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ON THE GAUSS MAP OF COMPLETE SPACE-LIKE HYPERSURFACES OF CONSTANT MEAN CURVATURE IN MINKOWSKI SPACE

By

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§ 1. Introduction.

Let $\boldsymbol{R}_1^{n+1}$ be the $(n+1)$-dimensional Minkowski space, that is, $\boldsymbol{R}^{n+1}$ with the Lorentz metric $\langle\, ,\rangle=(dx_1)^2+\cdots+(dx_n)^2-(dx_{n+1})^2$. It has been known that in $\boldsymbol{R}_1^{n+1}$ hyperplanes are the only complete space-like hypersurfaces whose mean curvatures are zero. This Bernstein type theorem was proposed by Calabi, and solved by him [3] (for $n\leqq 4$) and by Cheng and Yau [5] (for all n) (see also Ishihara [10] or Nishikawa [14]). On the other hand, for complete space-like hypersurfaces of nonzero constant mean curvature in $\boldsymbol{R}_1^{n+1}$, there are many nonlinear examples constructed by Treibergs [18], Hano and Nomizu [7], Ishihara and Hara [11] and others.

In his recent paper, Palmer [17] discussed the Gauss map of a complete space-like hypersurface of constant mean curvature in $\boldsymbol{R}_1^{n+1}$ and showed a condition for the hypersurface to be a hyperplane. This is a result analogous to the one obtained by Hoffman, Osserman and Schoen [9], who proved that the normals to a complete surface of constant mean curvature in the 3-dimensional Euclidean space $\boldsymbol{E}^3$ cannot lie in a closed hemisphere of $\boldsymbol{S}^2$, unless the surface is a plane or a right circular cylinder. Note that a right circular cylinder is the simplest example of a complete non-umbilical surface of constant mean curvature in $\boldsymbol{E}^3$.

In $\boldsymbol{R}_1^{n+1}$ the simplest example of a complete non-umbilical space-like hypersurface of constant mean curvature is given by the following:

$$\boldsymbol{H}^k(c)\times\boldsymbol{R}^{n-k}$$
$$=\left\{(x_1,\cdots,x_n,x_{n+1})\in\boldsymbol{R}_1^{n+1}\,;\,(x_{n-k+1})^2+\cdots+(x_n)^2-(x_{n+1})^2=\frac{1}{c},\ x_{n+1}>0\right\},$$

where c is a negative number and $k=1, 2, \cdots, n-1$. In particular, $\boldsymbol{H}^1(c)\times\boldsymbol{R}^{n-1}$ is called a *hyperbolic cylinder*.

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Recently, Ki, Kim and Nakagawa [12] characterized hyperbolic cylinders as the only complete space-like hypersurfaces of non-zero constant mean curvature in $\boldsymbol{R}_1^{n+1}$ for which the norm of the second fundamental form is maximal. Moreover, when $n=2$, K. Milnor [13] and Yamada [19] showed that the hyperbolic cylinder $\boldsymbol{H}^1(c)\times\boldsymbol{R}^1$ is the only "uniformly" non-umbilical surface among complete space-like surfaces of non-zero constant mean curvature, and the author gave another proof of this theorem [2].

In this paper, we shall improve the Palmer's theorem and characterize the hyperbolic cylinder in $\boldsymbol{R}_1^{n+1}$ by a method similar to the one employed by Hoffman et al [9]. In fact, we shall make use of the distance function of the hyperbolic space constructed by Cecil and Ryan [4].

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§2. The theorems.

Throughout this paper, we assume manifolds to be connected and geometric objects to be smooth.

Let M be a complete space-like hypersurface of constant mean curvature H in $\boldsymbol{R}_1^{n+1}$ and η be the time-like unit normal field of M. For each point p in M we regard $\eta(p)$ as a point in the n-dimensional hyperbolic space $\boldsymbol{H}^n=\boldsymbol{H}^n(-1)$ in $\boldsymbol{R}_1^{n+1}$. Then Palmer's theorem (in [17]) can be improved in the following fashion:

THEOREM 1. *Let M be a complete space-like hypersurface of constant mean curvature $\boldsymbol{R}_1^{n+1}$. If $\eta(M)$ is contained in a geodesic ball in $\boldsymbol{H}^n$, then M is a hyperplane in $\boldsymbol{R}_1^{n+1}$.*

A geodesic ball of radius r centered at $\bar{\eta}$ in $\boldsymbol{H}^n$ is denoted by $B_r(\bar{\eta})$. The distance in $\boldsymbol{H}^n$ from $\bar{\eta}$ to x is given by

$$L_{\bar{\eta}}(x)=\cosh^{-1}(-\langle\bar{\eta}, x\rangle).$$

This distance function $L_{\bar{\eta}}$ on $\boldsymbol{H}^n$ has, as level sets, compact totally unbilic hypersurfaces (geodesic spheres), and $B_r(\bar{\eta})$ is given by

$$B_r(\bar{\eta})=\{x\in\boldsymbol{H}^n\,;\,L_{\bar{\eta}}(x)<r\}.$$

It is clear that hyperplanes are the only space-like hypersurfaces for which $\eta(M)$ coincide with one point.

On the other hand, $\eta(\boldsymbol{H}^k(c)\times\boldsymbol{R}^{n-k})$ is a complete totally geodesic k-dimentional submanifold in $\boldsymbol{H}^n$, which is called a k-plane in $\boldsymbol{H}^n$. In particular, an $(n-1)$-plane in $\boldsymbol{H}^n$ is called a hyperplane in $\boldsymbol{H}^n$ and a parametrized 1-plane in $\boldsymbol{H}^n$ is a maximal geodesic in $\boldsymbol{H}^n$.

We can define a tubular neighborhood $U_r(\pi)$ of radius r around a k-plane π in $\boldsymbol{H}^n$. For each x in $\boldsymbol{H}^n$, there is a unique shortest geodesic γ in $\boldsymbol{H}^n$ from x to π. Let $L_\pi(x)$ denote the length of γ and define $U_r(\pi)$ by

$$U_r(\pi)=\{x\in\boldsymbol{H}^n\,;\ L_\pi(x)<r\}.$$

Then a characterization of the hyperbolic cylinder is obtained as follows.

THEOREM 2. *Let M be a complete space-like hypersurface of non-zero constant mean curvature in $\boldsymbol{R}_1^{n+1}$. If $\eta(M)$ is contained in $U_r(\beta)$ for some $r>0$ and for some maximal geodesic β on $\boldsymbol{H}^n$, then M is congruent to a hyperbolic cylinder $\boldsymbol{H}^1(c)\times\boldsymbol{R}^{n-1}$.*

This theorem is an immediate consequence of the next proposition.

PROPOSITION. *Let M be a complete space-like hypersurface of constant mean curvature in $\boldsymbol{R}_1^{n+1}$. If $\eta(M)$ is contained in $U_r(\pi)$ for some $r>0$ and for some k-plane π of $\boldsymbol{H}^n$, then $\eta(M)$ is contained in π and at least $(n-k)$-principal curvatures of M are zero at any point of M.*

REMARK. Theorem 2 can be proved by a theorem obtained by Choi and Treibergs [6], if we note that complete space-like hypersurfaces in $\boldsymbol{R}_1^{n+1}$ are entire. Furthermore, Theorem 1 can also follow from the Liouville theorem for harmonic mappings of Riemannian manifolds, which is proved by Hildebrandt, Jost and Widman in [8]. But our proofs do not depend on these facts, and we shall consistently make use of the generalized maximum principle on a complete Riemannian manifold.

§ 3. Preliminaries.

As in § 2, let M be a complete space-like hypersurface of constant mean curvature H in $\boldsymbol{R}_1^{n+1}$, η be the time-like unit normal field of M.

We choose a local field of orthonormal frames $e_1, e_2, \cdots, e_n$ on M and let $\omega_1, \omega_2, \cdots, \omega_n$ denote the dual coframes on M. We shall use the summation convention with Roman indices in the range $1\leqq i, j, \cdots \leqq n$. The second fundamental form on M is given by the quadratic form

$$\alpha=-\sum h_{ij}\omega_i\otimes\omega_j\otimes\eta$$

with values in the normal bundle of M. Let D (resp. ∇) denote the Levi-Civita connection of $\boldsymbol{R}_1^{n+1}$ (resp. M). Then the Gauss formula and the Weingarten formula are given respectively by

$$D_{e_i}e_j=\nabla_{e_i}e_j-h_{ij}\eta \quad \text{and} \quad D_{e_i}\eta=-\sum_j h_{ij}e_j .$$

Let h_{ijk} denote the covariant derivative of h_{ij}. Then we obtain the Coddazi equation

$$h_{ijk}=h_{ikj} .$$

Since the mean curvature H of M is defined by $\sum h_{ii}/n$, the norm of α satisfies

$$|\alpha|^2 \geqq nH^2 . \tag{1}$$

LEMMA. *The Gauss map η is a harmonic map of M into $\boldsymbol{H}^n \subset \boldsymbol{R}_1^{n+1}$, that is, if $\eta=(\eta_1, \cdots, \eta_n, \eta_{n+1})$ then a Laplacian of each component η_A $(A=1, \cdots, n+1)$ satisfies the following equation;*

$$\Delta\eta_A=|\alpha|^2\eta_A . \tag{2}$$

PROOF. Let p be any fixed point in M. Let $\{E_1, \cdots, E_n\}$ be an orthonormal local frames about p such that $(\nabla_{E_i}E_j)(p)=0$ $(i, j=1, \cdots, n)$. Then we have

$$E_i(h_{ij})_p=(h_{iji})_p=(h_{iij})_p , \qquad (D_{E_i}E_j)_p=-(h_{ij}\eta)_p$$

and, since H is constant,

$$\begin{aligned}(\Delta\eta_1, \cdots, \Delta\eta_{n+1})(p) &= (\sum_i E_iE_i\eta_1, \cdots, \sum_i E_iE_i\eta_{n+1})(p) \\ &= (\sum_i D_{E_i}D_{E_i}\eta)_p=(\sum_i D_{E_i}(-\sum_i h_{ij}E_j))_p \\ &= (-\sum_{i,j} E_i(h_{ij})E_j-h_{ij}D_{E_i}E_j)_p \\ &= (-\sum_j E_j(nH)E_j+\sum_{i,j}(h_{ij})^2\eta)_p \\ &= (|\alpha|^2\eta)_p . \qquad \blacksquare\end{aligned}$$

In order to prove the theorems, we need the following generalized maximum principle theorem due to Omori [15] and Yau [20].

THE GENERALIZED MAXIMUM PRINCIPLE. *Let N be a complete Riemannian manifold whose Ricci curvature is bounded from below and let F be a function of class C^2 on N. If F is bounded from above, then for any $\varepsilon>0$ there exists a point q such that*

(3) $$|\nabla F(q)|<\varepsilon, \quad \Delta F(q)<\varepsilon, \quad F(q)>\sup F-\varepsilon,$$

where $|\nabla F|$ *denotes the norm of the gradient* ∇F *of* F.

In the present case, the Ricci curvature is given by

$$S_{ij}=-nHh_{ij}+\sum_k h_{ik}h_{kj},$$

and hence is bounded from below by $-n^2H^2/4$. So we can apply the generalized maximum principle for any C^2-function on M which is bounded from above.

§ 4. Proof of the theorems.

In this section, we give the proofs of the previous theorems.

PROOF OF THEOREM 1. The condition $\eta(M)\subset B_r(\bar{\eta})$ is equivalent the following inequality valid everywhere on M;

$$1\leqq -\langle\eta, \bar{\eta}\rangle<\cosh r.$$

We may assume $\bar{\eta}=(0, 0, \cdots, 0, 1)$, by applying, if necessary, a Lorentz transformation to M. Then the condition reads

(4) $$1\leqq \eta_{n+1}<\cosh r,$$

and in particular, η_{n+1} is a smooth function on M which is bounded from above.

From the equation (2) combined with the relation (1), we have

(5) $$\Delta\eta_{n+1}=|\alpha|^2\eta_{n+1}\geqq nH^2\eta_{n+1}.$$

Let $\{\varepsilon_n\}$ be a convergent sequence such that $\varepsilon_m>0$ and $\varepsilon_m\to 0$ $(m\to\infty)$. Then, by the generalized maximum principle, there is a sequence of points $\{q_n\}$ such that η_{n+1} satisfies (3) at each $q_m\in M$ for ε_m, i.e.,

(3′) $$|\nabla\eta_{n+1}(q_m)|<\varepsilon_m, \quad \Delta\eta_{n+1}(q_m)<\varepsilon_m, \quad \eta_{n+1}(q_m)>\sup\eta_{n+1}-\varepsilon_m.$$

Then by the inequality (5),

$$nH^2\eta_{n+1}(q_m)<\varepsilon_m.$$

Furthermore, because the sequence $\{\eta_{n+1}(q_m)\}$ converges to $\sup\eta_{n+1}$, we have

$$nH^2\sup\eta_{n+1}\leqq 0.$$

Since (4) implies $\sup\eta_{n+1}\geqq 1$, it follows from this inequality that the mean curvature H must be zero.

Hence, by the result of Cheng and Yau, M must be a hyperplane. ■

PROOF OF PROPOSITION. For the k-plane π in $\boldsymbol{H}^n$, we can choose space-like orthonormal vectors $\{\sigma_1, \cdots, \sigma_{n-k}\}$ in $\boldsymbol{R}_1^{n+1}$ such that

$$\pi=\{x\in\boldsymbol{H}^n;\ \langle x, \sigma_a\rangle=0\ (a=1, \cdots, n-k)\}.$$

Let π_a $(a=1, \cdots, n-k)$ be the hyperplane in $\boldsymbol{H}^n$ defined by

$$\pi_a=\{x\in\boldsymbol{H}^n;\ \langle x, \sigma_a\rangle=0\}.$$

The distance in $\boldsymbol{H}^n$ from x to a hyperplane π_a is then given by

$$L_{\pi_a}(x)=L_{\sigma_a}(x)=|\sinh^{-1}(-\langle x, \sigma_a\rangle)|.$$

Since $U_r(\pi)$ is contained in $U_r(\pi_a)$ for every a, it follows from the assumption $\eta(M)\subset U_r(\pi)$ that the inequalities

$$-\sinh r<-\langle\eta, \sigma_a\rangle<\sinh r \qquad (a=1, \cdots, n-k)$$

are valid everywhere on M. We may assume

$$\sigma_a=(0, \cdots, 0, \overset{ath}{1}, 0, \cdots, 0) \qquad (a=1, \cdots, n-k),$$

by applying a Lorentz transformation to M if necessary. Let F_a be a smooth function on M defined by $F_a=(\langle\eta, \sigma_a\rangle)^2=(\eta_a)^2$. Then the above inequalities imply

$$0\leqq F_a<\sinh^2 r \qquad (a=1, \cdots, n-k). \tag{6}$$

and, in particular, F_a is bounded from above.

From the equation (2) combined with the relation (1), we have

$$\Delta\eta_a=|\alpha|^2\eta_a\,,$$

$$\Delta F_a=2\{|\nabla\eta_a|^2+|\alpha|^2(\eta_a)^2\}\geqq|\alpha|^2(\eta_a)^2\geqq 2nH^2F_a\,. \tag{7}$$

Let $\{\varepsilon_m\}$ be a convergent sequence such that $\varepsilon_m>0$ and $\varepsilon_m\to 0$ $(m\to\infty)$. Then, by the generalized maximum principle, there is a sequence of points $\{q_m\}$ such that F_a satisfies (3) at each q_m for ε_m, i.e.,

$$|\nabla F_a(q_m)|<\varepsilon_m, \qquad \Delta F_a(q_m)<\varepsilon_m, \qquad F_a(q_m)<\sup F_a-\varepsilon_m. \tag{3''}$$

Then by the inequality (7),

$$2nH^2F_a(q_m)<\varepsilon_m\,.$$

Furthermore, because the sequence $\{F_a(q_m)\}$ converges to $\sup F_a$, we have

$$2nH^2\sup F_a\leqq 0\,.$$

Since H is non-zero and (6) implies that $\sup F_a$ is non-negative, it follows from this inequality that $F_a=0$ for each $a=1, \cdots, n-k$. Hence we get $\eta_1=\cdots=\eta_{n-k}=0$ and $\eta(M)\subset\pi$.

Let p be a point in M and choose a local field of orthonormal frames $\{e_i\}$ on a neighborhood of p in such a way that $h_{ij}=\lambda_i\delta_{ij}$, where $\{\lambda_i\}$ are the principal curvatures of M. Note that, since $\eta=(0, \cdots, 0, \eta_{n-k+1}, \cdots, \eta_{n+1})$, the Weingarten formula is written as

$$\lambda_i e_i=(0, \cdots, 0, -e_i\eta_{n-k+1}, \cdots, -e_i\eta_{n+1}) \qquad (i=1, \cdots, n). \tag{8}$$

Let l denote the number of zero principal curvatures at p. We may assume $\lambda_1=\cdots=\lambda_l=0$, $\lambda_{l+1}, \cdots, \lambda_n\neq 0$ by changing the indices if necessary. Let $T_l^{\perp}$ be the subspace of the tangent space $T_p(M)$ at p of M, which is spanned by the vectors $e_{l+1}, \cdots, e_n$. The dimension of $T_l^{\perp}$ is $n-l$. On the other hand, it follows from (8) and simple calculation that $T_k^{\perp}$ is contained in the vector space spanned by the following k-independent vectors

$$(0, \cdots, 0, \overset{(n-k+1)th}{1}, 0, \cdots, 0, \eta_{n-k+1}/\eta_{n+1}), \cdots, (0, \cdots, 0, \overset{nth}{1}, \eta_n/\eta_{n+1}).$$

Then we get that $n-l\leqq k$.

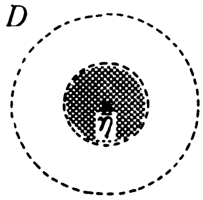
Hence, at least $(n-k)$-principal curvatures are zero at p. ■

PROOF OF THEOREM 2. Under the assumption, it follows from the proposition that the principal curvatures of M are 0 and nH with multiplicity $n-1$ and 1 respectively. Hence, from the congruence theorem due to Abe, Koike and Yamaguchi [1], M is congruent to a hyperbolic cylinder. ■

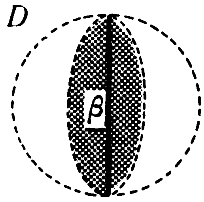
§ 5. Remarks.

In order to illustrate our results, we make a few remarks on the Gauss map images of a complete space-like surface M of constant mean curvature H in 3-dimensional Minkowski space $\boldsymbol{R}_1^3$. In this case, the Gauss map η is a map of M into $\boldsymbol{H}^2$.

It is well-known that a hyperbolic space $\boldsymbol{H}^2$ is isometric to the Poincaré disk (D, ds^2), where $D=\{z=u+iv\in\boldsymbol{C}; |z|<1\}$ and ds^2 is the Poincaré metric $ds^2=4dzd\bar{z}/(1-|z|^2)^2$. In the Poincaré disk, by choosing suitable isometries, we can regard a geodesic ball $B_r(\bar{\eta})$ and a tublar neighborhood $U_r(\beta)$ around a maximal geodesic β in $\boldsymbol{H}^2$ as the following regions respectively.



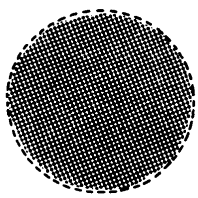
$B_r(\bar{\eta})$



$U_r(\beta)$

It is easy to see that the Gauss map image of a plane and a hyperbolic cylinder is the one point set $\{\bar{\eta}\}$ and the maximal geodesic β, respectively.

On the other hand, we know other examples of complete space-like surface with non-zero constant mean curvature, which are constructed by Treibergs and others. These examples are space-like surfaces of revolution in $\boldsymbol{R}_1^3$. The Gauss map images of these are classified into the following two types.



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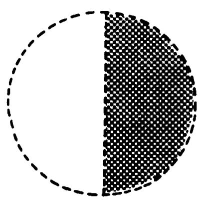

The domain rounded by ∂D and a geodesic in D

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