PROPERTIES OF NORMAL EMBEDDINGS CONCERNING STRONG SHAPE THEORY, II

By

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Abstract. We show that any topological pair with normally embedded subspace has the strong shape of a pair, such that the inclusion map of the subspace into the total space is a cofibration. Furthermore we prove that a strong shape morphism of pairs is a strong shape equivalence if and only if it operates as strong shape equivalence of the total spaces and of the subspaces considered seperately.

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1. Preliminaries.

This paper is a continuation of [2] dealing with strong shape theory of pairs as defined by Lisica and Mardešić in [4]. One major property of strong shape theory is the existence of a functor $T : \mathbf{ssh}^2 \to \mathbf{HTop}^2$ right adjoint to the strong shape functor $\eta : \mathbf{HTop}^2 \to \mathbf{ssh}^2$. The authors do not state this explicitly, but in case of spaces it is a consequence of [5, theorem 6, p. 371]. We only give a brief description of T:

To a given pair (X, A) we choose a resolution in ANR-pairs $\{f_\lambda\} : (X, A) \to \{g^\mu_\lambda : (P_\mu, Q_\mu) \to (P_\lambda, Q_\lambda) \mid \mu \geq \lambda \in \Lambda\}$ and form the simplicial complex $\mathcal{K}$, whose vertices are the indices $\lambda \in \Lambda$, and whose simplices are the finite, linearly ordered subsets of Λ. For a *finite* subcomplex $\mathcal{L} \subseteq \mathcal{K}$ we denote by $\mathcal{L}_\lambda \subseteq \mathcal{L}$ the full subcomplex spanned by all vertices μ with $\mu \leq \lambda$ and consider the subspace

$$\tilde{P}_{\mathcal{L}} \subseteq \prod_{\lambda \in \Lambda} \{\omega_\lambda : |\mathcal{L}_\lambda| \to P_\lambda\}$$

consisting of all families of maps on the geometric realizations $|\mathcal{L}_\lambda|$ subject to the condition $g^\mu_\lambda \omega_\mu = \omega_\lambda$ on $|\mathcal{L}_\mu|$ for $\mu \geq \lambda$. By replacing P_λ with Q_λ we get a closed subspace $\tilde{Q}_{\mathcal{L}} \subseteq \tilde{P}_{\mathcal{L}}$. The pairs $(\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$ form an inverse system over a cofinite directed index set, whose bonding maps are the restriction maps

$r_{\mathcal{L}}^{\mathcal{M}}: (\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}}) \to (\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$. We set $T(X, A) := \lim (\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$, and it follows directly from the definitions that there is a natural equivalence

$$\mathbf{ssh}^2(-, -; X, A) \approx \mathbf{HTop}^2(-, -; T(X, A)).$$

This allows the definition of induced mappings turning T into a right adjoint functor,

We need to take a closer look at the restriction maps $r_{\mathcal{L}}^{\mathcal{M}}$; $(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}}) \to (\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$ in the particular case, where $\mathcal{M}$ is obtained from $\mathcal{L}$ by attaching one additional simplex σ with boundary $\dot{\sigma} \subseteq \mathcal{L}$. If λ is the lowest element of σ we have a pullback diagram:

$$
\begin{array}{ccc}
\tilde{P}_{\mathcal{M}} & \longrightarrow & P_\lambda^{|\sigma|} \\
\downarrow{\scriptstyle r_{\mathcal{L}}^{\mathcal{M}}} & & \downarrow \\
\tilde{P}_{\mathcal{L}} & \longrightarrow & P_\lambda^{|\dot{\sigma}|},
\end{array}
$$

similarly for $\tilde{Q}_{\mathcal{M}}$. We see that $r_{\mathcal{L}}^{\mathcal{M}}: \tilde{P}_{\mathcal{M}} \to \tilde{P}_{\mathcal{L}}$ and $r_{\mathcal{L}}^{\mathcal{M}}: \tilde{Q}_{\mathcal{M}} \to \tilde{Q}_{\mathcal{L}}$ are fibrations, and by induction on the number of elements of $\mathcal{L}$ observing [1, Lemma 3.6] we see that $\tilde{P}_{\mathcal{L}}$ and $\tilde{Q}_{\mathcal{L}}$ are ANR-spaces.

2. Strong shape equivalences of pairs.

LEMMA 1. *We suppose that* $\{f_\lambda\}$; $X \to \{g_\lambda^\mu: X_\mu \to X_\lambda\}$ *is a strong expansion and that* $\pi: E \to B$ *is a fibration between ANR-spaces.*

a) *If we are given maps* $\alpha: X \to E$ *and* $\beta: X_\lambda \to B$ *for some index* λ, *and a homotopy* $H: \pi\alpha \cong \beta f_\lambda$, *then we can find some index* $\mu \geq \lambda$, *a map* $\gamma: X_\mu \to E$ *with* $\pi\gamma = \beta g_\lambda^\mu$ *and a homotopy* $G: \alpha \cong \gamma f_\mu$ *with* $\pi G = H$.

b) *If two maps* $\alpha_0, \alpha_1: X_\lambda \to E$ *and two homotopies* $\Gamma: \pi\alpha_0 \cong \pi\alpha_1$ *and* $C: \alpha_0 f_\lambda$ $\cong \alpha_1 f_\lambda$ *with* $\pi C = \Gamma(f_\lambda \times \mathrm{id})$ *are given, then for a suitable index* $\mu \geq \lambda$ *there is a homotopy* $\Sigma: \alpha_0 g_\lambda^\mu \cong \alpha_1 g_\lambda^\mu$ *with* $\pi\Sigma = \Gamma(g_\lambda^\mu \times \mathrm{id})$ *and* $\Sigma(f_\mu \times \mathrm{id}) \cong C$ *relative* $X \times \dot{I}$ *and relative* π.

POOOF. a) At first we take a map $\gamma': X_\nu \to E$ and a homotopy $\Gamma: \gamma' f_\nu \cong \alpha$ for some suitable index $\nu \geq \lambda$. This provides us with two maps $\pi\gamma'$, $\beta g_\lambda^\nu: X_\nu \to B$, whose compositions with f_ν are connected by the homotopy $\pi\Gamma \circ H: \pi\gamma' f_\nu \cong \beta f_\lambda$. We conclude the existence of a homotopy $\Sigma: \pi\gamma' g_\lambda^\mu \cong \beta g_\lambda^\mu$ for some $\mu \geq \nu$ with $\Sigma(f_\mu \times \mathrm{id}) \cong \pi\Gamma \circ H$ relative $X \times \dot{I}$. The fibration property of π ensures the existence of a homotopy $\Omega: X_\mu \times I \to E$ with $\Omega_0 = \gamma' g_\lambda^\mu$ and $\pi\Omega = \Sigma$. If we set $\gamma := \Omega_1$ then we have $\pi\gamma = \beta g_\lambda^\mu$, and there is a homotopy $G' := \Gamma^{-1} \circ \Omega(f_\mu \times \mathrm{id})$:

$\alpha \cong \gamma f_\mu$ with $\pi G' \cong H$ relative $X \times \dot{I}$. Using the fibration property of π again we can replace G' with another homotopy $G : \alpha \cong \gamma f_\mu$ with $\pi G = H$.

To prove (b) we take a look at the following diagram, where the square is formed as pullback diagram:
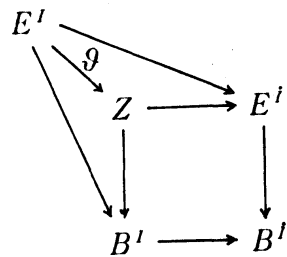
It is elementary to check that $\vartheta : E^I \to Z$ is a fibration, and by [1, Lemma 3.6] Z is an ANR-space. Therefore we can replace the fibration π in (a) by ϑ, and this proves (b).   q.e.d.

REMARK. If the fibration π in Lemma 1.a has the form either of the zero map from an ANR-space P to a one point space or of the restriction map $P^I \to P^{\dot{i}}$, then we rediscover the defining conditions of a strong expansion (c) and (d) in part 1 of this paper. Therefore Lemma 1.a characterizes strong expansions.

We consider a pair (X, A) consisting of an arbitrary topological space X and a normally embedded subspace A.

As usual $\Delta^n = \{(t_0, \cdots t_n) \in I^{n+1} \mid \sum_{k=0}^n t_k = 1\}$ is the topological standard-n-simplex with its faces $\Delta_k^n = \{(t_0, \cdots t_n) \in \Delta^n \mid t_k = 0\}$; by $k \in \Delta^n$ we mean the vertex determined by $t_k = 1$.

We set:

$$Y := X \times \Delta_0^2 \cup A \times \Delta^2$$

$$Y_1 := X \times \{1\} \cup A \times \Delta_2^2 \subseteq Y$$

$$Y_2 := A \times \Delta_1^2 \subseteq Y$$

$$Y_3 := A \times \{0\} = Y_1 \cap Y_2.$$

The space Y carries the weak topology determined by the subspaces $X \times \Delta_0^2$ and $A \times \Delta^2$.

An ANR-pair is a topological pair consisting of two ANR-spaces such that the subspace is closed. A fibration between ANR-pairs $\pi : (E, E_0) \to (B, B_0)$ is a map between these two ANR-pairs such that $\pi : E \to B$ and its restriction $\pi' : E_0 \to B_0$ are Hurewicz-fibrations.

LEMMA 2.   a) *For any fibration of ANR-pairs* $\pi : (E, E_0) \to (B, B_0)$ *the following lifting problem has a solution:*

$$
\begin{array}{ccc}
(Y_1, Y_3) & \xrightarrow{\ f\ } & (E, E_0) \\[4pt]
i\ \big\downarrow & \nearrow^{\ h} & \big\downarrow\ \pi \\[4pt]
(Y, Y_2) & \xrightarrow[\ g\ ]{} & (B, B_0)
\end{array}
$$

b) *If in* (a) *two maps* $h_0, h_1 : (Y, Y_2) \to (E, E_0)$ *and two homotopies* $H : h_0 i \cong h_1 i$ *and* $G : \pi h_0 \cong \pi h_1$ *(as maps of pairs!) with* $\pi H = C(i \times \mathrm{id})$ *are prescribed, then there exists* $\Gamma : h_0 \cong h_1$ *with* $\Gamma(i \times \mathrm{id}) = H$ *and* $\pi \Gamma = G$.

PROOF.   a) There is a deformation $D : B \times I \to B$ relative B_0 with $D_0 = \mathrm{id}$ and a neighborhood V of B_0 in B with $D_1(V) \subseteq B_0$. Since B is a normal space the set $U_1 := \{x \in X \,|\, g(x, 2) \in V\}$ is a normal neighborhood of A in X. We observe that by corollary 2.5 in part 1 of this paper the system of normal neighborhoods of A in X forms a strong expansion of A and apply Lemma 1.a to the maps $\alpha := f_{|A \times \{0\}} : A \to E_0$, $\beta := D_1 g_{|U_1 \times \{2\}} : U_1 \to B_0$ and the homotopy $H := g_{|A \times \Delta_1^2}$; we get a normal neighborhood U_2 of A in X contained in U_1, a map $\gamma : U_2 \times \{2\} \to E_0$ with $\pi\gamma = D_1 g_{|U_2 \times \{2\}}$ and a homotopy $G : A \times \Delta_1^2 \to E_0$ with $G = f$ on $A \times \{0\}$, $G = \gamma$ on $A \times \{2\}$ and $\pi G = g$ on $A \times \Delta_1^2$.

We set $g' := g_{|U_2 \times \{2\}} : U_2 \to B$ and consider the homotopy $D(g' \times \mathrm{id}) : U_2 \times I \to B$; it is stationary on $A \times I$ and ends at $\pi\gamma$. Since the fibration $\pi : E \to B$ has a metrizable base space it is regular and we can find a homotopy $\Phi : U_2 \times I \to E$ with $\Phi_1 = \gamma$ and $\pi\Phi = D(g' \times \mathrm{id})$, which is stationary on $A \times I$. In particular we have $\Phi_0 = \gamma$ on A and $\pi\Phi_0 = g'$.

Since the inclusion map $A \times (\Delta_1^2 \cup \Delta_2^2) \hookrightarrow A \times \Delta^2$ is a cofibration and a homotopy equivalence we can find a map $\Psi : A \times \Delta^2 \to E$ with $\Psi = f$ on $A \times \Delta_2^2$, $\Psi = G$ on $A \times \Delta_1^2$ and $\pi\Psi = g$ on $A \times \Delta^2$. Applying Lemma 1.b to the maps $\alpha_0 := f_{|U_2 \times \{1\}} : U_2 \to E$, and $\alpha_1 := \Phi_0 : U_2 \to E$ and the homotopies $\Gamma := g_{|U_2 \times \Delta_0^2}$ and $C := \Psi_{|A \times \Delta_0^2}$ we get a normal neighborhood U_3 of A in X contained in U_2 and a homotopy $\Sigma : U_3 \times \Delta_0^2 \to E$ with $\Sigma = f$ on $U_3 \times \{1\}$, $\Sigma = \Phi_0$ on $U_3 \times \{2\}$, $\pi\Sigma = g$ on $U_3 \times \Delta_0^2$ and $\Sigma_{|A \times \Delta_0^2} \cong \Psi_{|A \times \Delta_0^2}$ relative $A \times \partial\Delta_0^2$ and relative π. The last condition allows to replace Ψ with a map $\Psi' : A \times \Delta^2 \to E$ with $\Psi' = f$ on $A \times \Delta_2^2$, $\Psi' = G$ on $A \times \Delta_1^2$, $\Psi' = \Sigma$ on $A \times \Delta_0^2$ and $\pi\Psi' = g$ on $A \times \Delta^2$.

We now take an Urysohn function $\varphi : X \to I$ with $\varphi = 1$ on A and $\varphi = 0$ on $X \setminus U_3$ and set $\psi := \min(1, 2(1 - \varphi))$; furthermore we consider the map $\omega : X \times \Delta_0^2$

$\rightarrow E$ and the homotopy $\Omega: X \times \Delta_0^2 \times I \rightarrow B$ given by

$$\omega(x, 0, t_1, t_2) := \begin{cases} f(x, 0, 1, 0) & \text{for } \varphi(x) \leqq 1/2 \\ \Sigma(x, 0, t_1+t_2\phi(x), t_2(1-\phi(x))) & \text{for } \varphi(x) \geqq 1/2, \end{cases}$$

$$\Omega(x, 0, t_1, t_2, s) := g(x, 0, t_1+st_2\phi(x), t_2(1-s\phi(x))).$$

We observe: $\Omega_1 = \pi\omega$, and Ω is stationary on $(X \times \{1\} \cup A \times \Delta_0^2) \times I$. Since the fibration $\pi: E \rightarrow B$ is regular there is a homotopy $\Theta: X \times \Delta_0^2 \times I \rightarrow E$ with $\Theta_1 = \omega$ and $\pi\Theta = \Omega$, which is stationary on $(X \times \{1\} \cup A \times \Delta_0^2) \times I$. In particular we have $\pi\Theta_0 = g_{|X \times \Delta_0^2}$, $\Theta_0 = f$ on $X \times \{1\}$ and $\Theta_0 = \Psi'$ on $A \times \Delta_0^2$. Therefore we can define a map $h: Y \rightarrow E$ meeting all requirements by $h := \Theta_0$ on $X \times \Delta_0^2$ and $h := \Psi'$ on $A \times \Delta^2$.

(b) is deduced from (a) by replacing the fibration $\pi: (E, E_0) \rightarrow (B, B_0)$ with $\vartheta: (E^I, E_0^I) \rightarrow (Z, Z_0)$, constructed as in the proof of Lemma 1.b.   q.e.d.

PROPOSITION 1. *We consider a map of pairs* $f: (Y, \tilde{Y}) \rightarrow (Z, \tilde{Z})$, *such that* $f: Y \rightarrow Z$ *and the restricted map* $\tilde{f}: \tilde{Y} \rightarrow \tilde{Z}$ *are homotopy equivalences. If* (X, A) *is a pair such that the inclusion map* $i: A \hookrightarrow X$ *is a cofibration, then* $f_*: \mathbf{HTop}^2(X, A; Y, Y) \rightarrow \mathbf{HTop}^2(X, A; Z, \tilde{Z})$ *is bijective.*

PROOF. In this proof a homotopy of the form $H(g \times \mathrm{id}_I)$ will simply be denoted Hg, and homotopies between homotopies will always be understood relative to the boundary. Furthermore we will repeatedly make use of the following two statements, taken from [6]:

i) For any homotopy equivalence $f: Y \rightarrow Z$ there exist a map $g: Z \rightarrow Y$ and homotopies $\Phi: fg \cong \mathrm{id}$, $\Psi: gf \cong \mathrm{id}$ with $f\Psi \cong \Phi f$. ("f is a strong homotopy equivalence.")

ii) Suppose we are given maps $\alpha_0, \alpha_1: Y \rightarrow Z$, $\beta_0, \beta_1: Z \rightarrow T$ and homotopies $A: \alpha_0 \cong \alpha_1$, $B: \beta_0 \cong \beta_1$. Then: $\beta_0 A \circ B\alpha_1 \cong B\alpha_0 \circ \beta_1 A$. ("Godement interchange law" or "commutativity lemma".)

Now we consider a map $f: (Y, \tilde{Y}) \rightarrow (Z, \tilde{Z})$ with the property stated in the proposition. By (i) we can find maps and homotopies as follows:

$$g: Z \longrightarrow Y \qquad \tilde{g}: \tilde{Z} \longrightarrow \tilde{Y}$$

$$\Phi: fg \cong \mathrm{id} \qquad \tilde{\Phi}: \tilde{f}\tilde{g} \cong \mathrm{id}$$

$$\Psi: gf \cong \mathrm{id} \qquad \tilde{\Psi}: \tilde{g}\tilde{f} \cong \mathrm{id}$$

$$f\Psi \cong \Phi f \qquad \tilde{f}\tilde{\Psi} \cong \tilde{\Phi}\tilde{f}.$$

Let $j: \tilde{Y} \hookrightarrow Y$ and $k: \tilde{Z} \hookrightarrow Z$ be the inclusion maps. We observe $fj = k\tilde{f}$ and define a homotopy:

$$G := \Psi^{-1} j\tilde{g} \circ gk\tilde{\Phi} : j\tilde{g} \cong gk .$$

Applying (ii) to the homotopies $\tilde{\Phi}: \tilde{f}\tilde{g} \cong \mathrm{id}$ and $\Phi k: fgk \cong k$ we get $fgk\tilde{\Phi} \circ \Phi k$
$\cong \Phi k\tilde{f}\tilde{g} \circ k\tilde{\Phi} = \Phi f j\tilde{g} \circ k\tilde{\Phi} \cong f\Psi j\tilde{g} \circ k\tilde{\Phi}$ and hence:

$$fG \cong g\tilde{\Phi} \circ \Phi^{-1} k .$$

Then we apply (ii) to the homotopies $\tilde{\Psi}: \tilde{g}\tilde{f} \cong \mathrm{id}$ and $\Psi j: gfj \cong j$ and conclude
$\Psi j\tilde{g}\tilde{f} \circ j\tilde{\Psi} \cong gfj\tilde{\Psi} \circ \Psi j = gk\tilde{f}\tilde{\Psi} \circ \Psi j \cong gk\tilde{\Phi}\tilde{f} \circ \Psi j$ and therefore:

$$G\tilde{f} \cong j\tilde{\Psi} \circ \Psi^{-1} j .$$

Now we consider our pair (X, A) with cofibration inclusion $i: A \hookrightarrow X$, and we suppose that a map of pairs $\alpha: (X, A) \to (Z, \tilde{Z})$ is prescribed. Let $\tilde{\alpha}: A \to \tilde{Z}$ be the restricted map and observe: $k\tilde{\alpha} = \alpha i$. The cofibration property allows the construction of a homotopy $H: X \times I \to Y$ with $H_1 := g\alpha$ and $H_{|A \times I} = G\tilde{\alpha}$. This means in particular $H_0 = j\tilde{g}\tilde{\alpha}$ on A, so that $\beta := H_0$ may be considered as map of pairs $\beta: (X, A) \to (Y, \tilde{Y})$, and we claim $f\beta \cong \alpha$. Forgetting the subspaces for the moment these two maps are connected by the homotopy $\Gamma' := fH \circ \Phi\alpha$: $f\beta \cong \alpha$; but on $A \times I$ we have $\Gamma' i = fG\tilde{\alpha} \circ \Phi k\tilde{\alpha} \cong k\tilde{\Phi}\tilde{\alpha} \circ \Phi^{-1} k\tilde{\alpha} \circ \Phi k\tilde{\alpha} \cong k\tilde{\Phi}\tilde{\alpha}$, and the cofibration property ensures the existence of a homotopy $\Gamma: f\beta \cong \alpha$, $\Gamma \cong \Gamma'$, with $\Gamma = k\tilde{\Phi}\tilde{\alpha}$ on $A \times I$, in particular $\Gamma(A \times I) \subseteq \tilde{Z}$. This proves f_* to be surjective.

Let us suppose two maps $\alpha, \beta: (X, A) \to (Y, \tilde{Y})$ and a homotopy $H: f\alpha \cong f\beta$ with $H(A \times I) \subseteq \tilde{Z}$ are given; we denote by $\tilde{\alpha}, \tilde{\beta}: A \to \tilde{Y}$ and $\tilde{H}: A \times I \to \tilde{Z}$, $\tilde{H}: \tilde{f}\tilde{\alpha} \cong \tilde{f}\tilde{\beta}$, the restrictions and observe: $\alpha i = j\tilde{\alpha}$, $\beta i = j\tilde{\beta}$ and $Hi = k\tilde{H}$. On the total space we can define a homotopy $\Gamma': \alpha \cong \beta$ by $\Gamma' := \Psi^{-1}\alpha \circ gH \circ \Psi\beta$; then we have on $A \times I: \Gamma' i = \Psi^{-1} j\tilde{\alpha} \circ gk\tilde{H} \circ \Psi j\tilde{\beta}$. We apply (ii) to the homotopies $\tilde{H}: \tilde{f}\tilde{\alpha} \cong \tilde{f}\tilde{\beta}$ and $G: j\tilde{g} \cong gk$ and get $j\tilde{g}\tilde{H} \circ G\tilde{f}\tilde{\beta} \cong G\tilde{f}\tilde{\alpha} \circ gk\tilde{H}$, hence $j\tilde{g}\tilde{H} \circ j\tilde{\Psi}\tilde{\beta} \circ \Psi^{-1} j\tilde{\beta} \cong j\tilde{\Psi}\tilde{\alpha} \circ \Psi^{-1} j\tilde{\alpha} \circ gk\tilde{H}$ and finally $\Gamma' i \cong j(\Psi^{-1}\tilde{\alpha} \circ \tilde{g}\tilde{H} \circ \tilde{\Psi}\tilde{\beta})$. The cofibration property allows us to replace Γ' by a homotopy $\Gamma: \alpha \cong \beta$ with $\Gamma i = j$ $(\Psi^{-1}\tilde{\alpha} \circ \tilde{g}\tilde{H} \circ \tilde{\Psi}\tilde{\beta})$ and in particular $\Gamma(A \times I) \subseteq \tilde{Y}$. This shows that f_* is injective.   q.e.d.

Let (X, A) be a topological pair with normally embedded subspace and consider the space $X \times \{1\} \cup A \times I$ with the mapping cylinder topology. In general the projection map $p: (X \times \{1\} \cup A \times I, A \times \{0\}) \to (X, A)$ is not a homotopy equivalence of pairs, although it operates as homotopy equivalence on the total spaces and as homeomorphism of the subspaces. From Lemma 2.9 in part 1 of our paper we know that the situation improves in the ordinary shape category,

and the following theorem extends this result to the strong shape category:

THEOREM 1. *If A is normally embedded in X then $p: (X \times \{1\} \cup A \times I,\ A \times \{0\}) \to (X, A)$ is a strong shape equivalence of pairs.*

COROLLARY 1. *Every topological pair with normally embedded subspace has the strong shape of a pair (Y, B), such that the inclution map $B \hookrightarrow Y$ is a closed cofibration.*

PROOF OF THEOREM 1. We consider our spaces Y, Y_1, Y_2, Y_3 constructed above. The natural projection map $q: (Y, Y_2) \to (X, A)$ is a homotopy equivalence of pairs, because $j: (X, A) \to (Y, Y_2)$, $j(x) := (x, 2)$ is an inverse up to homotopy. The pair (Y_1, Y_3) coincides with the mapping cylinder $(X \times \{1\} \cup A \times I,\ A \times \{0\})$, and the restriction of $q: (Y, Y_2) \to (X, A)$ to (Y_1, Y_3) equals $p: (X \times \{1\} \cup A \times I,\ A \times \{0\}) \to (X, A)$. Hence it suffices to show that the inclusion map $i: (Y_1, Y_3) \hookrightarrow (Y, Y_2)$ is a strong shape equivalence of pairs, wherefore we make use of the right adjoint functor $T: \mathbf{ssh}^2 \to \mathbf{HTop}^2$ described in the introduction. We have to show that for every topological pair (Z, C) the induced map $i^*: \mathbf{HTop}^2(Y, Y_2;\ T(Z, C)) \to \mathbf{HTop}^2(Y_1, Y_3;\ T(Z, C))$ is bijective. We recall that $T(Z, C)$ is the limit of an inverse system of pairs $T(Z, C) = \lim (\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$ and consider a map $g: (Y_1, Y_3) \to \lim (\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$. We are going to construct a family of maps $h_{\mathcal{M}}: (Y, Y_2) \to (\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$ with $r_{\mathcal{L}}^{\mathcal{M}} h_{\mathcal{M}} = h_{\mathcal{L}}$ for $\mathcal{M} \supseteq \mathcal{L}$ and $h_{\mathcal{M}} = r_{\mathcal{M}} g$ on (Y_1, Y_3), $r_{\mathcal{M}}: \lim (\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}}) \to (\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$ being the projection map. The induction is by number of simplices of $\mathcal{M}$, where we distinguish two cases: If $\mathcal{M}$ does not contain a largest proper subcomplex, then $\mathcal{M}$ equals the union of all its proper subcomplexes $\mathcal{L}$ and therefore $(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}}) = \lim(\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$ and $h_{\mathcal{M}} = \lim h_{\mathcal{L}}$. If on the other hand there is a largest subcomplex $\mathcal{L}$, then we have to apply Lemma 2.a to the fibration of ANR-pairs $r_{\mathcal{L}}^{\mathcal{M}}: (\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}}) \to (\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$ to get $h_{\mathcal{M}}$. This completes the induction and determines a map $h: (Y, Y_2) \to \lim (\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$ with $r_{\mathcal{M}} h = h_{\mathcal{M}}$; and we necessarily have $hi = g$. This shows that $i^*: \mathbf{HTop}^2(Y, Y_2;\ T(Z, C)) \to \mathbf{HTop}^2(Y_1, Y_3;\ T(Z, C))$ is surjective. In a similar way, using Lemma 2.b instead of Lemma 2.a, we see that i^* is injective.

$$\text{q.e.d.}$$

It is obvious that every morphism in the homotopy category of pairs $[f] \in \mathbf{HTop}^2(X, A;\ Y, B)$ determines a morphism between the total spaces $[f_1] \in \mathbf{HTop}(X, Y)$ and another one between the relative spaces $[f_2] \in \mathbf{HTop}(A, B)$. We are going to explain that this carries over to the strong shape category, provided the subspaces are normally embedded. Let $\{f_\lambda\}: (X, A) \to \{g_\lambda^\mu: (P_\mu, Q_\mu)\}$

$\to(P_\lambda, Q_\lambda) \mid \mu\geqq\lambda\in\Lambda\}$ be a resolution of (X, A) in ANR-pairs; *if A is normally embedded in X* then $\{f_\lambda\} : X\to\{g_\lambda^\mu : P_\mu\to P_\lambda\}$ and $\{f_\lambda'\} : A\to\{g_\lambda'^\mu : Q_\mu\to Q_\lambda\}$ are ANR-resolutions. Hence for every strong shape morphism $\alpha\in\mathbf{ssh}^2(X, A; Y, B)$ between pairs with normally embedded subspaces there are unique strong shape morphisms $\alpha_1\in\mathbf{ssh}^2(X, \emptyset; Y, \emptyset)$ and $\alpha_2\in\mathbf{ssh}^2(A, A; B, B)$ fitting commutatively into the following diagram:

$$
\begin{array}{ccccc}
(X, \emptyset) & \xrightarrow{\ \ i_1\ \ } & (X, A) & \xleftarrow{\ \ i_2\ \ } & (A, A) \\
\downarrow{\alpha_1} & & \downarrow{\alpha} & & \downarrow{\alpha_2} \\
(Y, \emptyset) & \xrightarrow{\ \ j_1\ \ } & (Y, B) & \xleftarrow{\ \ j_2\ \ } & (B, B)
\end{array}
$$

α_1 is called the total part of α, α_2 is the relative part; we observe that these two morphisms can be identified with strong shape morphisms $\alpha_1\in\mathbf{ssh}(X, Y)$ and $\alpha_2\in\mathbf{ssh}(A, B)$ in an obvious way. The uniqueness statement made above implies in particular that the assignments $\alpha\mapsto\alpha_1$ and $\alpha\mapsto\alpha_2$ are functorial.

THEOREM 2. *A strong shape morphism $\alpha\in\mathbf{ssh}^2(X, A; Y, B)$ between pairs with normally embedded subspace is a strong shape equivalence of pairs if and only if its total part $\alpha_1\in\mathbf{ssh}(X, Y)$ and its relative part $\alpha_2\in\mathbf{ssh}(A, B)$ are strong shape equivalences.*

PROOF. Clearly the condition is necessary; we show that it is also sufficient. Our right adjoint functor $T: \mathbf{ssh}^2\to\mathbf{HTop}^2$ transforms the diagram above into the following form:

$$
\begin{array}{ccccc}
T(X, \emptyset) & \xrightarrow{\ T(i_1)\ } & T(X, A) & \xleftarrow{\ T(j_2)\ } & T(A, A) \\
\downarrow{T(\alpha_1)} & & \downarrow{T(\alpha)} & & \downarrow{T(\alpha_2)} \\
T(Y, \emptyset) & \xrightarrow{\ T(j_1)\ } & T(Y, B) & \xleftarrow{\ T(j_2)\ } & T(B, B)
\end{array}
$$

We observe that $T(i_1)$ and $T(j_1)$ induce homotopy equivalences of the total spaces and that $T(i_2)$ and $T(j_2)$ induce homotopy equivalences of the subspaces. If α_1 and α_2 are strong shape equivalences, then $T(\alpha_1)$ and $T(\alpha_2)$ are homotopy equivalences and therefore $T(\alpha)$ operates as homotopy equivalence on the total spaces and of the relative spaces of the pairs $T(X, A)$ and $T(Y, B)$. Proposition 1 implies that for every pair (Z, C), such that the inclusion map $C\hookrightarrow Z$ is is a cofibration, the induced function $T(\alpha)_* : \mathbf{HTop}^2(Z, C : T(X, A))\to \mathbf{HTop}^2(Z, C; T(Y, B))$ is bijective, and hence that $\alpha_* : \mathbf{ssh}^2(Z, C; X, A)\to$

ssh$^2(Z, C; Y, B)$ is bijective. By corollary 1 (X, A) and (Y, B) have the strong shape of pairs, such that the inclusion maps of the relative space into the total space are cofibrations, and therefore α must be a strong shape equivalence.

$$q.e.d.$$

COROLLARY 2. *A continuous map* $f: (X, A) \rightarrow (Y, B)$ *between pairs with normally embedded subspace is a strong shape equivalence if and only if* $f: X \rightarrow Y$ *and the restricted mapping* $f': A \rightarrow B$ *have the properties* (a) *and* (b) *from the introduction to part 1 of this paper. A pointed map* $f: (X, *) \rightarrow (Y, *)$ *is a strong shape equivalence if and only if the unpointed map* $f: X \rightarrow Y$ *is a strong shape equivalence.*

COROLLARY 3. *Let* (X, A) *be a pair with normally embedded subspace* A, *such that* A *has the strong shape of a point. Then the quotient map* $p: X \rightarrow X/A$ *is a strong shape equivalence.*

REMARK. By [3] a space has the strong shape of a point if and only if it has the ordinary shape of a point.

COROLLARY 4. *For every pair* (X, A) *with normally embedded subspace and every homology or cohomology theory* H *factoring over the strong shape category the quotient map* $p: (X, A) \rightarrow (X/A, *)$ *induces isomorphisms of the homology respectively cohomology groups* $H(p): H(X, A) \rightarrow H(X/A, *)$.

PROOF OF COROLLARY 3. Since A has strong shape of a point it cannot be empty, so choose a point $a \in A$. By corollary 2 the inclusion map $i: (X, a) \hookrightarrow (X, A)$ is a strong shape equivalence of pairs and therefore the induced function $i^*: \mathbf{HTop}^2(X, A; T(Y, B)) \rightarrow \mathbf{HTop}^2(X, a; T(Y, B))$ is bijective for every pair (Y, B). From the description of $T(Y, B)$ given in the preliminaries it is readily seen that for a pointed space $(Y, *)$ the classifying space $T(Y, *)$ can be chosen to be a pointed space too. Then the function $p^*: \mathbf{HTop}^2(X/A, *; T(Y, *)) \rightarrow \mathbf{HTop}^2(X, A; T(Y, *))$ is bijective, and the same holds for the composed function $(pi)^*: \mathbf{HTop}^2(X/A, *; T(Y, *)) \rightarrow \mathbf{HTop}^2(X, a; T(Y, *))$. Making use of the right adjoint property this shows that $\eta(pi)^*: \mathbf{ssh}^2(X/A, *; Y, *) \rightarrow \mathbf{ssh}^2(X, a; Y, *)$ is a bijection. Since $(Y, *)$ was arbitrary $pi: (X, a) \rightarrow (X/A, *)$ must be strong shape equivalence and hence $p: (X, A) \rightarrow (X/A, *)$ is a strong shape equivalence.

$$q.e.d.$$

Corollary 4 follows from 3 applied to the pair $(X \cup CA, CA)$.

References

[1] J. Dydak and S. Nowak, Strong shape for topological spaces, Trans. Amer. Math. Soc., **323** (1991), 765–796.

[2] B. Günther, Properties of normal embeddings concerning strong shape theory, I. Tsukuba J. Math., **15** (1991), 261–274.

[3] A. Koyama, S. Mardešić and T. Watanabe, Spaces which admit AR-resolutions, Proc. Amer. Math. Soc., **102** (1988), 749–752.

[4] Ju T. Lisica and S. Mardešić, Coherent prohomotopy and strong shape for pairs, Glasnik Mathematički, **20**(40) (1985), 419–343.

[5] Ju T. Lisica and S. Mardešić, Coherent prohomotopy and strong shape theory, Glasnik Matematički, **19**(39) (1984), 335–399.

[6] R.M. Vogt, A note on homotopy equivalences, Proc. Amer. Math. Soc., **32** (1972), 627–629.

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