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FOURTH ORDER SEMILINEAR PARABOLIC EQUATIONS

By

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1. Introduction.

The aim of this paper is to give a simple proof of the existence of a smooth solution to the semilinear parabolic equation with fourth order elliptic operator:

$$u_t=-\varepsilon^2\Delta^2 u+f(t, x, u, u_x, u_{xx})=: L(t, x, u), \tag{1}$$

$x\in\Omega\subset R^n$, Ω is a bounded domain, $t\in[0, T_{\max})$, $T_{\max}\leqq+\infty$, where $\Delta^2=\Delta\circ\Delta$, u_x is a vector of partial derivatives $(u_{x_1}, \cdots, u_{x_n})$ and u_{xx} stands for the Hessian matrix $[u_{x_i x_j}]$, $i, j=1, \cdots, n$. We consider (1) together with initial-boundary conditions

$$u(0, x)=u_0(x), \qquad x\in\Omega, \tag{2}$$

$$\frac{\partial u}{\partial n}=\frac{\partial(\Delta u)}{\partial n}=0 \qquad \text{when } x\in\partial\Omega. \tag{3}$$

Schematically we may write (3) as $B_1u=B_2u=0$.

In recent years a rapidly growing interest has been evinced in special problems such as the Cahn-Hilliard or the Kuramoto-Sivashinsky equations covered by our general form (1). Recently weak solutions for these special problems were considered in Temam's monograph [12]. The methods used here are an extension of those in previous papers [5, 6] devoted to the study of second order equations. General scheme of our proof of local existence (construction of the set X, considerations following (19)) is similar to the classical proof of the Picard theorem for solutions of ordinary differential equations.

2. Motivation.

We have two tasks in this paper. In Part I we prove local in time classical solvability of (1)-(3). We cannot expect global (that is in an arbitrarily large time interval) solvability of (1)-(3) under the weak assumption of local Lipschitz continuity of the nonlinear term f only (because of the possible rapid growth

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of f with respect to u, u_x or u_{xx}). However, the technique and estimates developed in reaching our first task allow immediate verification in Part II for a special problem (Cahn-Hilliard or Kuramoto-Sivashinsky equations) of global Lipschitz continuity of its specific nonlinearities, which in turn guarantees global solvability of this problem. Using our technique it is possible (see e.g. [6]) to find effective estimates of the life time of solutions to various problems with blowing-up solutions, blowing-up derivatives, etc.. The last may be of special interest for the numerical calculations as an indication of how long the solution of the approximated problem exists.

3. Assumptions.

Let us assume that $\partial\Omega\in C^{4+\mu}$ with some $\mu\in(0, 1)$, the function f is locally Lipschitz continuous with respect to its arguments t, u, u_{x_i}, $u_{x_i x_j}$ $(i, j=1, \cdots, n)$ and locally Hölder continuous with respect to x (exponent μ) in the set $[0, T]\times\bar{\Omega}\times R^{1+n+n^2}$. When $n>3$, for existence of the Hölder solution to (1)-(3) we need additionally to assume that the partial derivatives $f_t, f_u, f_{u_{x_i}}, f_{u_{x_i x_j}}$ fulfill the assumptions just mentioned for f (here and in what follows we use the simplified notation for partial derivatives, e.g. f_t denotes $\partial f/\partial t$). By "Hölder solution" of (1)-(3) we mean the classical solution of the problem being Hölder continuous together with all the derivatives appearing in (1). The initial function $u_0\in C^{4+\mu}(\bar{\Omega})$ fulfills the compatibility conditions required for a smooth solution:

$$\frac{\partial u_0}{\partial n}=\frac{\partial(\Delta u_0)}{\partial n}=0 \qquad \text{for } x\in\partial\Omega,$$

moreover, when $n>3$

$$\frac{\partial L(0, x, u_0)}{\partial n}=\frac{\partial(\Delta L(0, x, u_0))}{\partial n}=0 \qquad \text{for } x\in\partial\Omega,$$

4. Basic estimates and inequalities.

It is well known that a system $(\Delta^2, \{B_1, B_2\}, \Omega)$ defines a "regular elliptic boundary value problem" in the sense of [7], p. 76 also [11], pp. 165, 221, 273. Moreover, our considerations will remain valid for boundary conditions other than (3); e.g. for the Dirichlet condition:

$$(3') \qquad B_1'u\equiv u=0, \qquad B_2'u\equiv\frac{\partial u}{\partial n}=0 \quad \text{on } \partial\Omega,$$

The system $(\Delta^2, \{B_1', B_2'\}, \Omega)$ also defines a regular elliptic boundary value problem. It is known ([7], p. 75), that for such problems the Calderon-Zygmund estimates are valid i.e.:

(4) $$\forall_{1<p<\infty} \exists_{c>0} \|v\|_{W^{4,p}(\Omega)} \leqq c(\|\Delta^2 v\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}),$$

where v is an arbitrary $C^4(\bar{\Omega})$ function satisfying homogeneous boundary conditions; $B_1 v = B_2 v = 0$ on $\partial\Omega$. We need a version of such an estimate valid for second order elliptic operators (known also [9] as "the second fundamental inequality for elliptic operators"):

(5) $$\|v\|_{W^{2,p}(\Omega)} \leqq c_{p,r}(\|\Delta v\|_{L^p(\Omega)} + \|v\|_{L^r(\Omega)}),$$

where $q \geqq 1$, $p > 1$, $v \in W^{2,p}(\Omega)$ and $\partial v/\partial n = 0$ on $\partial\Omega$. The second terms on the right sides of (4), (5) will be replaced by $|\bar{v}| = \left| |\Omega|^{-1} \int_\Omega v(x)dx \right|$.

Further, we need a version of the interpolation inequality for intermediate derivatives [1], p. 75: For $\Omega \subset R^n$ having the uniform cone property, $\varepsilon_0 > 0$ fixed, there exists a constant $K = K(\varepsilon_0, m, \Omega)$ for every $v \in W^{m,2}(\Omega)$, such that

(6) $$\forall_{0<\varepsilon\leq\varepsilon_0} \forall_{0\leq j\leq m-1} |v|_{j,2}^2 \leqq \varepsilon' |v|_{m,p}^2 + C_{\varepsilon'} |v|_{0,p}^2$$

where $|v|_{j,2} = \left\{ \sum_{|\alpha|=j} \int_\Omega |D^\alpha v|^2 dx \right\}^{1/2}$, $\varepsilon' = 2K^2\varepsilon^2$ and $C_{\varepsilon'} = 2K^2\varepsilon^{-(2j/m-j)}$. We also claim an estimate ([1], p. 108);

(7) $$\exists_{C>0} \forall_{v\in W^{1,p}(\Omega)} \|v\|_{L^\infty(\Omega)} \leqq C\|v\|_{W^{1,p}(\Omega)}, \qquad pl > n,$$

where $\Omega \subset R^n$ has the cone property. Finally ([8], p. 37), when $\partial\Omega \in C^m$, then

(8) $$\|v\|_{W^{k,p}(\Omega)} \leqq C' \|v\|_{W^{m,q}(\Omega)}^\theta \|v\|_{L^r(\Omega)}^{1-\theta},$$

with $p \geqq q$, $p \geqq r$, $0 \leqq \theta \leqq 1$ and $k - n/p \leqq \theta(m - n/q) - (1-\theta)n/r$.

Part I. General theory.

5. Local solvability of the problem (1)-(3).

Let us note that, due to Lipschitz continuity of f, uniqueness of the Hölder (and weaker) solution of (1)-(3) is guaranteed. The proof, in which we consider the difference of two solutions, is very similar to that of Lemma 2 and will be omitted.

We define the range of arguments of the nonlinear function f; let $t \geqq 0$, $x \in \bar{\Omega}$, $v \in R$, $p = (p_1, \cdots, p_n) \in R^n$, $q = [q_{ij}] \in R^{n^2}$ and set

$$(9)\qquad X:=\left\{(t,\ x,\ v,\ p,\ q);\ t\in[0,\ T],\ x\in\bar{\Omega},\ \left(|v|^2+\sum_i|p_i|^2+\sum_{i,j}|q_{ij}|^2\right)^{1/2}\leqq R\right\}$$

where T and R are fixed positive numbers. The expression bounded in (9) by R corresponds, for the composite function $f(t,\ x,\ u,\ u_x,\ u_{xx})$ in (1), to $W^{2,\infty}(\Omega)$ norm of u. Let us denote the Lipschitz constants, inside X, for f with respect to $t,\ v,\ p_i,\ q_{ij}$ $(i,\ j=1,\cdots,n)$ by $L_1,\ L_3,\ L_4,\ L_5$ respectively (e.g. L_5 is suitable for each $q_{ij},\ i,\ j=1,\cdots,n$). Also let $|f(t,\ x,\ 0,\ 0,\ 0)|\leqq N$ for $t\in[0,\ T],\ x\in\bar{\Omega}$.

We shall start with the formulation of Lemma 1 necessary to present the main result of Part I; Theorem 1. Because the proof of this lemma is very technical, it will be left until the Appendix.

LEMMA 1. *As long as the Hölder solution of* (1)-(3) *remains in* X, *the following estimates hold; when the dimension* $n\leqq 3$, *then*

$$(10)\qquad \|u(t,\ \cdot)\|^2_{W^{2,\infty}(\Omega)}\leqq\nu\left(\int_\Omega u_t^2dx+N^2|\Omega|\right)+C_\nu\int_\Omega u^2dx,$$

also

$$(11)\qquad \|u(t,\ \cdot)\|^2_{W^{2,(2n/n-2)}(\Omega)}\leqq\nu\left(\int_\Omega u_t^2dx+N^2|\Omega|\right)+C_\nu\int_\Omega u^2dx$$

for the space dimension $n\geqq 4$. *Here* $\nu\in(0,\ \nu_0]$ (ν_0 *given in* (55)), C_ν *increases when* ν *decreases and* $|\Omega|$ *denotes the Lebesgue measure of* Ω.

We are now ready to formulate:

THEOREM 1. *For two arbitrary positive numbers* r, R *and initial function* u_0 *satisfying the condition*

$$(12)\qquad \nu\left[\int_\Omega L^2(0,\ x,\ u_0)dx+N^2|\Omega|\right]+C_\nu\int_\Omega u_0^2(x)dx\leqq r^2<R^2$$

(*the constants* ν *and* C_ν *were chosen in Lemma* 1) *there is a time* T_0, $0<T_0\leqq T$, *such that the Hölder solution of* (1)-(3) *corresponding to* u_0 *exists at least until the time* T_0.

COMMENT. Condition (12) defines certain neighbourhood of the zero function in $W^{2,\infty}(\Omega)$ to which u_0 should belong. When u_0 has too large norm we shall transform the problem (1)-(3) onto equivalent one for the new unknown function $v:=u-u_0$;

$$(1')\qquad v_t=-\varepsilon^2\Delta^2v+\bar{f}(t,\ x,\ v,\ v_x,\ v_{xx})$$

with $\bar{f}(t, x, v, v_x, v_{xx}) := -\varepsilon^2\Delta^2 u_0 + f(t, x, v+u_0, (v+u_0)_x, (v+u_0)_{xx})$ and homogeneous (zero) initial and boundary conditions corresponding to (2), (3). The estimate (12) for the transformed problem reads

$$\nu\left\{\int_\Omega [-\varepsilon^2\Delta^2 u_0 + f(0, x, u_0, u_{0x}, u_{0xx})]^2 dx + N^2|\Omega|\right\} \leqq r^2 < R^2, \tag{12'}$$

and is evidently fulfilled, provided $\nu>0$ is chosen sufficiently small. All the results obtained for u and (1)-(3) stay valid for v and the transformed problem.

The proof of Theorem 1 is divided into several steps. We start with two simple a priori estimates for $\|u(t, \cdot)\|_{L^2(\Omega)}$ and $\|u_t(t, \cdot)\|_{L^2(\Omega)}$ valid while u remains in X.

LEMMA 2 (First a priori estimate). *As long as u remains in X, we have an estimate*

$$\int_\Omega u^2(t, x) dx \leqq e^{ct}\left[\int_\Omega u_0^2(x) dx + \frac{N|\Omega|}{c}(1-e^{-ct})\right], \tag{13}$$

$c=c(L_3, L_4, L_5, N, \varepsilon)$ being a constant.

PROOF. Multiplying (1) by u and integrating over Ω, we get:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx = -\varepsilon^2\int_\Omega \Delta^2 u u dx + \int_\Omega f u dx.$$

Integreting by parts, noting (3)

$$-\varepsilon^2\int_\Omega \Delta^2 u u dx = -\varepsilon^2\int_\Omega (\Delta u)^2 dx,$$

from the Lipschitz continuity of f inside X and the Cauchy inequality we find:

$$\begin{aligned} &\int_\Omega f(t, x, u, u_x, u_{xx}) u dx \\ &= \int_\Omega [f(t, x, u, u_x, u_{xx}) - f(t, x, 0, u_x, u_{xx}) + f(t, x, 0, u_x, u_{xx}) \\ &\quad - f(t, x, 0, 0, u_{xx}) + \cdots + f(t, x, 0, 0, 0)] u dx \\ &\leqq \frac{\gamma}{2}\left[L_4\int_\Omega \sum_i u_{x_i}^2 dx + L_5\int_\Omega \sum_{i,j} u_{x_i x_j}^2 dx\right] \\ &\quad + \left[L_3 + \frac{N}{2} + \frac{L_4 n}{2\gamma} + \frac{L_5 n^2}{2\gamma}\right]\int_\Omega u^2 dx + \frac{N}{2}|\Omega|. \end{aligned} \tag{14}$$

Estimating the first term on the right side of (14) through (5) with $p=r=2$, and choosing $\gamma=\gamma_0$ sufficiently small that

$$c_{2,2}^2\gamma_0 \max\{L_4, L_5\} = \varepsilon^2,$$

we finally get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx \leqq \left[L_3 + \frac{N}{2} + \frac{L_4 n}{2\gamma_0} + \frac{L_5 n^2}{2\gamma_0} + \varepsilon^2\right]\int_\Omega u^2 dx + \frac{N}{2}|\Omega|,$$

which is equivalent to (13). The proof is completed.

We proceed to the next a priori estimate:

LEMMA 3 (Second a priori estimate). *As long as the solution u remains in X;*

$$\int_\Omega u_t^2(t, x)dx \leqq \left[\int_\Omega L^2(0, x, u_0)dx + \frac{c_2}{c_1}(1 - e^{-c_1 t})\right]e^{c_1 t}, \tag{15}$$

where $c_1 = c_1(L_3, L_4, L_5, \varepsilon)$ and $c_2 = c_2(L_1, \varepsilon)$ is proportional to c_1^{-1}.

PROOF. The difference quotient $u_h(t, x) = h^{-1}(u(t+h, x) - u(t, x))$ ($h>0$ is fixed) solves the equation:

$$u_{ht} = -\varepsilon^2\Delta^2 u_h + h^{-1}[f|_{t=t+h} - f|_{t=t}]. \tag{16}$$

Multiplying (16) by u_h, integrating over Ω and by parts, we find that:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx &= -\varepsilon^2\int_\Omega(\Delta u_h)^2 dx \\
&\quad + h^{-1}\int_\Omega[f(t+h, x, u(t+h, x), u_x(t+h, x), u_{xx}(t+h, x)) \\
&\quad - f(t, x, u(t+h, x), u_x(t+h, x), u_{xx}(t+h, x)) + \cdots \\
&\quad - f(t, x, u(t, x), u_x(t, x), u_{xx}(t, x))]u_h dx \\
&\leqq -\varepsilon^2\int_\Omega(\Delta u_h)^2 dx + \frac{\gamma}{2}\left[L_4\int_\Omega\sum_i u_{hx_i}^2 dx + L_5\int_\Omega\sum_{i,j}u_{hx_ix_j}^2 dx + L_1^2|\Omega|\right] \\
&\quad + \frac{1}{2\gamma}(1 + L_3 + L_4 + L_5)\int_\Omega u_h^2 dx,
\end{aligned}$$

making use of the Lipschitz conditions and Cauchy inequality and, in particular, an estimate:

$$\begin{aligned}
&h^{-1}\int_\Omega[f(t+h, x, u(t+h, x), u_x(t+h, x), u_{xx}(t+h, x)) \\
&\quad - f(t, x, u(t+h, x), u_x(t+h, x), u_{xx}(t+h, x))]u_h dx \\
&\leqq \frac{\gamma}{2}\int_\Omega L_1^2 dx + \frac{1}{2\gamma}\int_\Omega u_h^2 dx = \frac{\gamma}{2}L_1^2|\Omega| + \frac{1}{2\gamma}\int_\Omega u_h^2 dx.
\end{aligned}$$

Noting that u_h fulfils the same boundary conditions as u did, by (5), for $\gamma=\gamma_0$ we find that

$$\frac{d}{dt}\int_\Omega u_h^2(t, x)dx \leqq \gamma_0^{-1}(1+L_3+L_4+L_5+\gamma_0\varepsilon^2)\int_\Omega u_h^2(t, x)dx+\gamma_0 L_1^2|\Omega|, \tag{17}$$

which leads to an estimate

$$\int_\Omega u_h^2(t, x)dx \leqq \left[\int_\Omega u_h^2(0, x)dx+\frac{c_2}{c_1}(1-e^{-c_1 t})\right]e^{c_1 t}, \tag{18}$$

with $c_1=\gamma_0^{-1}(1+L_3+L_4+L_5+\gamma_0\varepsilon^2)$, $c_2=\gamma_0 L_1^2|\Omega|$. Passing in (18) with h to 0^+, noting that for the smooth solution we consider u_h tends to u_t when $h\to 0^+$ and $u_t(0, x)$ will be found from (1) with $t=0$, we justify (15). The proof is completed.

For the time being we restrict our considerations to space dimension $n\leqq 3$, higher dimensions will be treated in the Appendix. For $n\leqq 3$ we will now specify the value T_0 mentioned in the formulation of Theorem 1.

In the definition (9) of X we have introduced the time interval $[0, T]$, for which the Lipschitz constants for f were chosen. Next, from Lemmas 2, 3 we have increasing with t estimates (13), (15), which together with (10) in Lemma 1 give:

$$\begin{aligned} \|u(t, \cdot)\|^2_{W^{2,\infty}(\Omega)} &\leqq \nu\left[\int_\Omega u_t^2 dx+N^2|\Omega|\right]+C_\nu\int_\Omega u^2 dx \\ &\leqq \nu\left[\left(\int_\Omega L^2(0, x, u_0)dx+\frac{c_2}{c_1}(1-e^{-c_1 t})\right)e^{c_1 t}+N^2|\Omega|\right] \\ &\quad +C_\nu e^{ct}\left[\int_\Omega u_0^2(x)dx+\frac{N|\Omega|}{c}(1-e^{-ct})\right]. \end{aligned} \tag{19}$$

The estimate (19) is valid as long as u remains in X. But the right side of (19) increases with t, starting for $t=0$ from a value not exceeding r^2 (compare (12)). Defining T_0 as equal to $\min\{T, \tau\}$, where τ is the time for which the right side of (19) reaches the value R^2, we are sure that $u(t, \cdot)$ remains in X for $t\leqq T_0$ and $n\leqq 3$. Moreover, the composite function $f(t, x, u, u_x, u_{xx})$ is uniformly Lipschitz continuous (constants L_1, L_3, L_4, L_5) and bounded in $Q_{T_0}=[0, T_0]\times\bar{\Omega}$.

The remaining part of the proof of Theorem 1 for $n\leqq 3$ is based on estimates of solutions of linear $2b$-parabolic equations (here $b=2$) in $W_q^{m,2bm}(Q_{T_0})$ space (see [10], Chapt. VII, § 10). As a consequence of Theorem 10.4 reported there (with $m=1$, $b=2$, $t=4$, $s=0$, $l=0$; hence $l+t=4$), we have:

$$u\in W_q^{1,4}(Q_{T_0}) \quad \text{with arbitrary } q\in(1, \infty), \tag{20}$$

which means boundedness of the $W_q^{1,4}$ norm of u;

$$\sum_{j=0}^{4} \sum_{4r+s=j} \|D_t^r D_x^s u\|_{L^q(Q_{T_0})} < +\infty . \tag{21}$$

In particular $u_t \in L^q(Q_{T_0})$ and $u_{x_i x_j x_k x_l} \in L^q(Q_{T_0})$ for any $q \in (1, \infty)$.

To obtain a priori estimates for the Hölder solution of (1)-(3) we shall use the following:

LEMMA 4. *For $n \leqq 3$, under our basic assumption that f is locally Lipschitz continuous with respect to $t, u, u_{x_i}, u_{x_i x_j}$ $(i, j=1, \cdots, n)$ and Hölder continuous (exponent μ) with respect to x and that $u_0 \in C^{4+\mu}(\bar{\Omega})$ satisfies compatibility conditions*

$$\frac{\partial u_0}{\partial n} = \frac{\partial(\Delta u_0)}{\partial n} = 0 \qquad \text{for } x \in \partial\Omega , \tag{22}$$

the solution u will be estimated a priori in the Hölder space $C^{1+(\bar{\mu}/4), 4+\bar{\mu}}(Q_{T_0})$, $\bar{\mu} = \min\{2/9, \mu\}$.

OUTLINE OF THE PROOF. As a consequence of (20) with $q=2n+2$ we find that $u, u_t, u_{x_i} \in L^{2n+2}(Q_{T_0})$ which, with the use of the Sobolev theorem, ensures that

$$u \in C^{1/2, 1/2}(Q_{T_0}) . \tag{23}$$

Since as a consequence of (15) $u_t \in L^\infty(0, T_0; L^2(\Omega))$, then by (19) and (1)

$$\varepsilon^2 \Delta^2 u = -u_t + f(\cdot, \cdot, u, u_x, u_{xx}) \in L^\infty(0, T_0; L^2(\Omega))$$

and further, by the elliptic regularity [7, 11], $u \in L^\infty(0, T_0; W^{4,2}(\Omega))$. Again by the Sobolev theorem (in dimension $n \leqq 3$) $W^{4,2}(\Omega) \subset C^{2+(1/2)}(\bar{\Omega})$, hence

$$u \in L^\infty(0, T_0; C^{2+(1/2)}(\bar{\Omega})) . \tag{24}$$

Using Lemma 3.1, Chapt. II of [10] subsequently to u_{x_i} and then to $u_{x_i x_j}$ $(i, j = 1, \cdots, n)$, in the presence of (23), (24) we find that $u_{x_i} \in C^{1/6, 1/2}(Q_{T_0})$, moreover

$$u_{x_i x_j} \in C^{1/18, 1/2}(Q_{T_0}) . \tag{25}$$

Finally, from the Lipschitz, Hölder continuity of f inside X and (25) the composite function $f(t, x, u, u_x, u_{xx})$ belongs to $C^{1/18, \mu'}(Q_{T_0})$ for $\mu' = \min\{1/2, \mu\}$. From Theorem 10.1, Chapt. VII of [10] (with $l-s=\bar{\mu}$, $t+s=4$ and $l+t=4+\bar{\mu}$):

$$u \in C^{1+(\bar{\mu}/4), 4+\bar{\mu}}(Q_{T_0}) , \qquad \bar{\mu} = \min\left\{\frac{2}{9}, \mu'\right\} ,$$

(here the letter C is used instead of H in [10]), and we have the required estimate of u in the Hölder space. The proof of Lemma 4 is completed.

Until now a number of a priori estimates for the hypothetical solution of (1)-(3) have been given. With these estimates, however, the proper proof of existence of the Hölder solution to (1)-(3) based on the Leray-Schauder Principle ("method of continuity") is standard and will be omitted here (compare e.g. [10, 5]). The proof of Theorem 1 for $n\leqq 3$ is thus finished.

Part II. Applications.

6. Global existence of solution for the Cahn-Hilliard equation.

It is simple to conclude from the considerations of Part I, that if we are able to assure global in a time interval $[0, T_1]$ Lipschitz continuity of the function $f(t, x, u, u_x, u_{xx})$ (and its derivatives when $n>3$), then the solution (being as smooth as the data allow) exists at least for $t\in[0, T_1]$. Obviously we cannot expect such global Lipschitz continuity for general f in (1) (perhaps of a very complicated nature), but we may prove it for a number of special problems such as the Cahn-Hilliard equation. Here we will follow the presentation of this equation in [12], p. 147. Let us consider;

$$u_t=-\varepsilon^2\Delta^2 u+\Delta(F(u)), \tag{26}$$

$x\in\Omega\subset R^n$, $n\leqq 3$, together with conditions (2), (3). Here F is a polynomial of the order $2p-1$ (moreover $p=2$ if $n=3$),

$$F(u)=\sum_{j=1}^{2p-1} a_j u^j, \qquad p\in N,\ p\geqq 2, \tag{27}$$

with positive leading coefficient; $a_{2p-1}>0$. The prototype was $\bar{F}(u)=\beta u^3-\alpha u$ with $\beta, \alpha>0$.

Since $\Delta(F(u))=F'(u)\Delta u+F''(u)|\nabla u|^2$ is locally Lipschitz continuous (F', F'' are locally bounded), then the assumptions of Part I are satisfied (provided that u_0, $\partial\Omega$ are smooth and (22) is filfilled) and we have free local in time existence of the Hölder solution to (26), (2), (3). However, if we can justify, using a priori estimates, Lipschitz continuity of

$$f(t, x, u, u_x, u_{xx})=\Delta(F(u))=F'(u)\Delta u+F''(u)|\nabla u|^2 \tag{28}$$

in $[0, T_1]$ ($T_1>0$ will be fixed from now on), we will have proved the existence of the global Hölder solution to the Cahn-Hilliard equation. We need to estimate a priori $\|u(t, \cdot)\|_{L^\infty(\Omega)}$ and $\|\Delta u(t, \cdot)\|_{L^\infty(\Omega)}$ for $t\in[0, T_1]$. These two estimates are in order simple consequence of the one given in [12], p. 156:

(29) $$\|\Delta(F(u))\|^2_{L^2(\Omega)} \leqq k(1+\|\Delta^2 u\|^{2\sigma}_{L^2(\Omega)}),$$

where $k>0$ and $\sigma \in [0, 1)$ are constants independent of u (dependent on the special form (27) of F, k also on $\|\nabla u_0\|_{L^2(\Omega)}$). We have:

LEMMA 5. *For a sufficiently regular solution of the Cahn-Hilliard equation* ($n \leqq 3$) *the two a priori estimates are valid:*

(30) $$\|u(t, \cdot)-\bar{u}\|_{L^\infty(\Omega)} \leqq c(\|\Delta u_0\|^2_{L^2(\Omega)}+mt)^{1/2},$$

with $\bar{u}=|\Omega|^{-1}\int_\Omega u_0(x)dx$, also

(31) $$\|\Delta u(t, \cdot)\|_{L^\infty(\Omega)} \leqq C(\|u_0\|_{W^{4,2}(\Omega)}, T_1)$$

where C is a positive function increasing with respect to both arguments.

PROOF. We start with the proof of (30). Because of (3), integrating (26) over Ω we find that

$$\frac{d}{dt}\int_\Omega u(t, x)dx=0,$$

hence the mean value $\bar{u}$ is preserved in time. Multiplying (26) by $\Delta^2 u$ and integrating over Ω we get:

(32) $$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(\Delta u)^2 dx &= -\varepsilon^2\int_\Omega(\Delta^2 u)^2 dx+\int_\Omega \Delta(F(u))\Delta^2 u\, dx \\ &\leqq \left(-\varepsilon^2+\frac{\varepsilon^2}{2}\right)\int_\Omega(\Delta^2 u)^2 dx+\frac{1}{2\varepsilon^2}\int_\Omega[\Delta(F(u))]^2 dx \\ &\leqq -\frac{\varepsilon^2}{2}\int_\Omega(\Delta^2 u)^2 dx+\frac{k}{2\varepsilon^2}\left[1+\left(\int_\Omega(\Delta^2 u)^2 dx\right)^\sigma\right],\end{aligned}$$

where (29) was also used. The right side of (32) is a function of $z:=\int_\Omega(\Delta^2 u)^2 dx$, having the form $(-\varepsilon^2 z+(k/\varepsilon^2)z^\sigma+(k/\varepsilon^2))$ and therefore must be bounded from above, say by m, for $z \geqq 0$. Hence:

(33) $$\int_\Omega(\Delta u)^2 dx \leqq \int_\Omega(\Delta u_0)^2 dx+2mt.$$

Since, for $n \leqq 3$, as a consequence of (7) and (5)

(34) $$\|u(t, \cdot)-\bar{u}\|_{L^\infty(\Omega)} \leqq c\|\Delta u(t, \cdot)\|_{L^2(\Omega)},$$

we have (30). Note the slow growth of the right side of (30) of the order $t^{1/2}$.

To obtain (31) we shall consider first u_t in $L^2(\Omega)$. Formally we proceed as in the proof of Lemma 3, but now without using implicit Lipschitz constants.

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_h^2 dx = -\varepsilon^2\int_\Omega (\Delta u_h)^2 dx + \int_\Omega [\Delta(F(u))]_h u_h dx$$

$$= -\varepsilon^2\int_\Omega (\Delta u_h)^2 dx + \int_\Omega (F(u))_h \Delta u_h dx$$

$$\leqq -\varepsilon^2\int_\Omega (\Delta u_h)^2 dx + \int_\Omega F'(\tilde{u}) u_h \Delta u_h dx .$$

As a consequence of (30); $|F'(u)|\leqq K$, hence

$$\frac{d}{dt}\int_\Omega u_h^2 dx \leqq -\varepsilon^2\int_\Omega (\Delta u_h)^2 dx + (K/\varepsilon)^2\int_\Omega u_h^2 dx ,$$

and, for $h\to 0^+$

$$\int_\Omega u_t^2(t, x) dx \leqq \int_\Omega [-\varepsilon^2\Delta^2 u_0 + \Delta(F(u_0))]^2 dx \exp[(K/\varepsilon)^2 t] . \tag{35}$$

Finally, from (26)

$$\varepsilon^2\Delta^2 u = -u_t + F'(u)\Delta u + F''(u)|\nabla u|^2 , \tag{36}$$

where from (30), $F'(u)$ and $F''(u)$ are in $L^\infty([0, T_1]\times\bar{\Omega})$, Δu is in $L^\infty(0, T_1; L^2(\Omega))$ as a result of (33), u_t is in $L^\infty(0, T_1; L^2(\Omega))$ as follows from (35). Hence, as a consequence of the Sobolev inequality and (5)

$$\|\nabla u\|_{L^4(\Omega)} \leqq \text{const.}(\|\Delta u\|_{L^2(\Omega)} + |\bar{u}|), \qquad n\leqq 3 ,$$

also $|\nabla u|^2 \in L^\infty(0, T_1; L^2(\Omega))$. We have now verified that the right side of (36) belongs to $L^\infty(0, T_1; L^2(\Omega))$, thus $\Delta^2 u \in L^\infty(0, T_1; L^2(\Omega))$, which from (7), (4) for $n\leqq 3$ means that $\Delta u \in L^\infty([0, T_1]\times\bar{\Omega})$. Also $|\nabla u|$ is bounded in $[0, T_1]\times\bar{\Omega}$. The proof of Lemma 5 is completed.

For $n\leqq 3$ we have thus verified existence of the global Hölder solution to (26), (2), (3).

REMARK 1. The polynomial form of F in [12] is rather restricting. Under a weak assumption only;

$$\underset{M>0}{\exists}\ \underset{\gamma\in R}{\forall}\ -\int_0^\gamma F(z) dz \leqq M , \tag{37}$$

evidently satisfied by any F admitted by other authors [2, 3], we have the time independent estimate

$$\|u(t, \cdot) - \bar{u}\|_{L^2(\Omega)}^2 \leqq c_3\|\nabla u(t, \cdot)\|_{L^2(\Omega)}^2 \leqq \text{const.} \tag{38}$$

$$= c_3\left\{\|\nabla u_0\|_{L^2(\Omega)}^2 + \frac{2}{\varepsilon^2}\left[\int_\Omega\int_0^{u_0(x)} F(z) dz dx + M|\Omega|\right]\right\} ,$$

c_3 being a constant in the Poincaré inequality. Estimate (38) is a simple consequence of (37) and the existence of a Liapunov functional for the solution of (26), (2), (3);

$$\frac{d}{dt}\left[\frac{\varepsilon^2}{2}\int_\Omega \sum_i u_{x_i}^2(t, x)dx+\int_\Omega \int_0^{u(t,x)} F(z)dzdx\right]\leqq 0\,. \tag{39}$$

7. Kuramoto-Sivashinsky equation.

Considering [12], p. 137, let us study the problem

$$u_t=-\nu u_{xxxx}-u_{xx}-\frac{1}{2}(u_x)^2\,, \tag{40}$$

$t\geqq 0$, $x\in[-L/2, L/2]$, equipped by the space-periodic boundary conditions

$$\frac{\partial^j u}{\partial x^j}\left(t, -\frac{L}{2}\right)=\frac{\partial^j u}{\partial x^j}\left(t, \frac{L}{2}\right), \qquad j=0, 1, 2, 3, \tag{41}$$

$$u(0, x)=u_0(x) \qquad \text{for } x\in\left[-\frac{L}{2}, \frac{L}{2}\right]. \tag{42}$$

We note that as a consequence of (41) (all the unspecified integrals here are taken over $[-L/2, L/2]$);

$$\int u_x(t, x)dx=\int u_{xx}(t, x)dx=\int u_{xxx}(t, x)dx=\int u_{xxxx}(t, x)dx=0$$

since, e.g.

$$\int u_x(t, x)dx=u\left(t, \frac{L}{2}\right)-u\left(t, -\frac{L}{2}\right)=0\,. \tag{43}$$

With this observation it is easy to check that the expression

$$\left[\int(\varphi^{(k)}(x))^2dx+\left|\int\varphi(x)dx\right|\right]^{1/2}, \qquad k=1, 2, 3, 4 \tag{44}$$

define equivalent norms in $H^k(-L/2, L/2)$ for functions satisfying (41) (or first k conditions in (41) when $k<4$). For space-periodic boundary conditions (41) the last observation replaces the Calderon-Zygmund estimates (4), (5).

For the problem (40)-(42) the term f has the form:

$$f(t, x, u, u_x, u_{xx})=-u_{xx}-\frac{1}{2}(u_x)^2\,, \tag{45}$$

hence, to show global existence of the solution, we shall find a global in time L^∞ a priori estimate of u_x. This estimate will be obtained in two steps.

First step. Estimate of $\int(u_x)^2dx$.

Multiplying (40) by u_{xx} and integrating over $[-L/2, L/2]$ we find that:

$$-\frac{1}{2}\frac{d}{dt}\int(u_x)^2dx=\nu\int(u_{xxx})^2dx-\int(u_{xx})^2dx-\frac{1}{2}\int(u_x)^2u_{xx}dx,$$

but

$$\int(u_x)^2u_{xx}dx=\frac{1}{3}\int[(u_x)^3]_x dx=0$$

because of (41), hence applying (6) we obtain

$$\frac{d}{dt}\int(u_x)^2dx=-2\nu\int(u_{xxx})^2dx-\int(u_{xx})^2dx$$

$$\leqq(-2\nu+2\nu)\int(u_{xxx})^2dx+2C_\nu\int(u_x)^2dx$$

or

$$\int(u_x)^2(t,\ x)dx\leqq\int(u_{0x})^2dx\exp(2C_\nu t) \tag{46}$$

$$\leqq\int(u_{0x})^2dx\exp(2C_\nu T_1)=:m_0^2\,.$$

Second step. Estimate of $\int(u_{xx})^2dx$.

Multiplying (40) by u_{xxxx} and integrating over $[-L/2,\ L/2]$ we find:

$$\frac{1}{2}\frac{d}{dt}\int(u_{xx})^2dx=-\nu\int(u_{xxxx})^2dx+\int(u_{xxx})^2dx-\frac{1}{2}\int(u_x)^2u_{xxxx}dx\,, \tag{47}$$

next, using (46) and the Poincaré inequality we have

$$\left|\int(u_x)^2u_{xxxx}dx\right|\leqq\|u_x\|_{L^\infty}\|u_x\|_{L^2}\|u_{xxxx}\|_{L^2}$$

$$\leqq m_0\Big(\frac{\delta}{2}\|u_{xxxx}\|_{L^2}^2+\frac{c_3}{2\delta}\|u_{xx}\|_{L^2}^2\Big)\,.$$

Choosing $m_0(\delta/2)=\nu$ (hence $(m_0c_3/2\delta)=(\nu c_3/\delta^2)$, and using (6) to estimate the third derivative in (47), we obtain

$$\frac{1}{2}\frac{d}{dt}\int(u_{xx})^2dx\leqq\Big(-\nu+\frac{\nu}{2}+\frac{\nu}{2}\Big)\int(u_{xxxx})^2dx+\Big[C_{\nu/2}+\frac{\nu c_3}{2\delta^2}\Big]\int(u_{xx})^2dx,$$

which together with (46) and the inequality following from (7) and (43)

$$\|u_x(t,\ \cdot)\|_{L^\infty}^2\leqq c\int(u_{xx})^2(t,\ x)dx \qquad (n=1) \tag{48}$$

justify the required $L^\infty([0,\ T_1]\times[-L/2,\ L/2])$ estimate of u_x. From our general result it is clear that there exists a global Hölder solution of the problem (40)–(42). Our considerations are completed.

Part III. Appendix.

8. Proof of Lemma 1.

Since in fact the proof of (11) coincides with that of

$$\|u(t,\cdot)\|^2_{W^{2,\infty}(\Omega)} \leqq \nu\Big(\int_\Omega u_t^2 dx + N^2|\Omega|\Big) + C_\nu \int_\Omega u^2 dx\,, \tag{10}$$

we will present only the first proof. For $w := u_{x_t x_j}$, as a consequence of (7) with $p=4$, $l=1$, $n \leqq 3$:

$$\|w\|_{L^\infty(\Omega)} \leqq C\|w\|_{W^{1,4}(\Omega)} \leqq CC'\|w\|^{7/8}_{W^{2,2}(\Omega)}\|w\|^{1/8}_{L^2(\Omega)}\,, \tag{49}$$

where the inequality (8) has also been used. Now, from the Young inequality

$$\|w\|_{L^\infty(\Omega)} \leqq \frac{\delta}{2}\|w\|_{W^{2,2}(\Omega)} + C(\delta)\|w\|_{L^2(\Omega)} \tag{50}$$

(with $C(\delta) = \text{const.}\ \delta^{-7}$), hence from (6) we may claim

$$\|u_{x_t x_j}\|_{L^\infty(\Omega)} \leqq \delta\|u\|_{W^{4,2}(\Omega)} + \bar{C}_\delta\|u\|_{L^2(\Omega)} \qquad (n \leqq 3)\,. \tag{51}$$

As a consequence of (1), when u remains in X

$$\int_\Omega (\Delta^2 u)^2 dx = \varepsilon^{-4}\int_\Omega [u_t - f(t, x, u, u_x, u_{xx})]^2 dx \tag{52}$$

$$\leqq 3\varepsilon^{-4}\int_\Omega [u_t^2 + f^2(t, x, 0, 0, 0) + (f(t, x, 0, 0, 0) - f(t, x, u, u_x, u_{xx}))^2] dx$$

$$\leqq 3\varepsilon^{-4}\int_\Omega [u_t^2 + N^2] dx + c_4\varepsilon^{-4}\|u\|^2_{W^{2,2}(\Omega)}\,,$$

where $c_4 = c_4(L_3, L_4, L_5)$. As a result of (4), (51)

$$\|u_{x_t x_j}\|^2_{L^\infty(\Omega)} \leqq 2\delta^2\|u\|^2_{W^{4,2}(\Omega)} + 2(\bar{C}_\delta)^2\|u\|^2_{L^2(\Omega)}$$

$$\leqq 2c^2\delta^2(\|\Delta^2 u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})^2 + 2(\bar{C}_\delta)^2\|u\|^2_{L^2(\Omega)}\,.$$

Next, from (52)

$$\|u_{x_t x_j}\|^2_{L^\infty(\Omega)} \leqq 12\varepsilon^{-4}c^2\delta^2\Big[\int_\Omega (u_t^2 + N^2) dx + \frac{c^4}{3}\|u\|^2_{W^{2,2}(\Omega)}\Big] + (4c^2\delta^2 + 2(\bar{C}_\delta)^2)\|u\|^2_{L^2(\Omega)}\,. \tag{53}$$

As a consequence of (7) with $p=n+1$ and (5) with $p=n+1$, $r=2$, we may show that

$$\|u\|^2_{W^{1,\infty}(\Omega)} \leqq c(\|\Delta u\|^2_{L^{n+1}(\Omega)} + \|u\|^2_{L^2(\Omega)})$$

$$\leqq c_5(\|\Delta u\|^2_{L^\infty(\Omega)} + \|u\|^2_{L^2(\Omega)})\,.$$

Summing (53) with respect to i, j or with respect to i, i to get the bound for $\sum_{i,j}\|u_{x_i x_j}\|^2_{L^\infty(\Omega)}$ or $\|\Delta u\|^2_{L^\infty(\Omega)}$, respectively, we finally have

$\|u\|^2_{W^{2,\infty}(\Omega)}\leqq(n^2+c_5 n)\cdot$(right side of (53))$+c_5\|u\|^2_{L^2(\Omega)}$, which, for $\nu:=24\varepsilon^{-4}c^2\delta^2(n^2+c_5 n)$ and δ taken so small that

(54) $$12\varepsilon^{-4}c^2\delta^2(n^2+c_5 n)\frac{c_4}{3}|\Omega|\leqq\frac{1}{2}$$

gives (10). Condition (54) defines the value ν_0 mentioned in Lemma 1 ($\nu\in(0,\nu_0]$) in such a way, that

(55) $$\frac{1}{3}\nu_0 c_4|\Omega|=1\,.$$

The proof of (11) is similar to that of (10) with one exception, instead of (49) our starting point is an estimate (valid for $n\geqq 4$);

(56) $$\|w\|_{L^{2n/n-2}(\Omega)}\leqq C\|w\|_{W^{1,2}(\Omega)}\leqq CC'\|w\|^{1/2}_{W^{2,2}(\Omega)}\|w\|^{1/2}_{L^2(\Omega)}\,,$$

used for $w=u_{x_i x_j}$ as previously. The proof of Lemma 1 is completed.

9. Space dimensiens $n>3$.

We have now complete information necessary to obtain the a priori estimates of u in $W^{2,\infty}(\Omega)$ for arbitrary n. To simplify notation we denote by T_2 a positive time such that

(57) $$\|u\|_{L^\infty(0,T_2;W^{2,\infty}(\Omega))}\leqq R\,,$$

which is equivalent to saying that u remains in X until a time T_2 (such $T_2>0$ exists due to continuity of the Hölder solution and (12); we need to estimate it). The key idea of our further proof is that estimates obtained for u will be valid as well for u_t solving the equation

$$u_{tt}=-\varepsilon^2\Delta^2 u_t+f_t+f_u u_t+\sum_i f_{u_{x_i}}u_{tx_i}+\sum_{i,j}f_{u_{x_i x_j}}u_{tx_i x_j}\,.$$

From (11) and Lemmas 2, 3 we have

(58) $$u\in L^\infty(0,T_2;W^{2,2n/n-2}(\Omega))\,,$$

and from an estimate similar to (11), valid for u_t (we need our supplementary assumptions on f, u_0 to justify it):

(59) $$u_t\in L^\infty(0,T_2;W^{2,2n/n-2}(\Omega))\,,$$

and, as a consequence of (1), (58) and (59)

$$\varepsilon^2\Delta^2 u=-u_t+f(t,x,u,u_x,u_{xx})\in L^\infty(0,T_2;L^{2n/n-2}(\Omega))\,.$$

Then from the elliptic regularity theory [7, 11]:

(60) $$u \in L^{\infty}(0, T_2; W^{4, 2n/n-2}(\Omega)).$$

For $n \leqq 5$, as a consequence of (7)

$$W^{2,\infty}(\Omega) \subset W^{4, 2n/n-2}(\Omega),$$

thus using (60) we have verified (57). At this point we will fix the time T_0 (for $n=4$, 5) in a similar way as previously for $n \leqq 3$ in considerations following (19).

Next, for $u=6, \cdots, 9$, using (60), the analogous estimate for u_t;

(61) $$u_t \in L^{\infty}(0, T_2; W^{4, 2n/n-2}(\Omega))$$

(requiring new assumptions on f, u_0) and (1) we justify that

$$u \in L^{\infty}(0, T_2; W^{6, 2n/n-2}(\Omega)) \subset L^{\infty}(0, T_2; W^{2,\infty}(\Omega)).$$

We shall continue this procedure for larger n.

REMARK 2. In spite of certain technical complications involved in our proofs, the general idea of Theorem 1 is simple. It is based on Lemmas 1, 2, 3 giving a priori estimates and on the theory of linear problems known in literature. Moreover, our a priori estimates technique offers the possibility of effective estimates (as in [6]) of the life time of solutions. As a competitive technique we should mention the semigroups theory and its generalizations; compare e.g. [7, 4, 13].

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