2.2 and 2.3 that

X is a $\mathcal{DC}$-like space. Define $f: \prod_{i\in\omega} Y_{i,0} \to \prod_{i\in\omega} Y_i$ by $f(y)=(f_i(y_i))_{i\in\omega}$ for $y=(y_i)_{i\in\omega}\in \prod_{i\in\omega} Y_{i,0}$. Then $id_Z\times f: Z\times \prod_{i\in\omega} Y_{i,0}\to Z\times \prod_{i\in\omega} Y_i$ is a perfect mapping from $Z\times \prod_{i\in\omega} Y_{i,0}$ onto $Z\times \prod_{i\in\omega} Y_i$. Since $Z\times \prod_{i\in\omega} Y_{i,0}$ is a closed subspace of $Z\times X^{\omega}$, in order to prove this theorem, it suffices to prove that $Z\times X^{\omega}$ is paracompact.

Let us denote by $\mathcal{B}$ the base of $Z\times X^{\omega}$ consisting of sets of the form $B=U_B\times \prod_{i\in\omega} B_i$, where U_B is an open subset of Z and there is an $n\in\omega$ such that for each $i\leqq n$, B_i is an open-and-closed subset of X and for each $i>n$, $B_i=X$. For each $B=U_B\times \prod_{i\in\omega} B_i\in \mathcal{B}$, let $n(B)=\inf\{i\in\omega: B_j=X \text{ for each } j\geqq i\}$.

Let $\mathcal{O}$ be an open covering of $Z\times X^{\omega}$ and let $\mathcal{O}^F$ be the set of all finite unions of elements of $\mathcal{O}$. Put $\mathcal{O}'=\{B\in\mathcal{B}: B\subset O \text{ for some } O\in\mathcal{O}^F\}$. Let $\mathcal{K}=\{\prod_{i\in\omega} K_i: K_i \text{ is a compact subset of } X \text{ for each } i\in\omega\}$. For each $z\in Z$ and $K\in\mathcal{K}$, let $K_{(z,K)}=\{z\}\times K$. Then there is an $O\in\mathcal{O}^F$ such that $K_{(z,K)}\subset O$. By Wallace theorem in R. Engelking [5], there is a $B\in\mathcal{B}$ such that $K_{(z,K)}\subset B\subset O$. Thus we have $B\in\mathcal{O}'$. Define $n(K_{(z,K)})=\inf\{n(O): O\in\mathcal{O}' \text{ and } K_{(z,K)}\subset O\}$. It suffices to prove that $\mathcal{O}'$ has a σ-locally finite open refinement.

Let s be a stationary winning strategy for Player I in $G(\mathcal{DC}, X)$. Let $B=U_B\times \prod_{i\in\omega} B_i\in\mathcal{B}$ such that for each $i\leqq n(B)$, we have already obtained a compact set $C_{\lambda(B,i)}$ of $B_i(C_{\lambda(B,n(B))}=\emptyset$. $C_{\lambda(B,i)}=\emptyset$ may be occur for $i<n(B))$. We define $\mathcal{G}_{m,j}(B)$ and $\mathcal{B}_{m,j}(B)$ of collections of elements of $\mathcal{B}$ for each $m, j\in\omega$. Fix $i\leqq n(B)$. If $C_{\lambda(B,i)}\neq\emptyset$, let $W_{\gamma(B,i)}=B_i$. Put $\Lambda(B,i)=\{\lambda(B,i)\}$ and $\Gamma(B,i)=\{\gamma(B,i)\}$. Let $\mathcal{C}(B,i)=\{C_\lambda: \lambda\in\Lambda(B,i)\}=\{C_{\lambda(B,i)}\}$, and $\mathcal{W}(B,i)=\{W_\gamma: \gamma\in\Gamma(B,i)\}=\{W_{\gamma(B,i)}\}$. Assume that $C_{\lambda(B,i)}=\emptyset$. Then there is a discrete collection $\mathcal{C}(B,i)=\{C_\lambda: \lambda\in\Lambda(B,i)\}$ of compact susbets of X such that $s(B_i)=\bigcup\mathcal{C}(B,i)$. Since B_i is an open-and-closed subspace of X, B_i is a paracompact space with $\dim B_i\leqq 0$. Then there is a pairwise disjoint collection $\mathcal{W}(B,i)=\{W_\gamma: \gamma\in\Gamma(B,i)\}$ of open subsets in B_i (and hence, in X), satisfying

(i) $\mathcal{W}(B,i)$ covers B_i,

(ii) Every member of $\mathcal{W}(B,i)$ meets at most one member of $\mathcal{C}(B,i)$.

In each case, for $\gamma\in\Gamma(B,i)$, $K_\gamma=W_\gamma\cap C_\lambda$ if W_γ meets some (unique) C_λ. If $W_\gamma\cap(\bigcup\mathcal{C}(B,i))=\emptyset$, then we take a point $p_\gamma\in W_\gamma$ and let $K_\gamma=\{p_\gamma\}$. Thus, if $C_{\lambda(B,i)}\neq\emptyset$, then $K_{\gamma(B,i)}=W_{\gamma(B,i)}\cap C_{\lambda(B,i)}=C_{\lambda(B,i)}$. Put $\Delta_B=\Gamma(B,0)\times\cdots\times\Gamma(B,n(B))$. For each $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta_B$, let $K(\delta)=K_{\gamma(\delta,0)}\times\cdots\times K_{\gamma(\delta,n(B))}\times\{a\}\times\cdots\times\{a\}\times\cdots$, and let $\mathcal{K}_B=\{K(\delta): \delta\in\Delta_B\}$. Then $\mathcal{K}_B\subset\mathcal{K}$. For each $z\in U_B$ and $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta_B$, let $r(K_{(z,K(\delta))})=\max\{n(K_{(z,K(\delta))}), n(B)\}$. Fix $z\in U_B$ and $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta_B$. Take an $O_{z,\delta}=U_{z,\delta}\times \prod_{i\in\omega} O_{z,\delta,i}\in\mathcal{O}'$ such that $K_{(z,K(\delta))}\subset O_{z,\delta}$ and $n(K_{(z,K(\delta))})=n(O_{z,\delta})$. Then we can take an $H_{(z,K(\delta))}=H_{z,\delta}\times\prod_{i\in\omega}H_{(z,K(\delta)),i}\in\mathcal{B}$ such that:

$$\text{(iii)}\quad H_{z,\delta}\times\prod_{i=0}^{n(K_{(z,K(\delta))})-1}H_{(z,K(\delta)),i}\times X\times\cdots\times X\times\cdots\subset O_{z,\delta}$$
$$\text{and}\quad z\in H_{z,\delta}\subset U_B\cap U_{z,\delta},$$

(iv-1) For each i with $n(K_{(z,K(\delta))})\leqq i\leqq r(K_{(z,K(\delta))})$, let $H_{(z,K(\delta)),i}=W_{\gamma(\delta,i)}$,

(iv-2) For each $i<n(K_{(z,K(\delta))})$ with $i\leqq n(B)$, let $H_{(z,K(\delta)),i}$ be an open-and-closed subset of $W_{\gamma(\delta,i)}$ such that $K_{\gamma(\delta,i)}\subset H_{(z,K(\delta)),i}\subset O_{z,\delta,i}$,

(iv-3) For each i with $n(B)<i<n(K_{(z,K(\delta))})$, let $H_{(z,K(\delta)),i}=\{a\}$,

(iv-4) In case of that $r(K_{(z,K(\delta))})=n(B)$, let $H_{(z,K(\delta)),i}=X$ for $n(B)<i$. In case of that $r(K_{(z,K(\delta))})=n(K_{(z,K(\delta))})>n(B)$, let $H_{(z,K(\delta)),i}=X$ for $n(K_{(z,K(\delta))})\leqq i$.

Then we have $K_{(z,K(\delta))}\subset H_{(z,K(\delta))}$. Fix $m\in\omega$ and let $V_m(K(\delta))=\{z\in U_B: n(K_{(z,K(\delta))})\leqq m\}$. Then $V_m(K(\delta))=\cup\{H_{z,\delta}: n(K_{(z,K(\delta))})\leqq m\}$. Since Z is a perfect paracompact space, there is a family $\mathcal{V}_{\delta,m}=\bigcup_{j\in\omega}\mathcal{V}_{\delta,m,j}$, where $\mathcal{V}_{\delta,m,j}=\{V_\alpha: \alpha\in\Xi_{\delta,m,j}\}$, of collections of open sets in $V_m(K(\delta))$ (and hence, in Z) satisfying

(v) Every member of $\mathcal{V}_{\delta,m}$ is contained in some member of $\{H_{z,\delta}: n(K_{(z,K(\delta))})\leqq m\}$,

(vi) $\mathcal{V}_{\delta,m}$ covers $V_m(K(\delta))$,

(vii) $\mathcal{V}_{\delta,m,j}$ is discrete in Z for each $j\in\omega$.

For $j\in\omega$ and $\alpha\in\Xi_{\delta,m,j}$, take a $z(\alpha)\in V_m(K(\delta))$ such that $V_\alpha\subset H_{z(\alpha),\delta}$. Put $W_\delta=\prod_{i=0}^{n(B)}W_{\gamma(\delta,i)}\times X\times\cdots\times X\times\cdots$ and $V_{\alpha,\delta}=V_\alpha\times W_\delta$. Then $\{V_{\alpha,\delta}: \delta\in\Delta_B,\ m,\ j\in\omega$ and $\alpha\in\Xi_{\delta,m,j}\}$ is a collection of elements of $\mathcal{B}$ such that for each $\delta\in\Delta_B$, $m,\ j\in\omega$ and $\alpha\in\Xi_{\delta,m,j}$, $V_{\alpha,\delta}\subset B$ and $\{V_{\alpha,\delta}: \delta\in\Delta_B,\ m,\ j\in\omega$ and $\alpha\in\Xi_{\delta,m,j}\}$ covers B.

(viii) For each $m,\ j\in\omega$, $\{V_{\alpha,\delta}: \delta\in\Delta_B$ and $\alpha\in\Xi_{\delta,m,j}\}$ is discrete in $Z\times X^\omega$.

Fix $m,\ j\in\omega$. Let $(z,x)\in Z\times X^\omega$ and $x=(x_i)_{i\in\omega}$. For each $i\leqq n(B)$, since B_i is an open-and-closed subset of X, we may assume that $x_i\in B_i$. There is a unique $\delta=(\gamma(\delta,0),\cdots,\gamma(\delta,n(B)))\in\Delta_B$ such that $x\in W_\delta$. Since $\mathcal{V}_{\delta,m,j}$ is discrete in Z, there is an open neighborhood U of z in Z such that U meets at most one member of $\mathcal{V}_{\delta,m,j}$. Then $U\times W_\delta\in\mathcal{B}$ and $U\times W_\delta$ meets at most one member of $\{V_{\alpha,\delta'}: \delta'\in\Delta_B$ and $\alpha\in\Xi_{\delta',m,j}\}$. Thus $\{V_{\alpha,\delta}: \delta\in\Delta_B$ and $\alpha\in\Xi_{\delta,m,j}\}$ is discrete in $Z\times X^\omega$.

For each $\delta\in\Delta_B$, $m,\ j\in\omega$ and $\alpha\in\Xi_{\delta,m,j}$, let $G_{\alpha,\delta}=V_\alpha\times\prod_{i\in\omega}H_{(z(\alpha),K(\delta)),i}\subset V_{\alpha,\delta}$ and $\mathcal{G}_{\delta,m,j}(B)=\{G_{\alpha,\delta}: \alpha\in\Xi_{\delta,m,j}\}$. Define $\mathcal{G}_{m,j}(B)=\cup\{\mathcal{G}_{\delta,m,j}(B): \delta\in\Delta_B\}$. Then we have

(ix) For each $m,\ j\in\omega$, every member of $\mathcal{G}_{m,j}(B)$ is contained in some member of $\mathcal{O}'$.

(x) For each $m, j\in\omega$, $\mathcal{G}_{m,j}(B)$ is discrete in $Z\times X^{\omega}$.

This is clear from (viii).

Fix $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$, $m, j\in\omega$ and $\alpha\in\Xi_{\delta,m,j}$. Let $A\subset\{0, 1, \cdots, r(K_{(z(\alpha),K(\delta))})\}$. In case of that $r(K_{(z(\alpha),K(\delta))})=n(B)$. For each $i\in A$, let $B_{\alpha,A,i}=W_{\gamma(\delta,i)}-H_{(z(\alpha),K(\delta)),i}$. For each $i\notin A$ with $i\leqq n(B)$, let $B_{\alpha,A,i}=H_{(z(\alpha),K(\delta)),i}$. For each $i>n(B)$, let $B_{\alpha,A,i}=X$. Put $B_{\alpha,A}=V_\alpha\times\coprod_{i\in\omega}B_{\alpha,A,i}$. In case of that $r(K_{(z(\alpha),K(\delta))})=n(K_{(z(\alpha),K(\delta))})>n(B)$. For each $i\in A$ with $i\leqq n(B)$, let $B_{\alpha,A,i}=W_{\gamma(\delta,i)}-H_{(z(\alpha),K(\delta)),i}$. For each $i\notin A$ with $i\leqq n(B)$, let $B_{\alpha,A,i}=H_{(z(\alpha),K(\delta)),i}$. Let $n(B)<i<n(K_{(z(\alpha),K(\delta))})$. If $i\in A$, let $B_{\alpha,A,i}=X-H_{(z(\alpha),K(\delta)),i}=\bigoplus_{i\in\omega}Y_{i,0}$. If $i\notin A$, let $B_{\alpha,A,i}=H_{(z(\alpha),K(\delta)),i}=\{a\}$. For $i\geqq n(K_{(z(\alpha),K(\delta))})$, let $B_{\alpha,A,i}=X$. Put $B_{\alpha,A}=V_\alpha\times\prod_{i\in\omega}B_{\alpha,A,i}$. In each case, $B_{\alpha,A,i}\subset B_i$ for each $i\in\omega$. Notice that if $B_{\alpha,A}\neq\emptyset$, then $n(B)<n(B_{\alpha,A})$. By the definition, $V_{\alpha,\delta}=G_{\alpha,\delta}\cup(\cup\{B_{\alpha,A}: A\subset\{0, 1, \cdots, r(K_{(z(\alpha),K(\delta))})\}\})$. Since $n(K_{(z(\alpha),K(\delta))})\leqq m$, for a subset $A\subset\{0, 1, \cdots, \max\{m, n(B)\}\}$, let $\mathcal{B}_{\delta,m,j,A}(B)=\{B_{\alpha,A}: \alpha\in\Xi_{\delta,m,j}, B_{\alpha,A}$ is defined and $B_{\alpha,A}\neq\emptyset\}$. For $m, j\in\omega$ and $A\subset\{0, 1, \cdots, \max\{m, n(B)\}\}$, define $\mathcal{B}_{m,j,A}(B)=\cup\{\mathcal{B}_{\delta,m,j,A}(B): \delta\in\Delta_B\}$. Then, by (viii), we have

(xi) Every $\mathcal{B}_{m,j,A}(B)$ is discrete in $Z\times X^{\omega}$.

Let $\mathcal{B}_{m,j}(B)=\cup\{\mathcal{B}_{m,j,A}(B): A\subset\{0, 1, \cdots, \max\{m, n(B)\}\}$. Then, by (xi),

(xii) For each $m, j\in\omega$, $\mathcal{B}_{m,j}(B)$ is locally finite in $Z\times X^{\omega}$.

Fix a $B_{\alpha,A}=V_\alpha\times\prod_{i\in\omega}B_{\alpha,A,i}\in\mathcal{B}_{\delta,m,j,A}(B)$ for $\delta=(\gamma(\delta, 0), \cdots, \gamma(\delta, n(B)))\in\Delta_B$, $m, j\in\omega$, $\alpha\in\Xi_{\delta,m,j}$ and $A\subset\{0, 1, \cdots, \max\{m, n(B)\}\}$.

(xiii) For each $i\in A$ with $i\leqq n(B)$ such that $C_{\lambda(B,i)}=\emptyset$, $s(B_i)\cap B_{\alpha,A,i}=\emptyset$.

Since $B_{\alpha,A,i}=W_{\gamma(\delta,i)}-H_{(z(\alpha),K(\delta)),i}$, $s(B_i)\cap B_{\alpha,A,i}=(\cup\mathcal{C}(B, i))\cap(W_{\gamma(\delta,i)}-H_{(z(\alpha),K(\delta)),i})=K_{\gamma(\delta,i)}-H_{(z(\alpha),K(\delta)),i}=\emptyset$.

For each $i\notin A$ with $i\leqq n(B)$, a compact set $K_{\gamma(\delta,i)}$ is contained in $B_{\alpha,A,i}=H_{(z(\alpha),K(\delta)),i}$. Let $C_{\lambda(B_{\alpha,A},i)}=K_{\gamma(\delta,i)}$. For each $i\notin A$ with $n(B)<i<n(K_{(z(\alpha),K(\delta))})$, let $C_{\lambda(B_{\alpha,A},i)}=\{a\}$. For each $i\in A$, let $C_{\lambda(B_{\alpha,A},i)}=\emptyset$.

Now we define $\mathcal{G}_\tau$ and $\mathcal{B}_\tau$ for each $\tau\in(\omega\times\omega)^{<\omega}$ with $\tau\neq\emptyset$. For each $m, j\in\omega$, let $\mathcal{G}_{(m,j)}=\mathcal{G}_{(m,j)}(Z\times X^{\omega})=\mathcal{G}_{m,j}(Z\times X^{\omega})$ and $\mathcal{B}_{(m,j)}=\mathcal{B}_{(m,j)}(Z\times X^{\omega})=\mathcal{B}_{m,j}(Z\times X^{\omega})$. Assume that for $\tau\in(\omega\times\omega)^{<\omega}$ with $\tau\neq\emptyset$, we have already obtained $\mathcal{G}_\tau$ and $\mathcal{B}_\tau$. For each $B\in\mathcal{B}_\tau$ and $m, j\in\omega$, we denote $\mathcal{G}_{m,j}(B)$ and $\mathcal{B}_{m,j}(B)$ by $\mathcal{G}_{\tau\oplus(m,j)}(B)$ and $\mathcal{B}_{\tau\oplus(m,j)}(B)$ respectively. Define $\mathcal{G}_{\tau\oplus(m,j)}=\cup\{\mathcal{G}_{\tau\oplus(m,j)}(B): B\in\mathcal{B}_\tau\}$ and $\mathcal{B}_{\tau\oplus(m,j)}=\cup\{\mathcal{B}_{\tau\oplus(m,j)}(B): B\in\mathcal{B}_\tau\}$.

Our proof is complete if we show

CLAIM. *$\cup\{\mathcal{G}_\tau: \tau\in(\omega\times\omega)^{<\omega}$ and $\tau\neq\emptyset\}$ is a σ-locally finite open refinement of $\mathcal{O}'$.*

PROOF OF CLAIM. Let $\tau\in(\omega\times\omega)^{<\omega}$ and $\tau\neq\emptyset$. By the construction, $\mathcal{G}_\tau\subset\mathcal{B}$. By (ix), every member of $\mathcal{G}_\tau$ is contained in some member of $\mathcal{O}'$. By (x), (xii) and induction, $\mathcal{G}_\tau$ is locally finite in $Z\times X^\omega$. Assume that $\cup\{\mathcal{G}_\tau: \tau\in(\omega\times\omega)^{<\omega}$ and $\tau\neq\emptyset\}$ does not cover $Z\times X^\omega$. Take a point $(z, x)\in Z\times X^\omega-\cup\{\cup\mathcal{G}_\tau: \tau\in(\omega\times\omega)^{<\omega}$ and $\tau\neq\emptyset\}$. Let $x=(x_i)_{i\in\omega}$. Take a unique $\delta(0)=\gamma(\delta(0), 0)\in\Delta_{Z\times X^\omega}=\Gamma(Z\times X^\omega, 0)$ such that $x\in W_{\delta(0)}$. Let $K(0)=K(\delta(0))\in\mathcal{K}_{Z\times X^\omega}$ and let $m(0)=n(K_{(z,K(0))})$. Choose a $j(0)\in\omega$ such that $(z, x)\in\cup\mathcal{G}_{m(0),j(0)}(Z\times X^\omega)\cup(\cup\mathcal{B}_{m(0),j(0)}(Z\times X^\omega))$. Let $\tau(0)=(m(0), j(0))\in\omega\times\omega$. Since $(z, x)\notin\cup\mathcal{G}_{\tau(0)}$, there are an $\alpha(0)\in\Xi_{\delta(0),m(0),j(0)}$ and $A(0)\subset\{0, 1, \cdots, m(0)\}$ such that $(z, x)\in B_{\alpha(0),A(0)}$ and $B_{\alpha(0),A(0)}\in\mathcal{B}_{\tau(0)}(Z\times X^\omega)$. We have $0=n(Z\times X^\omega)<n(B_{\alpha(0),A(0)})$. For $B_{\alpha(0),A(0)}$, take a unique $\delta(1)=(\gamma(\delta(1)), 0), \cdots, \gamma(\delta(1), n(B_{\alpha(0),A(0)})))\in\Delta_{B_{\alpha(0),A(0)}}$ such that $x\in W_{\delta(1)}$. Let $K(1)=K(\delta(1))\in\mathcal{K}_{B_{\alpha(0),A(0)}}$ and $m(1)=n(K_{(z,K(1))})$. Take a $j(1)\in\omega$ such that $(z, x)\in\cup\mathcal{G}_{m(1),j(1)}(B_{\alpha(0),A(0)})\cup(\cup\mathcal{B}_{m(1),j(1)}(B_{\alpha(0),A(0)}))$. Let $\tau(1)=((m(0), j(0)), (m(1), j(1)))\in(\omega\times\omega)^{<\omega}$. Since $(z, x)\notin\cup\mathcal{G}_{\tau(1)}$, there are an $\alpha(1)\in\Xi_{\delta(1),m(1),j(1)}$ and $A(1)\subset\{0, 1, \cdots, \max\{m(1), n(B_{\alpha(0),A(0)})\}\}$ such that $(z, x)\in B_{\alpha(1),A(1)}$ and $B_{\alpha(1),A(1)}\in\mathcal{B}_{\tau(1)}(B_{\alpha(0),A(0)})$. We have $n(B_{\alpha(0),A(0)})<n(B_{\alpha(1),A(1)})$. Continuing this matter, we can choose a sequence $\{\delta(k): k\in\omega\}$, a sequence $\{K(k): k\in\omega\}$ of compact subsets of X^ω, where $K(k)=\prod_{i\in\omega}K(k)_i\in\mathcal{K}$, sequences $\{m(k): k\in\omega\}$, $\{j(k): k\in\omega\}$ of natural numbers, a sequence $\{\tau(k): k\in\omega\}$ of elements of $(\omega\times\omega)^{<\omega}$, where $\tau(k)=((m(0), j(0)), \cdots, (m(k), j(k)))$, a sequence $\{\alpha(k): k\in\omega\}$, a sequence $\{A(k): k\in\omega\}$ of finite subsets of ω, a sequence $\{B_{\alpha(k),A(k)}: k\in\omega\}$ of elements of $\mathcal{B}$ containing (z, x), where $B_{\alpha(k),A(k)}=V_{\alpha(k)}\times\prod_{i\in\omega}B_{\alpha(k),A(k),i}$, satisfying the following: Let $k\in\omega$. Assume that we have already obtained sequences $\{\delta(i): i\leqq k\}$, $\{K(i): i\leqq k\}$, $\{m(i): i\leqq k\}$, $\{j(i): i\leqq k\}$, $\{\tau(i): i\leqq k\}$, $\{\alpha(i): i\leqq k\}$, $\{A(i): i\leqq k\}$ and $\{B_{\alpha(i),A(i)}: i\leqq k\}$. Then

(xiv) $\delta(k+1)=(\gamma(\delta(k+1), 0), \cdots, \gamma(\delta(k+1), n(B_{\alpha(k),A(k)})))\in\Delta_{B_{\alpha(k),A(k)}}$. $W_{\delta(k+1)}$ is a unique element of $\{W_\delta: \delta\in\Delta_{B_{\alpha(k),A(k)}}\}$ containing x,

(xv) $K(k+1)=K(\delta(k+1))\in\mathcal{K}_{B_{\alpha(k),A(k)}}$,

(xvi) $m(k+1)=n(K_{(z,K(k+1))})$, and $j(k+1)\in\omega$. Let $\tau(k+1)=((m(0), j(0)), \cdots, (m(k+1), j(k+1)))$,

(xvii) $\alpha(k+1)\in\Xi_{\delta(k+1),m(k+1),j(k+1)}$ and $A(k+1)\subset\{0, 1, \cdots, \max\{m(k+1), n(B_{\alpha(k),A(k)})\}\}$,

(xviii) $(z, x)\in B_{\alpha(k+1),A(k+1)}=V_{\alpha(k+1)}\times\prod_{i\in\omega}B_{\alpha(k+1),A(k+1),i}$, $B_{\alpha(k+1),A(k+1)}\in\mathcal{B}_{\tau(k+1)}(B_{\alpha(k),A(k)})$, and $n(B_{\alpha(k),A(k)})<n(B_{\alpha(k+1),A(k+1)})$,

(xix) For each $i\leqq n(B_{\alpha(k),A(k)})$ with $i\in A(k+1)$ such that $C_{\lambda(B_{\alpha(k),A(k)},i)}=\emptyset$, $s(B_{\alpha(k),A(k),i})\cap B_{\alpha(k+1),A(k+1),i}=\emptyset$,

(xx) For each $i\leqq n(B_{\alpha(k),A(k)})$ with $i\notin A(k+1)$ such that $C_{\lambda(B_{\alpha(k),A(k)},i)}\neq\emptyset$, $K(k+1)_i=C_{\lambda(B_{\alpha(k),A(k)},i)}$.

Assume that for each $i\in\omega$, $|\{k\in\omega: i\in A(k)\}|<\omega$, where $|A|$ denotes the cardinality of a set A. Then for each $i\in\omega$, there is a $k_i\in\omega$ such that $i\leqq k_i$ and if $k\geqq k_i$, then $i\notin A(k)$. Then, by (xx),

(xxi) For each $i\in\omega$ and $k\geqq k_i$, $K(k)_i=K(k_i)_i$.

Let $K=\prod_{i\in\omega}K(k_i)_i\in\mathcal{K}$. There is an $O\in\mathcal{O}'$ such that $K_{(z,K)}\subset O$. By (xviii) and (xxi), take a $k\geqq 1$ such that $n(O)\leqq n(B_{\alpha(k-1),A(k-1)})$ and if $i\leqq n(O)$, then $K(k)_i=K(k_i)_i$. Then we have $K_{(z,K(k))}\subset O$ and hence, $m(k)=n(K_{(z,K(k))})\leqq n(O)$. Since $\alpha(k)\in\Xi_{\delta(k),m(k),j(k)}$, $n(K_{(z(\alpha(k)),K(k))})\leqq m(k)$. For i with $n(O)\leqq i\leqq n(B_{\alpha(k-1),A(k-1)})$, by the definition, $H_{(z(\alpha(k)),K(k)),i}=W_{\gamma(\delta(k),i)}$. Hence $A_k\cap\{n(O),\cdots,n(B_{\alpha(k-1),A(k-1)})\}=\emptyset$. Since $(z,x)\in B_{\alpha(k),A(k)}$ and $B_{\alpha(k),A(k)}\in\mathcal{B}_{\tau(k)}(B_{\alpha(k-1),A(k-1)})$, there is an $i\in A(k)$ such that $x_i\notin H_{(z(\alpha(k)),K(k)),i}$. Thus $i<n(O)$ and $x_i\in B_{\alpha(k),A(k),i}=W_{\gamma(\delta(k),i)}-H_{(z(\alpha(k)),K(k)),i}$. Since $i\in A(k)$, $k<k_i$. For each $k'>k$, $K(k')_i\subset B_{\alpha(k),A(k),i}$. Thus $K(k_i)_i\subset B_{\alpha(k),A(k),i}$. Since $K(k)_i\subset H_{(z(\alpha(k)),K(k)),i}$, we have $K(k)_i\neq K(k_i)_i$. This is a contradiction. Therefore there is an $i\in\omega$ such that $|\{k\in\omega: i\in A(k)\}|=\omega$. Let $\{k\in\omega: i\in A(k)$ and $i\leqq n(B_{\alpha(k),A(k)})\}=\{k_t: t\in\omega\}$. Let $t\in\omega$. Since $C_{\lambda(B_{\alpha(k_t),A(k_t)},i)}=\emptyset$, if $k_{t+1}=k_t+1$, then, by (xix), $s(B_{\alpha(k_t),A(k_t),i})\cap B_{\alpha(k_{t+1}),A(k_{t+1}),i}=\emptyset$. Assume that $k_{t+1}>k_t+1$. Since $K_{\gamma(\delta(k_t+1),i)}=C_{\lambda(B_{\alpha(k_t+1),A(k_t+1)},i)}=C_{\lambda(B_{\alpha(k_{t+1}-1),A(k_{t+1}-1)},i)}\subset H_{(z(\alpha(k_{t+1}),K(k_{t+1})),i}$, we have $s(B_{\alpha(k_t),A(k_t),i})\cap B_{\alpha(k_{t+1}),A(k_{t+1}),i}=\emptyset$. Since s is a stationary winning strategy for Player I in $G(\mathcal{DC},X)$, $\bigcap_{t\in\omega}B_{\alpha(k_t),A(k_t),i}=\emptyset$. But $x_i\in\bigcap_{t\in\omega}B_{\alpha(k_t),A(k_t),i}$, which is a contradiction. It follows that $\bigcup\{\mathcal{G}_\tau:\tau\in(\omega\times\omega)^{<\omega}$ and $\tau\neq\emptyset\}$ is a covering of $Z\times X^\omega$. The proof is completed.

REMARK 3.3. Let M be the Michael line and let $\mathbf{P}$ be the space of irrational numbers. It is well known that $M\times\mathbf{P}$ is not normal. M is a hereditarily paracompact space. But M is not perfect. Since $\mathbf{P}$ is homeomorphic to ω^ω, we cannot omit the condition "Z is perfect" in Theorem 3.2. Furthermore we cannot replace "Z is a perfect paracompact space" by "Z is a hereditarily paracompact space" in Theorem 3.2.

THEOREM 3.4. *If Z is a perfect paracompact space and Y_i is a paracompact space with a σ-closure-preserving cover by compact sets for each $i\in\omega$, then $Z\times\prod_{i\in\omega}Y_i$ is paracompact.*

PROOF. This follows from Theorem 3.2 and Lemma 2.4 (a).

Similarly, by Theorem 3.2 and Lemma 2.4 (b),

THEOREM 3.5. *If Z is a perfect paracompact space and Y_i is a paracompact,*

σ-C-scattered space for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is paracompact.

For a space X, let $\mathcal{F}[X]$ denote the Pixley-Roy hyperspace of X. Every Pixley-Roy hyperspace has a closure-preserving cover by finite sets and is σ-scattered. For a space X, the following are equivalent (see H. Tanaka [15]): (a) $\mathcal{F}[X]$ is paracompact; (b) $\mathcal{F}[X^2]$ is paracompact; (c) $\mathcal{F}[X^n]$ is paracompact for each $n \in \omega$ and (d) $\mathcal{F}[X^n]^m$ is paracompact for each $n, m \in \omega$. T. Przymusinski [12] posed the following problem: If $\mathcal{F}[X]$ is paracompact, then is $\mathcal{F}[X]^\omega$ paracompact? We have

THEOREM 3.6. *If Z is a perfect paracompact space and Y_i is a space such that $\mathcal{F}[Y_i]$ is paracompact for each $i \in \omega$, then $Z \times \prod_{i \in \omega} \mathcal{F}[Y_i]$ is paracompact.*

It is well known that Z is a regular hereditarily Lindelöf space if and only if Z is a regular perfect Lindelöf space (R. Engelking [5]).

THEOREM 3.7. *If Z is a regular hereditarily Lindelöf space and Y_i is a regular Lindelöf $\mathcal{DC}$-like space for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is Lindelöf. Hence, if X is a regular Lindelöf $\mathcal{DC}$-like space, then $X^\omega \in \mathcal{L}'$.*

PROOF. By Lemmas 2.2 and 3.1, we may assume that for each $i \in \omega$, Y_i is a regular Lindelöf $\mathcal{DC}$-like space with $\dim Y_i \leqq 0$. Let $X = \bigoplus_{i \in \omega} Y_i \cup \{a\}$, where $a \notin \bigcup Y_i$. Define the topology of X as the proof of Theorem 3.2. It suffices to prove that $Z \times X^\omega$ is Lindelöf.

Let $\mathcal{B}$ be the base of $Z \times X^\omega$ defined in the proof of Theorem 3.2 and let $\mathcal{O}$ be an open covering of $Z \times X^\omega$. Define $\mathcal{O}'$ and $n(B)$ for each $B \in \mathcal{B}$ as before. We show that $\mathcal{O}'$ has a countable open refinement. By the proof of Theorem 3.2, $\mathcal{O}'$ has a σ-locally finite refinement $\mathcal{G} = \bigcup \{\mathcal{G}_n : n \in \omega\}$ such that $\mathcal{G} \subset \mathcal{B}$. For each $m \in \omega$, let $p_m : Z \times X^\omega \to Z \times X^m$ be the projection from $Z \times X^\omega$ onto $Z \times X^m$. For $n, m \in \omega$, let $\mathcal{G}_{n,m} = \{G \in \mathcal{G}_n : n(G) \leqq m\}$. Then $\mathcal{G}_n = \bigcup \{\mathcal{G}_{n,m} : m \in \omega\}$ for each $n \in \omega$. Put $\mathcal{H}_{n,m} = p_m(\mathcal{G}_{n,m}) = \{p_m(G) : G \in \mathcal{G}_{n,m}\}$ for $n, m \in \omega$. Then every $\mathcal{H}_{n,m}$ is locally finite in $Z \times X^m$. By Lemma 2.5, every $Z \times X^m$ is Lindelöf. Then for each $n, m \in \omega$, $\mathcal{H}_{n,m}$ is countable. Hence every $\mathcal{G}_{n,m}$ is countable. Thus $\mathcal{G} = \bigcup \{\mathcal{G}_n : n \in \omega\} = \bigcup \{\mathcal{G}_{n,m} : n, m \in \omega\}$ is countable. It follows that $Z \times X^\omega$ is Lindelöf. The proof is completed.

THEOREM 3.8. *If Z is a regular hereditarily Lindelöf space and Y_i is a regular Lindelöf space with a σ-closure-preserving cover by compact sets for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is Lindelöf.*

THEOREM 3.9. *If Z is a regular hereditarily Lindelöf space and Y_i is a regular Lindelöf, σ-C-scattered space for each $i \in \omega$, then $Z \times \prod_{i \in \omega} Y_i$ is Lindelöf.*

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